

Largest families of sets, under conditions defined by a given poset

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Let $[n] = \{1, 2, \dots, n\}$ and let $\mathcal{F} \subset 2^{[n]}$ be a family of its subsets. Sperner proved in 1928 that if \mathcal{F} contains no pair of members in inclusion that is $F_1, F_2 \in \mathcal{F}$ implies $F_1 \not\subset F_2$ then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$. Inspired by an application, Erdős generalized this theorem in the following way. Suppose that \mathcal{F} contains no $k + 1$ distinct members $F_1 \subset F_2 \subset \dots \subset F_{k+1}$ then $|\mathcal{F}|$ is at most the sum of the k largest binomial coefficients of order n . This bound is, of course, tight. We will survey results of this type: determine the largest family of subsets of $[n]$ under a certain condition forbidding a given configuration described solely by inclusion among the members. This maximum is denoted by $\text{La}(n, P)$ where P is the forbidden configuration. The final (hopelessly difficult) conjecture is that $\text{La}(n, P)$ is asymptotically equal to the sum of the k largest binomial coefficients where the k largest levels contain no configuration P , but $k + 1$ levels do.

Another related problem is the following one. A copy of the given poset P is a family \mathcal{F} of subsets of $[n]$ where the embedding $f : P \rightarrow \mathcal{F}$ maps comparable elements of P into comparable subsets. Two copies $\mathcal{F}_1, \mathcal{F}_2$ of P are incomparable if no member of \mathcal{F}_1 is a subset of a member of \mathcal{F}_2 and no member of \mathcal{F}_2 is a subset of a member of \mathcal{F}_1 . The maximum number of incomparable copies of P is denoted by $\text{LA}(n, P)$. This quantity is asymptotically determined for all P , unlike $\text{La}(n, P)$.