On distribution of algebraic numbers in the domains of small measure in the fields of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}_p$

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January 9, 2014

In the paper [1] it is shown that for any sufficiently large $Q \in \mathbb{N}$ there are intervals $I$ with length $|I| = Q^{-1}/2$ which do not contain real algebraic numbers $\alpha$ of degree $\deg \alpha = n$ and height $H(\alpha) \leq Q$. It was also shown that if $|I| > Q^{-1+\epsilon}$ for $\epsilon > 0$ then there is at least $c_0 Q^3 |I|$ of real algebraic numbers $\alpha \in I$ such that $\deg \alpha = 3$ and $H(\alpha) \leq Q$.

In the talk we will discuss a generalizations of the above results on the sets of algebraic numbers of arbitrary degree in the fields of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}_p$. In this abstract we formulate the results for the field of $p$-adic numbers.

Y. Bugeaud in [2] stated the problem of the length of the interval depends on the height of algebraic numbers, which form the regular system on this interval. In [2] it is shown that for a given finite interval $I$ in $[-1/2, 1/2]$ the value of $T_0(\Gamma, N(\alpha), I)$ in the definition of regular system is equal to

$$T_0(\mathbb{Q}, N(\alpha), I) = 10^4 |I|^{-2} \log^2 100 |I|^{-1}$$

for $n = 1$, and in [3] that

$$T_0(\mathbb{A}_2, N(\alpha), I) = 72^3 |I|^{-3} \log^3 72 |I|^{-1}$$

for $n = 2$, where $\mathbb{A}_k$ is the set of real algebraic numbers of degree $k$. In [1] it is shown that $T_0(\mathbb{A}_3, N(\alpha), I) = c_1 |I|^{-4-\epsilon}$, $0 < \epsilon < 1$. Probably, there is a more strong connection between $I$ and $T_0(\mathbb{A}_n, N(\alpha), I)$, namely $T_0(\mathbb{A}_n, N(\alpha), I) = c_2 I^{-(n+1)}$. In this paper, we address to Y. Bugeaud’s problem for the $p$-adic numbers with arbitrary $n$.

Throughout $c_1 = c_1(n)$, $c_2 = c_2(n)$, ... are constants depending only on $n$. The Haar measure of a measurable set $S \subset \mathbb{Q}_p$ is denoted by $\mu(S)$. Let $\mathbb{A}_p$ be the set of all algebraic numbers and $\mathbb{Q}_p^*$ is the extension of $\mathbb{Q}_p$ containing $\mathbb{A}_p$. Denote by $\mathbb{A}_{n,p}$ the set of algebraic numbers of degree $n$ lying in $\mathbb{Z}_p$. Let $K_0$ be a disc in $\mathbb{Z}_p$. The natural number $H(\alpha)$ denotes the height of $\alpha \in \mathbb{A}_p$, which is the absolute height of the minimal polynomial of $\alpha$.

**Theorem 1.** Let $K$ be a finite cylinder in $K_0$. Then there are positive constants $c_3, c_4$ and a positive number $T_0 = c_4 \mu(K)^{-(n+1)}$ such that for any $T \geq T_0$ there exist numbers

*Supported by grant CRC 701, University of Bielefeld
†Supported by grant CRC 701, University of Bielefeld
‡Supported by grant CRC 701, University of Bielefeld
\[ \alpha_1, \ldots, \alpha_t \in A_{n,p} \cap K \text{ such that} \]
\[ H(\alpha_i) \leq T^{1/(n+1)} \quad (1 \leq i \leq t), \]
\[ |\alpha_i - \alpha_j|_p \geq T^{-1} \quad (1 \leq i < j \leq t), \]
\[ t \geq c_3 T \mu(K). \]  

(1)

Note that from Theorem 1 it follows that the set \( A_{n,p} \) with the function \( N(\alpha) = H^{n+1}(\alpha) \) form a regular system in \( K_0 \).

