Diophantine Approximation
and
Related Topic

Aarhus, Denmark July 13 - July 17, 2015

Program and Abstract Book

List of Participants

Faustin Adiceam (Ireland) fadiceam@gmail.com
Cristoph Aistleitner (Austria) christoph.aistleitner@jku.at
Renat Akhunzhanov (Russia) akhunzha@mail.ru
Iskander Aliev (UK) AlievI@cardiff.ac.uk
Giedrius Alkauskas (Lithuania) giedrius.alkauskas@gmail.com
Jinpeng An (China) anjinpeng@gmail.com
Dzmitrii Badziahin (UK) dzmitry.badziahin@durham.ac.uk
Paloma Bengoechea (UK) paloma.bengoechea@york.ac.uk
Vasilii Bernik (Belarus) bernik.vasili@mail.ru
Dmitriy Bilyk (USA) bilyk.dmitriy@gmail.com
Sanda Bujacic (Croatia) sbujacic@math.uniri.hr
Yitwah Cheung (USA) ycheung.sfsu@gmail.com
Nicolas Chevallier (France) nicolas.chevallier@uha.fr
Henrietta Dickinson (Ireland) detta.dickinson@maths.nuim.ie
Edward Dobrowolski (Canada) Edward.Dobrowolski@unbc.ca
Lenny Fukshansky (USA) lenny@cmc.edu
Dmirty Gayfulin (Russia) gamak.57.msk@gmail.com
Oleg German (Russia) german.oleg@gmail.com
Stephen Harrap (UK) s.g.harrap@durham.ac.uk
Alan Haynes (UK) alan.haynes@york.ac.uk
Morten Hein Tiljeset (Denmark) s.g.harrap@durham.ac.uk
Thomas Jordan (UK) thomas.jordan@bristol.ac.uk
Henna Koivusalo (UK) henna.koivusalo@york.ac.uk
Simon Kristensen (Denmark) sik@math.au.dk
Michel Laurent (France) michel-julien.laurent@univ-amu.fr
Kalle Leppala (Denmark) kalle.leppala@math.au.dk
Antonie Marnat (France) amarnat@unistra.fr
Tapani Matala-aho (Finland) tapani.matala-aho@oulu.fi
Stephen McGuire (Ireland) stephenmcguire07@hotmail.com
Nikolay Moshchevitin (Russia) moshchevitin@gmail.com
Sara Munday (UK) magicdairyfairy@gmail.com
Radhakrishnan Nair (UK) nair@liverpool.ac.uk
Erez Nesharim (Israel) ereznesh@gmail.com
Yuri Nesterenko (Russia) nester@mi.ras.ru
Matthew Palmer (UK) mp12500@bristol.ac.uk
Steffen Hojris Pedersen (Denmark) steffenh@math.au.dk
Andrzej Schinzel (Poland) schinzel@impan.pl
Johannes Schleischitz (Austria) johannes.schleischitz@boku.ac.at
Denis Shatskov (Russia) studenthol@rambler.ru
David Simmons (USA) david9550@gmail.com
Fabian Sueess (UK) fs752@york.ac.uk
Leonhard Summerer (Austria) Leonhard.Summerer@univie.ac.at
Topi Torma (Finland) Topi.Torma@oulu.fi
Michel Waldschmidt (France) michel.waldschmidt@imj-prg.fr
Jamie Walton (UK) jamie.walton@york.ac.uk
Barak Weiss (Israel) barakw@cs.bgu.ac.il
Chris White (UK) Chris.White@bristol.ac.uk
Martin Widmer (UK) Martin.Widmer@rhul.ac.uk
Victoria Zhuravleva (Russia) victoria.zhuravleva@me.com

Schedule

Monday 13.07

09:30 - 10:10	Dzmitrii Badziahin	Cantor-winning sets and their applications in Diophantine Approximation
10:20 - 11.00	Barak Weiss	Badly approximable points on fractals
Coffee break		
11:30 - 12:10	Jinpeng An	Bounded orbits on the space of 3-dimensional unimodular lattices
12:20 - 13:00	Yitwah Cheung	Prescriptions for a diagonal flow on the space of lattices
Lunch		
15:30 - 16:00	Andrzej Schinzel	On the Diophantine equation $x^2 + x + 1 = yz$
16:10 - 16:40	Yuri Nesterenko	Diophantine approximations to Catalan's constant
Coffee break		
17:10 - 17:40	Tapani Matala-aho	On Siegel's lemma
17:50 - 18:20	Topi Torma	On irrationality exponents of generalized continued fractions with bounded partial coefficients
18:30 - 18:50	Steffen Pedersen	On a higher dimensional Mahler approximation

