COUNTING THE MINIMAL NUMBER OF INFLECTIONS OF A PLANE DOODLE

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Obviously, any plane curve which is diffeomorphic to the figure "‘eight"’ can not be drawn without at least two inflection points. In cite [Sh] B. Shapiro posed the problem of finding/estimating of the minimal number of inflection points of an immersed plane curve with only double points under the action of the group of plane diffemorphisms. He obtained a number of results for the class of the so-called tree-like curves characterized by the property that removal of any double point makes the curve disconnected. In particular, using an auxiliary tree he got lower and upper bounds for the number of inflections of tree-like curves and found a criterion when a tree-like curve can be drawn without inflections.

A *doodle* is the union of a finite number of closed immersed plane curves without triple intersections. When we say that a curve or a doodle γ can be drawn with a certain number of inflection points we mean that exists a plane diffeomorphism such that $diff(\gamma)$ has at most this number of inflection points.

The main result of this note is as follows.

Theorem 1. *Any doodle with n double points can be drawn with at most $2n$ inflection points.*

We conjecture the following stronger statement.

Conjecture 1. *Any closed plane curves with n double points can be drawn with at most $n+1$ inflection points.*

Furthermore, we present an infinite family of topologically distinct minimal fragments forcing an inflection point which implies that the problem of defining the exact minimal number of inflection points of a given doodle is algorithimically very hard. Our results seem to support the general principle that invariants of curves and knots of geometric origin are difficult to calculate algorithmically. Observe that algebraic invariants of doodles similar to Vassiliev invariants of knots were introduced by V. I. Arnold in [Ar] and considered by number of authors. In particular A. Merkov in [Me1], [Me2].

A *fragment* is the union of a finite number of immersed plane curves without triple intersections. Obviously, the lower bound on the minimum number of inflections of a given doodle is the number of fragments forcing an inflection in this doodle. A fragment is called *a minimal fragment forcing an inflection* if the following two conditions are satisfied (see Fig. 1): 1) any drawing of this fragment necessarily contains an inflection point. 2) removing any vertex or cutting any curve we obtain a fragment which can be drawn without inflection points. Obviously, any minimal fragment forcing an inflection must be connected.

Theorem 2. *There is $c > 0$ such that the number of fragments forcing an inflection with at most k double points is at least e^{ck}.*

The above theorem is true even for fragments consisting of curves without self-intersections. That fact in its turn makes it very hard not only to count the minimum number of inflections of a given doodle but also to find a criterion when a doodle can be drawn without inflections.

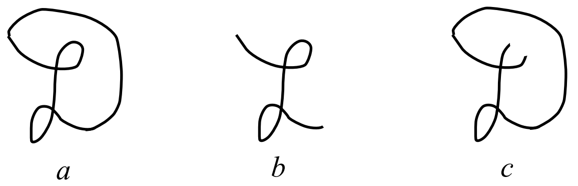

FIGURE 1. Fragments forcing an inflection point. a — non-minimal, b, c — minimal.

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