Distance graphs in $\mathbb{R}^4$

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This talk is devoted to the extremal properties of diameter graphs. For more details we refer the reader to the paper [8], on which this talk is based.

Definition 1. *A graph $G = (V, E)$ is a diameter graph in $\mathbb{R}^d$ if $V \subset \mathbb{R}^d$, V is finite,* $\operatorname{diam} V = 1$ *and $E \subseteq \{(x, y), x, y \in \mathbb{R}^d, |x - y| = 1\}$, where $|x - y|$ denotes the Euclidean distance between x and y.*

Analogously, we may define a diameter graph on the sphere S^d_r of radius r. We think of the sphere being embedded in $\mathbb{R}^{d+1}$, and the (unit) distance is induced from the ambient space.

Diameter graphs are closely related to the famous Borsuk problem. In 1933 Borsuk [1] asked, whether any set of diameter 1 in $\mathbb{R}^d$ can be partitioned into $(d + 1)$ parts of strictly smaller diameter. The positive answer to the question is known as Borsuk's conjecture. This was shown to be true in dimensions up to 3. For 60 years the question in higher dimensions remained open, until in 1993 Kahn and Kalai [7] constructed a finite set of points in dimension 1325 that does not admit a partition into 1326 parts of smaller diameter. The minimal dimension in which the counterexample is known was reduced by several authors, with a current record $d = 65$ due to Bondarenko.

Borsuk's problem for finite sets in $\mathbb{R}^d$ can be formulated in terms of diameter graphs. Namely, whether it is true that any diameter graph G in $\mathbb{R}^d$ satisfies $\chi(G) \leqslant d+1$? This and related problems were studied by several authors. In [6] Hopf and Pannwitz proved that the number of edges in any diameter graph in $\mathbb{R}^2$ is at most n, which easily implies Borsuk's conjecture for finite sets on the plane. Vázsonyi conjectured, that any diameter graph in $\mathbb{R}^3$ on n vertices can have at most $2n - 2$ edges. Again, it is not difficult to see that Vázsonyi's conjecture implies Borsuk's conjecture for finite sets in $\mathbb{R}^3$. Vázsonyi's conjecture was proved independently by Grünbaum [4], Heppes [5] and Straszewicz [10].

Almost 50 years later, two other papers on this topic appeared. In [2] Dol'nikov proved that in a diameter graph in $\mathbb{R}^3$ any two odd cycles must share a vertex. Out of this statement he derived Borsuk's conjecture for finite sets. The method he introduced was later developed by Swanepoel [12], who managed to give yet another proof of Vázsonyi's conjecture.

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In this work we generalize Dol'nikov's and Swanepoel's approaches to the case of the three-dimensional sphere:

Theorem 1. *Let X be a finite subset of diameter 1 on S_r^3, $|X| = n$, and $G = G(X)$ be diameter graph with X as a vertex set. If $r > 1/\sqrt{2}$, then:*

1. *G has at most $2n - 2$ edges.*

2. *$\chi(G) \leqslant 4$.*

3. *Any two odd cycles in G have a common vertex.*

We utilize this result to study some properties of diameter graphs in $\mathbb{R}^4$. Namely, we investigate the behaviour of the quantity $D_d(l, n)$, which is the maximum number of cliques of size l in a diameter graph on n vertices in $\mathbb{R}^d$. Erdős [3] studied $D_d(2, n)$ for different d. He showed that for $d \geqslant 4$ we have $D_d(2, n) = \frac{\lfloor d/2 \rfloor - 1}{2\lfloor d/2 \rfloor} n^2 + \bar{o}(n^2)$. Swanepoel [11] determined $D_d(2, n)$ for $d \geqslant 4$ and sufficiently large n. Functions $D_d(l, n)$ and similar functions were studied in several papers. In particular, the following conjecture was raised in [9]:

Conjecture 1 (Schur et. al., [9])**.** *We have $D_d(d, n) = n$ for $n \geqslant d + 1$.*

This was proved by Hopf and Pannwitz for $d = 2$ in [6] and for $d = 3$ by Schur et. al. in [9]. In the latter paper the authors also showed that $D_d(d + 1, n) = 1$. In this paper we find $D_4(3, n)$ for sufficiently large n and $D_4(4, n)$ for all n, thus completing the description of $D_4(l, n)$ for different n. We also refine the result of Swanepoel concerning $D_4(2, n)$ by giving a reasonable bound on n: we show that his result holds for $n \geqslant 52$.

Theorem 2.

1. *For $n \geqslant 52$ we have $D_4(2, n) = F_2(n)$, where*

$$F_2(n) = \begin{cases} \lfloor n/2 \rfloor \lceil n/2 \rceil + \lceil n/2 \rceil + 1, & \text{if } n \not\equiv 3 \bmod 4, \\ \lfloor n/2 \rfloor \lceil n/2 \rceil + \lceil n/2 \rceil, & \text{if } n \equiv 3 \bmod 4 \end{cases}$$

   (this part of the theorem in case of sufficiently large n is due to Swanepoel [11]).

2. *For all sufficiently large n we have $D_4(3, n) = F_3(n)$, where*

$$F_3(n) = \begin{cases} (n-1)^2/4 + n, & \text{if } n \equiv 1 \bmod 4, \\ (n-1)^2/4 + n - 1, & \text{if } n \equiv 3 \bmod 4, \\ n(n-2)/4 + n, & \text{if } n \equiv 0 \bmod 2. \end{cases}$$

3. *(Schur's conjecture in $\mathbb{R}^4$) For all $n \geqslant 5$ we have $D_4(4, n) = n$.*

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