Tuesday 14.07

09:30 - 10:10	Radhakrishnan Nair	Haas-Molnar Continued Fractions and Metric Diophantine Approximation
10:20 - 11.00	Dmitriy Bilyk	Small ball inequality and low discrepancy constructions
Coffee break		
11:30 - 12:00	Cristoph Aistleitner	On parametric Thue-Morse Sequences and Lacunary Trigonometric Products
12:10 - 12:40	Alan Haynes	Open questions about higher dimensional gaps problems
12:50 - 13:10	Denis Shatskov	Oscillation of irrational measure function in the multidimensional case
Lunch		
15:30 - 16:00	David Simmons	Unconventional height functions in Diophantine approximation
16:10 - 16:40	Erez Nesharim	Weighted badly approximable vectors in fractals
Coffee break		
17:10 - 17:50	Leonhard Summerer	Recent advances in simultaneous approximation and the generalization of Jarnik's identity
18:00 - 18:30	Johannes Schleischitz	Diophantine approximation on manifolds
18:40 - 19:00	Renat Akhunzhanov	On multidimensional Lagrange and Dirichlet spectrums

Wednesday 15.07

09:30-10:10	Thomas Jordan	Fourier transforms and continued fractions
10:20 - 11.00	Giedrius Alkauskas	Transfer operator for the Gauss' continued fraction map. Structure of the eigenvalues and trace formulas
Coffee break		
11:30 - 12:00	Nicolas Chevallier	Asymptotic behavior of best simultaneous Diophantine approximations
12:10 - 12:50	Michel Laurent	Diophantine approximation by primitive points. Metrical aspects
13:00 - 13:30	Iskander Aliev	Lattice programming gaps and Frobenius numbers
Lunch		
Free afternoon (guided tour to museum)		
Barbeque		

Thursday 16.07

09:30 - 10:10	Lenny Fukshansky	Height bounds for zeros of quadratic forms
10:20 - 11.00	Vasilii Bernik	On the distribution of algebraic numbers of xed degree and bounded height
Coffee break		
11:30 - 12:00	Henrietta Dickinson	The distribution of algebraic conjugate points
12:10 - 12:40	Martin Widmer	Asymptotic Diophantine approximation: The multiplicative case
Lunch		
15:30 - 16:00	Victoria Zhuravleva	Diophantine approximation with Pisot numbers
16:10-16:40	Fabian Sueess	A refinement of the Khinchin-Jarnik Theorem on affine coordinate subspaces of Euclidean space
Coffee break		
17:10 - 17:40	Matthew Palmer	The Duffin-Schaeffer theorem in number fields
17:50 - 18:20	Sanda Bujacic	A variation of a congruence of Subbarao for $n = 2^\alpha 5^\beta, \alpha, \beta \geq 0$
18:30 - 19:00	Kalle Leppala	Hausdorff dimension of generalized continued fraction Cantor sets

Friday 17.07

09:30-10.10	Faustin Adiceam	A problem in Diophantine geometry : how far can you see in a forest?
10:20-10:40	Henna Koivusalo	Quasicrystals and the Littlewood conjecture
Coffee break		
11:10 - 11:50	Antonie Marnat	Parametric Geometry of Numbers and uniform exponents
12:00 - 12:30	Oleg German	Can multiparametric geometry of numbers solve Oppenheim conjecture?
Lunch		

ABSTRACTS

Faustin Adiceam, **A problem in Diophantine geometry : how far can you see in a forest?**

We will be answering the following question raised by Christopher Bishop :

'Suppose we stand in a forest with tree trunks of radius $r > 0$ and no two trees centered closer than unit distance apart. Can the trees be arranged so that we can never see further than some distance $V < \infty$, no matter where we stand and what direction we look in? What is the size of V in terms of r?'

The methods used to study this problem involve Fourier analysis and sharp estimates of exponential sums.

Christoph Aistleitner, **On parametric Thue-Morse Sequences and Lacunary Trigonometric Products.**

We investigate the distribution of the fractional parts of $(n_k\alpha)$, where $(n_k) = (0, 3, 5, 6, 9, \dots)$ denotes the Thue-Morse sequence of integers, that is the sequence of those positive integers which have an even sum-of-digits in base 2. We show how the problem of estimating exponential sums of $n_k\alpha$ can be transformed into the problem of estimating certain larcunary trigonometric products, and obtain sharp metric results for exponential sums of $n_k\alpha$ and for the discrepancy of this sequence. We also discuss connections with the theory of continued fractions and with metric Diophantine approximation. This talk is based on joint work with Roswitha Hofer and Gerhard Larcher.

Renat Akhunzhanov, **On multidimensional Lagrange and Dirichlet spectrums.**

The one-dimensional Lagrange spectrum is the set

$$\mathbb{L} = \left\{ \lambda \in \mathbb{R} \mid \exists v \in \mathbb{R} : \liminf_{q\to\infty} q||qv|| = \lambda \right\}.$$

It is well known that the discrete part of the Lagrang spectrum has a sequence

$$\frac{1}{\sqrt{5}}, \frac{1}{2\sqrt{2}}, \frac{5}{\sqrt{221}}, \frac{13}{\sqrt{1517}}, \dots$$

This sequence converges to $\frac{1}{3}$. Also, the Lagrange spectrum has a segment $[0, \lambda^*]$, which calls Hall's ray (Hall first proved that $\lambda^* > 0$).

