On distribution of algebraic numbers in the domains of small measure in the fields of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}_p$

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In the paper [1] it is shown that for any sufficiently large $Q \in \mathbb{N}$ there are intervals I with length $|I| = Q^{-1}/2$ which do not contain real algebraic numbers α of degree $\deg \alpha = n$ and height $H(\alpha) \le Q$. It was also shown that if $|I| > Q^{-1+\epsilon}$ for $\epsilon > 0$ then there is at least $c_0Q^4|I|$ of real algebraic numbers $\alpha \in I$ such that $\deg \alpha = 3$ and $H(\alpha) \le Q$.

In the talk we will discuss a generalizations of the above results on the sets of algebraic numbers of arbitrary degree in the fields of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}_p$. In this abstract we formulate the results for the field of p-adic numbers.

Y. Bugeaud in [2] stated the problem of the length of the interval depends on the height of algebraic numbers, which form the regular system on this interval. In [2] it is shown that for a given finite interval I in $[-1/2, 1/2]$ the value of $T_0(\Gamma, N(\alpha), I)$ in the definition of regular system is equal to

$$T_0(\mathbb{Q}, N(\alpha), I) = 10^4|I|^{-2}\log^2 100|I|^{-1}$$

for $n = 1$, and in [3] that

$$T_0(A_2, N(\alpha), I) = 72^3|I|^{-3}\log^3 72|I|^{-1}$$

for $n = 2$, where A_k is the set of real algebraic numbers of degree k. In [1] it is shown that $T_0(A_3, N(\alpha), I) = c_1|I|^{-4-\epsilon}$, $0 < \epsilon < 1$. Probably, there is a more strong connection between I and $T_0(A_n, N(\alpha), I)$, namely $T_0(A_n, N(\alpha), I) = c_2|I|^{-(n+1)}$. In this paper, we address to Y. Bugeaud's problem for the p-adic numbers with arbitrary n.

Throughout $c_1 = c_1(n)$, $c_2 = c_2(n)$, ... are constants depending only on n. The Haar measure of a measurable set $S \subset \mathbb{Q}_p$ is denoted by $\mu(S)$. Let $\mathcal{A}_p$ be the set of all algebraic numbers and $\mathbb{Q}_p^*$ is the extension of $\mathbb{Q}_p$ containing $\mathcal{A}_p$. Denote by $\mathcal{A}_{n,p}$ the set of algebraic numbers of degree n lying in $\mathbb{Z}_p$. Let K_0 be a disc in $\mathbb{Z}_p$. The natural number $H(\alpha)$ denotes the height of $\alpha \in \mathcal{A}_p$, which is the absolute height of the minimal polynomial of α.

Theorem 1. *Let K be a finite cylinder in K_0. Then there are positive constants c_3, c_4 and a positive number $T_0 = c_4\mu(K)^{-(n+1)}$ such that for any $T \ge T_0$ there exist numbers*

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$\alpha_1, \ldots, \alpha_{\mathbf{t}} \in \mathcal{A}_{n,p} \cap K$ *such that*

$$\begin{gathered} H(\alpha_i) \le T^{1/(n+1)} \ (1 \le i \le \mathbf{t}), \\ |\alpha_i - \alpha_j|_p \ge T^{-1} \ (1 \le i < j \le \mathbf{t}), \\ \mathbf{t} \ge c_3 T \mu(K). \end{gathered} \tag{1}$$

Note that from Theorem 1 it follows that the set $\mathcal{A}_{n,p}$ with the function $N(\alpha) = H^{n+1}(\alpha)$ form a regular system in K_0.

For positive integer Q and $c_5, c_6 \in \mathbb{R}^+$, $c_7 \in \mathbb{R}^+ \cup \{0\}$ define the set of polynomials

$$\mathcal{P}_n(c_6 Q^{c_7}) = \{P \in \mathbb{Z}[x] : \deg P = n, \ \ c_5 Q^{c_7} < H(P) \le c_6 Q^{c_7}, \ \ c_5 < c_6\}. \tag{2}$$

Let $\delta, d_n, c_8 \in \mathbb{R}^+$ and $0 \le r_n \le 1$. Denote by $\bar{\mathcal{L}}_n = \bar{\mathcal{L}}_n(c_6 Q^{r_n}, \delta, K)$ the set of $w \in K$ for which the system of the inequalities

$$|P(w)|_p < c_8 Q^{-d_n}, \quad |P'(w)|_p \le \delta, \tag{3}$$

has a solution in polynomials $P \in \mathcal{P}_n(c_6 Q^{r_n})$.

The proof of Theorem 1 is based on the following metric result.

Theorem 2. *For any real number s, where $0 < s < 1$, and for any cylinder K in K_0 there exists a sufficiently large number $Q_0 = Q_0(K)$ such that for*

$$\mu(K) > c_9 Q_0^{-1}, \ \ d_n \ge n + r_n, \ \ \delta \le 2^{-n-5} c_6^{-n-1} c_8^{-1} s^2 (f(n))^{-2}$$

and sufficiently large constant c_9, which does not depend on Q_0, and for all $Q > Q_0$

$$\mu(\bar{\mathcal{L}}_n) < s\mu(K) \tag{4}$$

holds.

From above it follows that the cylinder K with $\mu(K) > c_9 Q^{-1}$ for sufficiently large c_9 contains $\gg Q^{n+1}\mu(K)$ algebraic p-adic numbers of degree n and $H(\alpha) \le Q$. Note that if $\mu(K) \le \frac{1}{2}Q^{-1}$ then the following result holds which is the complement of Theorem 1 in some sense.

Theorem 3. *For any $Q \in \mathbb{N}$ there exist the cylinders K with $\mu(K) \le \frac{1}{2}Q^{-1}$ which do not contain algebraic numbers $\alpha \in \mathbb{Q}_p$ of degree $\deg \alpha = n$, $n \ge 1$, and $H(\alpha) \le Q$.*

Proof. For the given Q choose $s \in \mathbb{N}$ satisfying the inequality $p^{-s} < \frac{1}{2}Q^{-1}$. Consider the cylinder $K = K(p^s, \frac{1}{2}Q^{-1})$. Let $\alpha \in K$ be an algebraic number of degree $\deg \alpha = n$, $n \ge 1$, and $H(\alpha) \le Q$. It means that $\alpha \in \mathbb{Q}_p$, $\alpha \ne 0$, is a root of irreducible polynomial $P(x) = \sum_{i=0}^{n} a_i x^i$. If we assume that $a_0 = 0$ then from $P(\alpha) = 0$ it follows that $\alpha(\sum_{i=1}^{n} a_i \alpha^{i-1}) = 0$. Last implies that α is a root of polynomial $P_1(x) = \sum_{i=1}^{n} a_i x^{i-1}$ of $\deg P_1 \le n-1$ which contradicts to the fact that $\deg \alpha = n$. Therefore, $a_0 \ne 0$ and from

$$a_0 = -\alpha \sum_{i=1}^{n} a_i \alpha^{i-1},$$

we obtain

$$Q^{-1} \le |a_0|_p \le |\alpha|_p \max_{1 \le i \le n} |a_i \alpha^{i-1}|_p \le \frac{1}{2} Q^{-1},$$

which is a contradiction. This completes the proof of Theorem 3. □

References

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