Irreducible triangulations of 2-manifolds with boundary

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Abstract

This is a talk on joint work with María José Chávez, Seiya Negami, Antonio Quintero, and María Trinidad Villar.

A *triangulation* of a 2-manifold M is a simplicial 2-complex with underlying space homeomorphic to M. The operation of *contraction of an edge* e in a triangulation T of M is the operation which consists in contracting e to a single vertex and collapsing each face that meets e to a single edge. T is called an *irreducible triangulations* if none of the edges of T can be contracted without leaving the category of simplicial complexes; in other words, no edge of T can be contracted without either creating multiple edges or changing the topological type of the underlying space. In fact, in the case of a closed M other than the 2-sphere there is only one impediment to edge contractibility: (1) an edge is non-contractible if and only if it appears in more than two cycles of length 3 (made up of edges of T). In case of a 2-manifold with boundary there are two additional impediments to contractibility—it is impossible to contract: (2) a chord (that is, a non-boundary edge with both end vertices in the boundary) and (3) a boundary edge if the length of the boundary is 3. *Vertex splitting* is the operation inverse to the edge contraction. Observe that all triangulations of M can be generated by repeatedly splitting the irreducible triangulations of M (see [1, 8, 10]).

Irreducible triangulations have proved to be an effective tool for solving problems in combinatorial topology of 2-manifolds and discrete geome-

try. For instance, irreducible triangulations contain minimal triangulations which in their turn contain neighborly triangulations.

Two triangulations are called *isomorphic* if there is a bijection between their vertex sets that preserves edges and faces. Here, we distinguish between triangulations only up to isomorphism.

While the case of closed 2-manifolds has been well studied (see [1, 2, 5, 6, 7, 9, 11]), the case of 2-manifolds with boundary is a new subject matter which has just now received attention: Boulch, Colin de Verdière, and Nakamoto [3] have proved that for any 2-manifold with boundary the base of irreducible triangulations is finite.

Let $S - D$ denote the (once-) punctured 2-manifold that results from a closed 2-manifold S by the removal of a disk D. The key idea of our approach to identification of all combinatorial types (up to isomorphism) of irreducible triangulations of $S - D$ can be stated as follows. We call contractible edges *cables*. A vertex of a triangulation T of $S - D$ is called a *pylonic vertex* if that vertex is incident with all cables of T. (Therefore, we can define an irreducible triangulation to be one with an empty cable-subgraph.) A *pylonic triangulation* is defined to be one that has a pylonic vertex. Observe that if a pylonic triangulation has at least two cables, it has a unique pylonic vertex. Our key lemma states that each irreducible triangulation of $S - D$ is obtainable either by removing a vertex (together with the incident edges faces) from an irreducible triangulation of S or by removing a pylonic vertex from a pylonic triangulation of S. We then generate triangulations of S from the irreducible ones and detect pylonic triangulations among them. A complete list of irreducible triangulations of the projective plane has been identified by Barnette [1]. Observe that if a reducible triangulation is not pylonic, it could never become such under any further vertex splitting.

We use this approach in [4] to identify all irreducible triangulations of the Möbius band (regarded as the projective plane with a disk removed); there are totally six irreducible triangulations (with the number of vertices varying from 5 to 7). In the trivial case of the 2-disk, there is only one irreducible triangulation—a single triangle.

A project has been launched aimed at identifying all irreducible triangulations of the punctured 2-torus (that is, the 2-torus with a disk removed). A complete list of irreducible triangulations of the (closed) 2-torus has been identified by the speaker [5]. We have already identified 293 irreducible triangulations of the punctured 2-torus (with the number of vertices varying from 6 to 10). However, some work remains to be done to check that our searching procedure has been implemented correctly.

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