We consider the multi-dimensional Lagrange spectrum

$$\mathbb{L}_s = \left\{ \lambda \in \mathbb{R} \mid \exists \mathbf{v} \in \mathbb{R}^s : \liminf_{q\to\infty} q^{1/s} \max_{1\leqslant i\leqslant s} ||qv_i|| = \lambda \right\}.$$

The one-dimensional Dirichlet spectrum $\mathbb{D}$ is defined as follows:

$$\mathbb{D} = \left\{ \lambda \in \mathbb{R} \mid \exists v \in \mathbb{R} : \limsup_{t\to\infty} t \cdot \min_{1\leqslant q\leqslant t} ||qv|| = \lambda \right\}.$$

The structure of the one-dimensional Dirichlet spectrum was studied by many mathematicians. In particular $\mathbb{D} \subset [1/2 + 1/2\sqrt{5}, 1]$ and there exists so-called "discrete part of the spectrum". Moreover it is known that for a certain d^* one has $[d^*, 1] \subset \mathbb{D}$.

We consider s-dimensional Dirichlet spectrum with respect to Euclidean norm defined as

$$\mathbb{D}_s = \left\{ \lambda \in \mathbb{R} \mid \exists \mathbf{v} \in \mathbb{R}^s : \limsup_{t\to\infty} t \cdot \min_{1\leqslant q\leqslant t} \left(\sum_{i=1}^{s} ||qv_i||^2 \right)^{s/2} = \lambda \right\}.$$

Multidimensional spectrum has rather different properties. In this talk we annonce the following results.

Theorem I(Akhunzhanov 2013). *For any* $t \leqslant \frac{1}{6\cdot 4^s\cdot R\cdot C(s)(2s+3)}$ *the* s*-dimensional Lagrange spectrum has* $\lambda \in \mathbb{L}_s$ *where* $t \le \lambda \le t(1+16B\cdot t^{1+1/s})$.

Theorem II(Akhunzhanov, Shatskov 2013).

$$\mathbb{D}_2 = \left[0, \frac{2}{\sqrt{3}}\right].$$

Iskander Aliev, **Lattice programming gaps and Frobenius numbers.**

Given a full-dimensional lattice $\Lambda \subset \mathbb{Z}^k$ and a vector $l \in \mathbb{Q}^k_{>0}$, we consider the family of the lattice problems

$$\min\{l\cdot x : x \equiv r \pmod{\Lambda}, x \ge 0\}\,, \quad r \in \mathbb{Z}^k. \tag{1}$$

The *lattice programming gap* $\mathrm{gap}(\Lambda, \mathrm{l})$ is the largest value of the minima in (1) as r varies over $\mathbb{Z}^k$. Hoşten and Sturmfels proved in 2007 (in a more general setting) that $\mathrm{gap}(\Lambda, l)$ can be computed in polynomial time when the dimension k is fixed. We obtain lower and upper bounds for $\mathrm{gap}(\Lambda, \mathrm{l})$ in terms of l and the determinant of Λ. We also show that computing the lattice programming gap is NP-hard when k is a part of input. The proofs are based on recent results on the Frobenius numbers.

Giedrius Alkauskas, **Transfer operator for the Gauss' continued fraction map. Structure of the eigenvalues and trace formulas.**

Let L be the transfer operator associated with the Gauss' continued fraction map, known also as the Gauss-Kuzmin-Wirsing operator, acting on the Banach space. Due to important contributions by R.O. Kuzmin, E. Wirsing, J. Lewis and D. Zagier, D. Mayer, K.I. Babenko, Ph. Flajolet, B. Vallee, and others, we know that this operator is deeply related to the Riemann and Selberg zeta functions, modular and Maass forms for the full modular group.

In this talk we present the recent result that provides two exact terms of the asymptotic formula for the eigenvalues of L. This settles, in a much stronger form, the conjectures raised by several mathematicians in 1988-1995. Further, we find an exact series for the eigenvalues, which also gives the canonical decomposition of the trace formulas due to D. Mayer and K.I. Babenko. This crystallizes the contribution of each individual eigenvalue in the trace formulas. Algebraic numbers and real numbers from the extended ring of periods enter the topic, and this gives an arithmetic point of view on the spectrum of L.

Jinpeng An, **Bounded orbits on the space of 3-dimensional unimodular lattices.**

We show that for any countably many one-parameter diagonalizable subgroups F_n of SL(3,R), the set of points in the space of 3-dimensional unimodular lattices such that all the F_n-orbits are bounded has full Hausdorff dimension. The proof involves hyperplane absolute winning (HAW) property and Diophantine approximation. This is a joint work with Lifan Guan and Dmitry Kleinbock.

Dzmitry Badziahin, Cantor-winning sets and their applications in Diophantine Approximation.

Winning sets were invented by W. Schmidt in 1960's. On one hand they share several remarkable properties: 1. They have full Hausdorff dimension. 2. Countable intersection of winning sets is again winning and therefore has full Hausdorff dimension. 3.The property of being winning is invariant under bi-Lipschitz homeomorphisms. On the other there are natural examples of Schmidt winning sets in the area of Diophantine approximation. For example the set of badly approximable real numbers and more generally the set of badly approximable points in R^n is winning. This immediately gives a lot of information about the structure of these sets.