For positive integer \( Q \) and \( c_5, c_6 \in \mathbb{R}^+ \), \( c_7 \in \mathbb{R}^+ \cup \{0\} \) define the set of polynomials
\[ P_n(c_6 Q^{c_7}) = \{ P \in \mathbb{Z}[x] : \deg P = n, \ c_5 Q^{c_7} < H(P) \leq c_6 Q^{c_7}, \ c_5 < c_6 \}. \]  

(2)

Let \( \delta, d_n, c_8 \in \mathbb{R}^+ \) and \( 0 \leq r_n \leq 1 \). Denote by \( \mathcal{L}_n = \mathcal{L}_n(c_6 Q^{r_n}, \delta, K) \) the set of \( w \in K \) for which the system of the inequalities
\[ |P(w)|_p < c_8 Q^{-d_n}, \quad |P'(w)|_p \leq \delta, \]  

(3)

has a solution in polynomials \( P \in P_n(c_6 Q^{r_n}) \).

The proof of Theorem 1 is based on the following metric result.

**Theorem 2.** For any real positive number \( s \), where \( 0 < s < 1 \), and for any cylinder \( K \) in \( K_0 \) there exists a sufficiently large number \( Q_0 = Q_0(K) \) such that for
\[ \mu(K) > c_9 Q_0^{-1}, \quad d_n \geq n + r_n, \quad \delta \leq 2^{-n-5} c_6^{-n-1} c_8^{-1} s^2 (f(n))^{-2} \]

and sufficiently large constant \( c_9 \), which does not depend on \( Q_0 \), and for all \( Q > Q_0 \)
\[ \mu(\mathcal{L}_n) < s \mu(K) \]  

(4)

holds.

From above it follows that the cylinder \( K \) with \( \mu(K) > c_9 Q^{-1} \) for sufficiently large \( c_9 \) contains \( Q^{n+1} \mu(K) \) algebraic \( p \)-adic numbers of degree \( n \) and \( H(\alpha) \leq Q \). Note that if \( \mu(K) \leq \frac{1}{2} Q^{-1} \) then the following result holds which is the complement of Theorem 1 in some sense.

**Theorem 3.** For any \( Q \in \mathbb{N} \) there exist the cylinders \( K \) with \( \mu(K) \leq \frac{1}{2} Q^{-1} \) which do not contain algebraic numbers \( \alpha \in \mathbb{Q}_p \) of degree \( \deg \alpha = n, n \geq 1, \) and \( H(\alpha) \leq Q \).

**Proof.** For the given \( Q \) choose \( s \in \mathbb{N} \) satisfying the inequality \( p^{-s} < \frac{1}{2} Q^{-1} \). Consider the cylinder \( K = K(p^s, \frac{1}{2} Q^{-1}) \). Let \( \alpha \in K \) be an algebraic number of degree \( \deg \alpha = n, n \geq 1 \), and \( H(\alpha) \leq Q \). It means that \( \alpha \in \mathbb{Q}_p, \alpha \neq 0 \), is a root of irreducible polynomial \( P(x) = \sum_{i=0}^{n} a_i x^i \). If we assume that \( a_0 = 0 \) then from \( P(\alpha) = 0 \) it follows that \( \alpha(\sum_{i=1}^{n} a_i \alpha^{i-1}) = 0 \). Last implies that \( \alpha \) is a root of polynomial \( P_1(x) = \sum_{i=1}^{n} a_i x^{i-1} \) of deg \( P_1 \leq n - 1 \) which contradicts to the fact that \( \deg \alpha = n \). Therefore, \( a_0 \neq 0 \) and from
\[ a_0 = -\alpha \sum_{i=1}^{n} a_i \alpha^{i-1}, \]
we obtain
\[ Q^{-1} \leq |a_0|_p \leq |\alpha|_p \max_{1 \leq i \leq n} |a_i \alpha^{i-1}|_p \leq \frac{1}{2} Q^{-1}, \]

which is a contradiction. This completes the proof of Theorem 3. \( \square \)
References


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