ON PLANARITY, MATROID DUALITY AND MATROID ISOMORPHISM OF GRAPHS

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In this talk I would like to discuss and outline various aspects of planarity, matroid duality, and matroid isomorphism of graphs and some related topics.

A graph G is *matroid dual* (*matroid isomorphic*) to a graph F if $M(G) = M^*(F)$ (resp., $M(G) = M(F)$), where $M(G)$ is the forest matroid of G and M^* is the matroid dual to matroid M. In other words, G is *matroid dual* (*matroid isomorphic*) to F if $E(G) = E(F)$ and $\mathcal{C}(G) = \mathcal{C}^*(F)$ (resp., $\mathcal{C}(G) = \mathcal{C}(F)$), where $E(G)$ is the set of edges of G, $\mathcal{C}(G)$ is the set of circuits of G, and $\mathcal{C}^*(G)$ is the set of cocircuits of G.

There are several planarity criteria of graphs. We will focus on the following three of them:

Kuratowski's planarity criterion (1930):
A graph G is planar if and only if G has a subdivision of K_5 or K_{33}.

Whitney's planarity criterion (1932):
A graph G is planar if and only if there exists a graph F matroid dual to G, i.e. such that $M(G) = M^*(F)$ or, the same, $\mathcal{C}(G) = \mathcal{C}^*(F)$.

Kelmans' planarity criterion (1976):
A 3-connected graph G is planar if and only if every edge of G belongs to exactly two non-separating circuits of G.

It turns out that each of these planarity criteria can be strengthened in a pretty natural and interesting way. It is easy to see that the planarity problem for graphs can be reduced to the problem for 3-connected graphs. It is also easy to show that a 3-connected graph G distinct from K_5 is non-planar if and only if G contains a subdivision of $K_{3,3}$.

One of possible strengthening of Kuratowski's theorem (Kelmans, 1981) is that a 3-connected non-planar graph distinct from K_5 always contains a special subdivision of $K_{3,3}$, namely, it contains a subdivision S of $K_{3,3}$ such that some three edges of $K_{3,3}$ forming a matching are not subdivided in S, i.e. S is a cycle with three overlapping edges-chords. Moreover, this result is "tight", namely, there are infinitely many 3-connected graphs having no subdivisions of $K_{3,3}$ with four non-subdivided edges or with two non-subdivided edges adjacent in $K_{3,3}$.

The above strengthening of Kuratowski's theorem can further be strengthened for so called quasi 4-connected non-planar graphs (Kelmans, 1997), namely, such a graph always contains a subdivision of $K_{3,3}$ with five non-subdivided edges forming a spanning tree in $K_{3,3}$. Again, this result is "tight", namely, there are infinitely many quasi 4-

connected non-planar graphs having no non-subdivided edges forming a cycle.

An immediate Corollary from the above result is that any bipartite quasi 4-connected non-planar graph B contains a subdivision S of $K_{3,3}$, in which every edge is subdivided into an odd number of edges, i.e. such that the bicoloring of $K_{3,3}$ in S can be extended to the bicoloring of B (Kelmans, 1997). This Corollary is not true for 3-connected bipartite graphs but remains true for cubic 3-connected bipartite graphs (Kelmans, 1997).

Given graph G and F, we say that G is *strongly isomorphic to* F if there exists an isomorphism α from G to F such that the renaming the vertices of F by the names of vertices in G according to bijection α results in a graph equal to G.

Whitney's planarity criterion is a part of an interesting Whitney's picture on "graphs verses matroids". This picture also includes (among other things) the following

Whitney's matroid isomorphism theorem:

Let G and F be graphs without isolated vertices. If G is 3-connected and F is circuit isomorphic to G, then G is strongly isomorphic to F.

A graph G is called *circuit semi-dual to* (*circuit semi-isomorphic to*) F if $E(G) = E(F)$ and $\mathcal{C}(G) \subseteq \mathcal{C}^*(F)$ (resp., $\mathcal{C}(G) \subseteq \mathcal{C}(F)$). The notions "*cocircuit semi-dual*" and "*cocircuit semi-isomorphic*" can be defined similarly. Natural questions are whether the conclusions of the above Whitney's theorems remain true if the *circuit duality* and *circuit isomorphism* conditions are replaced, respectively, by *circuit semi-duality* and *circuit semi-isomorphism* or by *cocircuit semi-duality* and *cocircuit semi-isomorphism.* In 1987 we gave complete answers to these questions. In particular, we proved that the conclusions of the above Whitney's theorems remain true if the *circuit duality* and *circuit isomorphism* conditions are replaced, respectively, by *circuit semi-duality* and *circuit semi-isomorphism.* Another natural direction to extend the Whitney picture is to replace the forest matroid of a graph by some other matroids related to a graph. We have some progress in this direction as well.

Further direction for extensions of the Whitney picture is to consider graphs G_1 and G_2 such that $E(G_1) = E(G_2)$ and $\mathcal{A}_1(G_1) \subseteq \mathcal{A}_2(G_2)$, where $\mathcal{A}_i(G_i)$ is the family of certain edge subsets of G_i not necessarily related with a matroid of G_i. For example, let $\mathcal{D}(F)$ be the family of the edge subsets of F inducing 2-regular subgraphs in F. In 1976 we were able to prove that if G is 3-connected, $E(G) = E(F)$, and $\mathcal{C}(G) \subseteq \mathcal{D}(F)$, then G is strongly isomorphic to F. In 1967 Halin and Jung raised the following question along this line. A *k-skein* is the union of k openly disjoint paths with common end-vertices. Let $\mathcal{S}_k(G)$ be the family of the edge-sets of k-skeins in G. Graphs G and F are called *k-skein isomorphic* if $E(G) = E(F)$ and $\mathcal{S}_k(G) = \mathcal{S}_k(F)$. The Halin-Jung question was: Is it true that k-connected k-skein isomorphic graphs are strongly isomorphic for $k \geq 3$ (except for K_4 when $k = 3$). It turns out that the claim is true for $k = 3$ (Kelmans, 1980 and Hemminger-Jung, 1982) and there is one counterexample for $k = 4$ (Kelmans, 1984).

At last we will see that Kelmans' planarity criterion can also be strengthened for quasi 4-connected graphs in a pretty interesting way (Kelmans, 1981).