Quite recently various sets were introduced in Diophantine approximation which share similar properties to 1 and 2 of Schmidt winning sets but can not be shown (at least straightforwardly) to be Schmidt winning. For example in 2014 it was shown by Badziahin, Velani and independently by Beresnevich that a countable intersection of the following sets $Bad(i, j)$ on any non-degenerate two-dimensional curve L has full Hausdorff dimension.

In the talk I shall introduce a notion of Cantor-winning sets and show that they share the same properties 1 – 3 as Schmidt winning sets. Finally we relate Cantor-winning sets with so called generalised badly approximable sets appeared in Diophantine approximation. In particular they include the example given above.

Vasily Bernik, On the distribution of algebraic numbers of fixed degree and bounded height.

Let $P(x)$ be an integer polynomial of degree n, let $H = H(P) = \max_{0\le j\le n} |a_j|$ be its height and let $\alpha_1, \dots, \alpha_n$ be the roots of the polynomial. For integer $Q > 1$ lets consider the following class of polynomials

$$\mathcal{P}_n(Q) = \{P(x) \in \mathbb{Z}[x] : \deg P = n, H(P) \le Q\}.$$

Denote by $\#A$ the number of elements in a finite set A and by $c_1 = c_1(n), c_2, \dots$ the values which depend on n only.

If $P(x) \in \mathcal{P}_n(Q)$ then some facts about the distribution of the roots of the polynomial $P(x)$ are known.

1. $\#\{\alpha_j, P \in \mathcal{P}_n(Q)\} \le n(2Q + 1)$
2. $|\alpha_j| < Q + 1$
3. $\#\{\alpha_j \cap R\} > c_1 Q^{n+1}$
4. If $\alpha_i \ne \alpha_j$ and α_i, α_j are roots of polynomial $P(x) \in \mathcal{P}_n(Q)$ then $|\alpha_i - \alpha_j| > c_2 Q^{-n+1}$ and if α is a root of polynomial $P_1(x) \in \mathcal{P}_n(Q)$, β is a root of polynomial $P_2(x) \in \mathcal{P}_n(Q)$ and $\alpha \ne \beta$ then $|\alpha - \beta| > c_3 Q^{-2n}$.

During the last 15 years more than ten articles have been published presenting more precise and more complex results concerning the distribution of algebraic numbers on the real line and on the complex plane. In particular, it was proved that the sequence of algebraic numbers is not uniformly distributed if ordered naturally (Mahler's hypothesis, 1985); algebraic numbers form a regular system on intervals I, $|I| = c_4 Q^{-1}$ (Bugeaud's hypothesis [1]); algebraic numbers form a regular system on special intervals I, $|I| = Q^{-\gamma}$ which don't contain rational points with small denominator for $\gamma > 1$ [2].

These results were obtained in the articles of H. Husakova, V. Bernik, F. Goetze, D. Kaliada.

Dmitriy Bilyk, Small ball inequality and low discrepancy constructions.

The so-called "small ball inequality" is a lower bound for the sup-norm of "hyperbolic" linear combinations of multiparameter Haar functions (or other wavelets). It bears resemblance to the classical Sidon theorem on lacunary Fourier series, and is closely connected to some questions in probability and approximation theory. A few years ago, it was discovered that this inequality is also related to discrepancy theory. Although the relation was only heuristic, it gave way to a significant improvement on the lower bounds of the sup-norm of the discrepancy function in higher dimensions. In recent joint work with N. Feldheim we have found some new proofs of the two-dimensional version of this inequality, which have revealed a new (and this time formal) connection between the small ball inequality and discrepancy theory: the extremal sets arising in the inequality are precisely the binary nets: sets with of $N = 2^n$ points, which are perfectly distributed with respect to dyadic rectangles, i.e. every binary box of volume $1/N$ contains exactly one point. Conversely, every binary net can be obtained in such a way. We shall survey recent and older results on this topic, and indicate some approaches to the higher-dimensional problems.

Sanda Bujačić, A variation of a congruence of Subbarao for $n = 2^\alpha 5^\beta, \alpha, \beta \geq 0$.

Euler's totient function of a positive integer $n > 1$, denoted as $\varphi(n)$, is defined to be the number of positive integers less than n that are coprime to n. Function σ of a positive integer n is defined to be the sum of all the positive integer divisors of n.

There are many open problems concerning the characterization of the positive integers n fulfilling certain congruences involving $\varphi(n)$ and $\sigma(n)$. M. V. Subbarao deals with the problem finding composite integers n such that

$$n\sigma(n) \equiv 2 \pmod{\varphi(n)}.$$

A. Dujella and F. Luca deal with the similar problem, or more precisely, they deal with the congruence which is similar to Subbarao's congruence, namely

$$n\varphi(n) \equiv 2 \pmod{\sigma(n)} \tag{1}$$

They prove that there are only finitely many such n whose prime factors belong to a fixed finite set $\mathcal{P}$.

