Eigenvalues of graph matrices and graph parameters

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Abstract

Let $G = (V, E)$ be a graph on vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set $E = E(G)$. Also let d_i be the degree of vertex v_i for $i = 1, 2, \dots, n$. For each $v_i \in V$, the set of neighbors of vertex v_i is denoted by $N_G(i)$ or simply N_i. The average degree $\overline{d}$ of G is defined as $\overline{d} = \frac{1}{n} \sum_{i=1}^{n} d_i$. If vertices v_i and v_j are adjacent, we denote that by $v_i v_j \in E(G)$. The adjacency matrix $A(G)$ of G is defined by its entries $a_{ij} = 1$ if $v_i v_j \in E(G)$ and 0 otherwise. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0$ denote the eigenvalues of $A(G)$. Let $D(G)$ be the diagonal matrix of vertex degrees. Then the Laplacian matrix of G is $L(G) = D(G) - A(G)$. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$ denote the eigenvalues of $L(G)$. They are usually called the Laplacian eigenvalues of G. Among all eigenvalues of the Laplacian of a graph, the most studied is the second smallest, called the algebraic connectivity of a graph. It is well known that a graph is connected if and only if $a = \mu_{n-1} > 0$. Besides the algebraic connectivity, μ_1 is the invariant that interested the graph theorists. The spectral graph theorists are increasingly interested in the largest eigenvalue μ_1 of $L(G)$ and this interest is mainly due to the numerous applications of μ_1. The signless Laplacian matrix of G is $Q(G) = D(G) + A(G)$. Let $q_1 \geq q_2 \geq \cdots \geq q_n$ denote the eigenvalues of $Q(G)$. They are usually called the signless Laplacian eigenvalues of G.

In this talk, we discuss the following problems ([2]-[8]):

Theorem 0.1. *Let G be a simple graph of order $n \geq 2$. If $\mu_1 = n - 1$ and $\mu_{n-1} = 1$, then both G and $\overline{G}$ have diameter 3.*

Theorem 0.2. *Let G be a simple connected graph of order $n \geq 2$. If $\mu_1 = n - 1$ then*

$$\max_{v_i v_j \in E(G)} |N_i \cup N_j| = n.$$

Theorem 0.3. *If G is a connected graph on $n \geq 7$ vertices with average degree $\overline{d}$, then*

$$q_2 - \overline{d} \geq -1$$

with equality holding if and only if $G \cong K_n$.

Theorem 0.4. *Let G be a connected graph of order $n \geq 9$. Then*

$$q_2 - \lambda_1 \geq 1 - \sqrt{n-1}$$

with equality if and only if G is the star $K_{1,n-1}$.

Theorem 0.5. *Let G be a conneted graph of order $n > 2$ with diameter D and algebraic connectivity $a(G)$. Then*

$$a(G) + D \geq 3 \tag{1}$$

with equality holding in (1) if and only if G is isomorphic to a graph $(H_1 \cup H_2) \vee \{\bullet\}$, where H_1 and H_2 are any two graphs such that $|V(H_1)| + |V(H_2)| = n - 1$.

Theorem 0.6. *Let G be a graph of order n, minimum degree $\delta(G)$ and algebraic connectivity $a(G)$. Then $a(G) - \delta(G)$ is minimum for a graph composed of 2 cliques on $\lceil \frac{n}{2} \rceil$ vertices with a common vertex if n is odd, and linked by an edge if n is even.*

Theorem 0.7. *Let G be a connected graph on $n \geq 4$ vertices with independence number α. Then*

$$q_1 + q_n + 2\alpha \leq 3n - 2$$

with equality holding if and only if $G \cong CS(n, n - \alpha)$.

Theorem 0.8. *Let G be a connected graph of order $n \geq 6$. Then $q_1 - q_n$ is minimum for a path P_n and for an odd cycle C_n.*

Key Words: Graph, Adjacency eigenvalues, Laplacian eigenvalues, Signless laplacian eigenvalues, Diameter, Maximum degree, Minimum degree, Independence number

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