Minimal k-connected graphs with minimal number of vertices of degree k

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Abstract

A (vertex) k-connected graph is called minimal, if it becomes not k-connected after deleting any edge. Let us denote by $V(G)$ the set of vertices of a graph G and $v(G) = |V(G)|$. Let $d_G(x)$ denote the degree of a vertex x in the graph G and $\Delta(G)$ denote the maximal vertex degree of the graph G. We denote by $v_k(G)$ the number of vertices of degree k of a graph G. Clearly, all vertices of a k-connected graph have degree at least k.

In 1967 minimal biconnected graphs were considered in the papers [1] and [2]. It can be deduced from the results of these papers that

$$v_2(G) \geq \frac{v(G)+4}{3}$$

for a minimal biconnected graph G.

In 1979 W. Mader [5, 6] has proved a very strong result that generalize for arbitrary k the one written above:

$$v_k(G) \geq \frac{(k-1)v(G)+2k}{2k-1} \tag{1}$$

for a minimal k-connected graph G. This bound is tight: there are infinite series of graphs for which the inequality (1) turns to equality. In what follows we consider such graphs. Let us call them *extremal* minimal k-connected graphs.

Definition 1. Let $k \geq 2$ and T be a tree with $\Delta(T) \leq k+1$. The graph $G_{k,T}$ is constructed from k disjoint copies $T_1, \ldots, T_k$ of the tree T. For any vertex $a \in V(G)$ we denote by a_i the correspondent vertex of the copy T_i. If $d_G(a) = j$ then we add $k+1-j$ new vertices of degree k that are adjacent to $\{a_1, \ldots, a_k\}$.

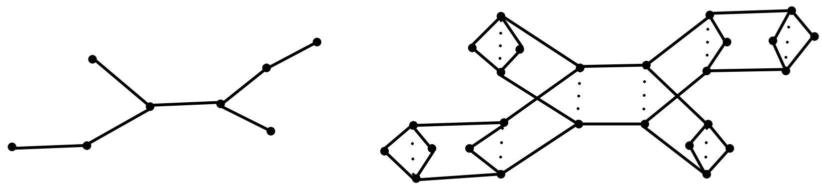

Figure 1: A tree T and correspondent extremal minimal biconnected graph $G_{2,T}$.

Clearly, if $v(T) = n$ then $v(G_{k,T}) = (2k-1)n + 2$. It is not difficult to verify that $G_{k,T}$ is a minimal k-connected graph, and, hence, it is an extremal graph. A tree T with $\Delta(T) = 3$ and the graph $G_{2,T}$ are shown on the picture.

In 1982 Oxley [7] presented an algorithm of constructing all extremal biconnected and triconnected graphs. It was proved here that every extremal minimal biconnected graph can be obtained from the complete bipartite graph $K_{2,3}$ by several operations of substituting a vertex of degree two by a graph $K_{2,2}$ (joint by two edges to two vertices of the neighborhood of the vertex that have been substituted). Any extremal minimal 3-connected graph can be obtained from the graph $K_{3,4}$ by several operations of substituting a vertex of degree three by a graph $K_{3,3}$.

In [12] the author have proved that every minimal biconnected graph is a graph of type $G_{2,T}$ for some tree T with $\Delta(T) \le 3$. Now we present a similar result for 3-connected and 4-connected graphs.

Theorem 1. *Let $k \in \{3, 4\}$. Then any minimal k-connected graph G with $v_k(G) = \frac{(k-1)v(G)+2k}{2k-1}$ is a graph $G_{k,T}$ for some tree T with $\Delta(T) \le k+1$.*

KEYWORDS: connectivity, minimal k-connected graph.

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