In this work, I deal with the congruence (1) and I prove that, if the fixed finite set of prime factors of a positive integer n is $\mathcal{P} = \{2, 5\}$, then there are only finitely many positive integers of the form $2^\alpha 5^\beta$, $\alpha, \beta \geq 0$, that satisfy the congruence (1). Those numbers are $n = 1, 2, 5, 8$.

Yitwah Cheung, Prescriptions for a diagonal flow on the space of lattices.

Given a lattice in R^{d+1}, we consider the possibilities for the evolution of the $(d+1)$-tuple of successive minima under the action of the one-parameter subgroup $diag(e^t, ..., e^t, e^{-dt})$. In particular, for orbits that are divergent, there is a notion of a "limit set" that is independent of choice of norm, and we are interested in the possibilities for this set. In joint work with Barak Weiss, we introduce the notion of a "prescription" which is a $(d+1)$-tuple of piecewise linear functions representing an idealized behavior of the logarithms of the successive minima, which under certain separation conditions we show is realized up bounded error by the evolution of the successive minima of an actual lattice. Our work may be thought as a refinement of a recent result of Damien Roy. As an application, we construct arbitrarily slowly divergent trajectories satisfying a conjecture of Schmidt proved by Moshchevitin in 2008.

Nicolas Chevallier, **Asymptotic behavior of best simultaneous Diophantine approximations.** It is a joint work with Yitwah Cheung. We extend to best simultaneous Diophantine approximations, two results about the ordinary continued fraction expansion. The first is the classical Levy-Khintchin result about the growth rate of the denominators and the second is a result of Bosma, Jager and Wiedijk about the distribution of $q_n|q_n\theta - p_n|$. In both cases, the idea is to state an intermediate result in the space of unimodular lattices.

Detta Dickinson, **The distribution of algebraic conjugate points.**
This is joint work with V. Bernik and N. Budarina.

Consider a point $(a_1, a_2, a_3) \in [0,1]^3$ such that $P(a_i) = 0$ for some polynomial $P \in \mathbf{Z}[x]$ of degree $n \geq 3$ and height Q. In this talk the distribution of such point will be discussed and in particular it will be shown that in "most" boxes in $\mathbf{R}^3$ of volume Q^{-1} there are at least cQ^{n-2} such points for some constant $c > 0$. An informal explanation of the proof will be given indicating some of the main problems and methods involved. Most of the work in the proof is in demonstrating that there exists a set of points (x_1, x_2, x_3) of measure at least $Q^{-1}/2$ in each such box which satisfy the inequalities

$$|P(x_i)| < Q^{-\tau_i} \quad \text{and} \quad |P'(x_i)| \gg Q$$

for some polynomial $P \in \mathbf{Z}[x]$ of degree n and height at most Q.

Lenny Fukshansky, **Height bounds for zeros of quadratic forms.**
A celebrated theorem of Cassels (1955) asserts that an isotropic rational quadratic form necessarily has a non-trivial integral zero of relatively small height. This important result opened a field of effective investigations with respect to height in the theory of quadratic forms, and has had numerous extensions and generalizations over the years by a variety of authors. In this talk, I will discuss some of the recent results on existence and distribution properties of small-height zeros of quadratic forms over global fields and the field of algebraic numbers.

Oleg German, **Can multiparametric geometry of numbers solve Oppenheim conjecture?**
The Oppenheim conjecture for the product of linear forms claims that for $n \geqslant 3$ the product of n linearly independent homogeneous linear forms $L_1(\mathbf{x}), \ldots, L_n(\mathbf{x})$ in n variables is bounded away from zero in nonzero integer points if and only if the lattice

$$\Lambda = \left\{ \big(L_1(\mathbf{x}), \ldots, L_n(\mathbf{x})\big) \,\middle|\, \mathbf{x} \in \mathbb{Z}^n \right\}$$

is diagonally equivalent to an algebraic one. This conjecture is known to imply the Littlewood conjecture.

The aim of the talk is to speculate on possible ways to construct a counterexample with the help of multiparametric.

Alan Haynes, **Open questions about higher dimensional gaps problems.**
We will present several open problems which arose in the study of higher dimensional versions of the classical 3-gaps (Steinhaus) theorem. The problems are very simple to state, but seem to be quite difficult to resolve. In fact, as we will explain, one of them turns out to be closely related to a famous open problem in Diophantine approximation.

Thomas Jordan, **Fourier transforms and continued fractions.**

Joint work with Tuomas Sahlsten. A result of Kaufman shows that there is a probability measure which gives full measure to the set of badly approximable numbers and for which the Fourier transform of the measure decays polynomially. For such a measure almost every point x will satisfy that for $m > 1$, an integer, $\{m^k x mod 1\}_{k\in\mathbb{N}}$ will be equidistributed in $[0, 1]$. We will show that the methods Kaufman used can be adapted to show that a class of invariant measures for the Gauss map also have Fourier transform which decay polynomially. This class includes the Fourier-Stieltjes measure for the Minkowski question mark function.

Henna Koivusalo, **Quasicrystals and the Littlewood conjecture.**

Cut and project sets give a way of defining discrete point patterns through a data of a linear subspace and an acceptance strip. The points in the pattern are given by first intersecting the integer lattice with the acceptance strip (cut) and then projecting these intersection point to the subspace (project). We establish a connection between finite patches in cut and project sets and an action of a toral rotation defined by the cut and project data. The existence of certain very well ordered cut and project sets turns out to be equivalent to the negation of the Littlewood conjecture. The work is joint with Alan Haynes and James Walton.

Michel Laurent, **Diophantine approximation by primitive points. Metrical aspects.**

We are interested in variants of the classical Khintchine-Groshev theorem in metrical diophantine approximation in which we impose to the integer solutions to be primitive in various meaning. In other words, we intend to solve systems of linear equations, homogeneous or non-homogeneous, in coprime integers, or satisfying finer coprimality conditions. We shall give several statements extending in this sense the Khintchine-Groshev theorem. We also link the topic with orbits of unimodular groups acting on spaces of real matrices, recalling first an early result of Dani-Raghavan. In this situation effective density results, even generic, are far from being complete. We shall present some known results arising from various techniques.

Kalle Leppälä, **Hausdorff dimension of generalized continued fraction Cantor sets.**

Let A and B be two finite sets of positive integers. We use Bowen's equation for calculating the Hausdorff dimension of the set of all possible values of the generalized continued fractions $a_1/(b_1 + a_2/(b_2+))$, where $a_n : s$ are from A and $b_n : s$ are from B.

Antoine Marnat, **Parametric Geometry of Numbers and uniform exponents.**

In this lecture, we introduce the parametric geometry of numbers and its recent application to the study of spectra of exponents of Diophantine approximation. We then focus on the generalization of Jarnik's relation $\hat{\lambda} + 1/\hat{\omega} = 1$, valid in dimension 2, to the higher dimension.

Tapani Matala-aho, **On Siegel's lemma.**

Bombieri-Vaaler in [Invent. math. 73 (1983), 11-32] proved a refined version of Siegel's lemma. We will talk about on a short proof of this refined version over rational integers.

Radhakrishnan Nair, **Haas-Molnar Continued Fractions and Metric Diophantine Approximation.**

A class of Möbius function maps of the unit interval were introduced by A. Haas and D. Molnar. These include the regular continued fraction map and A. Renyi's backward continued fraction map as important special cases. As shown by Haas and Molnar, it is possible to extend the theory of metric diophantine already well developed for the regular continued fraction map to the class of Haas-Molnar maps. In particular for a real number x, suppose $\left(\frac{p_n}{q_n}\right)_{n\geq 1}$ denotes its sequences of regular continued fraction convergents, and set $\theta_n(x) = q_n^2 \left|x - \frac{p_n}{q_n}\right|$ $(n = 1, 2 \ldots)$. Following a conjecture of H. W. Lenstra, the metric behaviour of the Cesàro averages of the sequence $(\theta_n(x))_{n\geq 1}$ has been studied by a number of authors. Haas and Molnar have extended this study to the analogues of the sequence $(\theta_n(x))_{n\geq 1}$ for the Haas-Molnar family of continued fraction expansions. In this talk we discuss the behavious of $(\theta_{k_n}(x))_{n\geq 1}$ for certain sequences of natural numbers $(k_n)_{n\geq 1}$, in the context of Haas-Molnar maps. The new work in this talk is joint with Liangang Ma

Erez Nesharim, **Weighted badly approximable vectors in fractals.**

Diophantine approximation on fractals is by now a legitimate area of research. One may ask whether approximations of elements in the fractal exhibit phenomena similar to those appearing in the classical treatment. Of course, this depends on the assumptions one is willing to make regarding the fractal. I will discuss the property of having "many" weighted badly approximable elements. For a wide class of fractals, this may be deduced from proving that the set of weighted badly approximable vectors is hyperplane absolute winning. Based on the work of Jinpeng An, we were able to prove this, by using yet another version of Schmidt's game called the "potential game". In other cases, such a claim might still be far from being proved. I will describe a more direct method of finding many badly approximable elements in fractals. I will indicate the application and the limits of this method in the context of badly approximable elements on manifolds. This work is part of my PhD research, done under the supervision of Prof. Barak Weiss. Part of this talk is based on a joint work with David Simmons.

Yuri Nesterenko, **Diophantine approximations to Catalan's constant.**

We will discuss a hypergeometric construction of rational approximations to Catalan's constant $G = \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$. These approximations are still insufficient to prove irrationality of G. This is a joint work with C. Viola.

Matthew Palmer, **The Duffin-Schaeffer theorem in number fields.**

In this talk, we develop a version of the Duffin-Schaeffer theorem in general number fields. This work builds on Nakada & Wagner's result in imaginary quadratic fields, and Cantor's version of Khintchine's theorem in general number fields.

Steffen Pedersen, **On a higher dimensional Mahler approximation.**

Mahler defined for $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$ a diophantine exponent

$$\omega_n(\alpha) = \sup\{\omega \in \mathbb{R} : 0 < |P(\alpha)| \leq H(P)^{-\omega}, \\ \text{for infinitely many } P \in \mathbb{Z}[X], \text{ with } \deg P \leq n\},$$

where $H(P)$ is the maximum of the absolute value of the coefficients of P, called the height of P. $\omega_n(\alpha)$ measures how well α can be approximated by algebraic numbers of degree at most n.

In this talk I will present several results on a higher dimensional version of Mahler's approximation, introduced by K. Yu, where we evaluate points $\alpha \in \mathbb{R}^d$ in polynomials $P \in \mathbb{Z}[X_1, \ldots, X_d]$ of total degree at most k.

This is ongoing research joint with Simon Kristensen and Barak Weiss.

Andrzej Schinzel, **On the Diophantine equation** $x^2 + x + 1 = yz$.

Theorem. All solutions of the equation $x^2 + x + 1 = yz$ in non-negative integers x, y, z such that $y \leq z$ and only those are given by the formulae $x = A_{(k-1)}A_k + B_{(k-1)}B_k + A_kB_{(k-1)}$,
$y = A_{(k-1)^2} + A_{(k-1)}B_{(k-1)} + B_{(k-1)^2}$,
$z = A_k^2 + A_kB_k + B_k^2$, where A_k, B_k are numerators and denominators, respectively, of the continued fraction $[b_0, b_1, ..., b_k]$, $k \geq 0$ is even, b_0 is an integer and $b_i (i = 1, ..., k)$ are positive integers,except if $k = 0$, $b_0 < 0$, one has to take $x = |b_0| - 1$.

Johannes Schleischitz, **Diophantine approximation on manifolds.**

The talk deals with the rational approximation properties of a special class of smooth manifolds in Euclidean space $\mathbb{R}^k$. Budarina, Dickinson and Levesley considered curves parametrized by polynomials in one variable with integral coefficients, i.e. of the form $\{(X, P_2(X), \ldots, P_k(X)) : X \in [0, 1]\}$ with $P_1(X) = X$ and $P_j \in Z[X]$ for $2 \leq j \leq k$. For sufficiently large parameters τ, depending (solely) on the degrees of the polynomials, they succeeded in determining the Hausdorff dimension of points on such a curve that are approximable to degree τ. The talk is dedicated to a recent improvement of this result by lowering the bound. A particularly good (in fact somehow best possible) reduction of the bound is obtained for the Veronese curve $P_j = X^j$ for $2 \leq j \leq k$, where the bound τ is decreased from $k - 1$ to 1. The method of the proof, which differs from the method in the mentioned paper, will be sketched in the special case of the Veronese curve. If time suffices, possible generalizations to the case of polynomials in more variables, which leads to more general manifolds, will be discussed.

Denis Shatskov, **Oscillation of irrational measure function in the multidimensional case.**

Let Θ is a matrix of size $m \times n$. We denote the irrational measure function, as

$$\psi_\Theta(t) = \min_{\substack{x_i \in \mathbb{Z} \\ 1 \leqslant \max\limits_{1 \leqslant i \leqslant n} |x_i| \leqslant t}} \max_{1 \leqslant j \leqslant m} \|\theta_j^1 x_1 + \ldots + \theta_j^n x_n\|.$$

We proved that difference function $\psi_\Theta - \psi_{\Theta'}$ for almost all pairs Θ, Θ' in cases $m = 1$, $n = 2$ or $m \geqslant 2$ and $n = 1$ changes its sign infinity many times as $t \to +\infty$.

David Simmons, **Unconventional height functions in Diophantine approximation.**

The standard height function $H(\mathbf{p}/q) = q$ of simultaneous approximation can be calculated by taking the LCM (least common multiple) of the denominators of the coordinates of the rational points: $H(p_1/q_1, \ldots, p_d/q_d) = \mathrm{lcm}(q_1, \ldots, q_m)$. If the LCM operator is replaced by another operator such as the maximum, minimum, or product, then a different height function and thus a different theory of simultaneous approximation will result. In this talk I will discuss some basic results regarding approximation by these nonstandard height functions, as well as mentioning their connection with intrinsic approximation on Segre manifolds using standard height functions. This work is joint with Lior Fishman.

Fabian Sueess, **A refinement of the Khinchin-Jarnik Theorem on affine coordinate subspaces of Euclidean space.**

Joint work with Jon Chaika, Felipe Ramirez and David Simmons We show that affine coordinate subspaces of dimension at least two are of Khinchin type for divergence. We also obtain partial results for lines in the plane, which provide evidence that the aforementioned statement might be true for one-dimensional subspaces. Furthermore, we get partial results towards a Jarnik-type analogue.

Leonhard Summerer, **Recent advances in simultaneous approximation and the generalization of Jarnik's identity.**

In this talk I will give an overview of some recent advances in simultaneous approximation related to the geometric approach introduced by Schmidt and the speaker. In particular, I will focus on several aspects of a Theorem of D. Roy which in some sense completes this geometric method. Its remarkable consequences will be illustrated by means of the example of O. German's improvement of Jarnik's Theorem on the relation between uniform approximation constants.

Topi Törmä, **On irrationality exponents of generalized continued fractions with bounded partial coefficients.**

In this talk I shall present a theorem for obtaining an upper bound for the asymptotic irrationality exponent of certain generalized continued fractions with bounded partial coefficients. I'll also show that for every $s \in \{1\} \cup [2, \infty]$ there exists a number in the set

$$\left\{ \tau \in \mathbb{R} \mid \tau = \underset{n=1}{\overset{\infty}{K}} \left(\frac{a_n}{1} \right),\ a_n \in \{1, 2\} \right\}$$

with asymptotic irrationality exponent s. Furthermore the size of this set will be discussed.

Barak Weiss, **Badly approximable points on fractals.**

We prove that for a large class of fractals, with respect to the natural Hausdorff measure, almost every point is not badly approximable. This is in contrast with results asserting that this set is winning for certain games, and generalizes results of Einsiedler, Fishman and Shapira for the middle-thirds Cantor set. The proof is based on a result classifying stationary measures for some random walks on the space of lattices. Joint work with David Simmons.

Martin Widmer, Asymptotic Diophantine approximation: The multiplicative case.

Let α and β be real irrational numbers and $0 < F < 1/30$. We discuss a precise estimate for the number of positive integers $q \leq Q$ that satisfy $\|q\alpha\|\|q\beta\| < F$. If we choose F as a function of Q we get asymptotics as Q gets large, provided FQ grows quickly enough in terms of the (multiplicative) Diophantine type of (α, β), e.g., if (α, β) is a counterexample to Littlewood's conjecture then we only need that FQ tends to infinity. Our result yields a new upper bound on sums of reciprocals of products of fractional parts, and sheds some light on a recent question of Lê and Vaaler.

Victoria Zhuravleva, Diophantine approximation with Pisot numbers.

Let $X = (x_n)_{n=1}^{\infty}$ be a sequence of real numbers satisfying linear recurrence

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \ldots + a_k x_{n-k} + \ldots + a_{d-1} x_{n-d+1} + a_d x_{n-d}.$$

We consider the case when the characteristic polynomial of X is an irreducible polynomial in $\mathbb{Z}[x]$ which is the minimal polynomial of a Pisot number α.

Let

$$L(X) := \sup_{\xi \in \mathbb{R}} \liminf_{n \to \infty} ||\xi x_n||, \qquad L(\alpha) := \sup_{\xi \in \mathbb{R}} \liminf_{n \to \infty} ||\xi x_n||.$$

It's easy to see that

$$L(X) = \begin{cases} 0, & \text{if } x_n \to 0 \text{ as } n \to \infty, \\ L(\alpha) > 0, & \text{if } x_n \nrightarrow 0 \text{ as } n \to \infty. \end{cases}$$

The inequality $L(\alpha) > 0$ holds since the sequence X is lacunary when $x_n \nrightarrow 0$ as $n \to \infty$. We call a sequence X lacunary if there exists $\lambda > 1$ such that for all sufficiently large n we have $|x_{n+1}| > \lambda |x_n|$. For any lacunary secuence X A.Y. Khintchin proved that there exist $\xi \in \mathbb{R}$ and $\gamma > 0$ such that for all $n \in \mathbb{N}$ we have $||\xi x_n|| \geq \gamma$.

Until recently, all known results were just estimates for $L(\alpha)$ if $\alpha \notin \mathbb{Z}$. The first exact value was obtained for $\alpha = \frac{\sqrt{5}+1}{2}$ in [3]. It turned out that in this case $L(\alpha) = \frac{1}{5}$. Then the value of $L(\alpha)$ was calculated explicitly in several other cases ([1], [2], [4]).

Here is a list of the main results.

- If $\sum_{i=1}^{d} a_i$ is odd, then
$$L(\alpha) = \frac{1}{2};$$
- If $\sum_{i=1}^{d} a_i$ is even, then
$$L(\alpha) \leq \frac{\sum_{i=1}^{d} |a_i|}{2 \sum_{i=1}^{d} |a_i| + 2};$$
- If α is a Pisot number of degree ≤ 3, then $L(\alpha) \geq \frac{1}{5}$;
- If α is a Pisot number of degree ≤ 4, then $L(\alpha) \geq \frac{3}{17}$;
- If α is a Pisot number less than $\frac{\sqrt{5}+1}{2}$, then $L(\alpha) \geq \frac{3}{17}$.

References

[1] Dubickas A. *Distribution of some quadratic linear recurrence sequences modulo 1*, Carpathian Journal of Mathematics 30 (1) (2014), 79–86.

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[3] Zhuravleva V. *Diophatine approximations with Fibonacci numbers*. J. de Theor. des Nombres de Bordeaux 25 (2013), 499–520.

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