Erdős's problems on distinct distances: Recent developments

In his seminal paper in the Mathematical Monthly in 1946, Erdős raised two problems, in some sense "dual" to each other: 1. At most how many times can the same distance occur among n points in the plane or in other metric spaces? 2. At least how many distinct distances must occur among n points?

In spite of the dramatic developments related to the latter problem, due to Guth and Katz, both problems are open even for the Euclidean norm in the plane.

We report on some further progress. Let P be a set of n points in the plane, contained in an algebraic curve C of degree d. We prove that the number of distinct distances determined by P is at least $c_d n^{4/3}$, unless C contains a line or a circle. We also prove the lower bound $c'_d \min\{m^{2/3}n^{2/3}; m^2; n^2\}$ for the number of distinct distances between m points on one irreducible plane algebraic curve and n points on another, unless the two curves are parallel lines, orthogonal lines, or concentric circles. (Here c_d and c'_d are suitable positive constants.) This generalizes a result on distances between lines of Sharir, Sheffer, and Solymosi, and can be considered as a step towards characterizing those point sets that determine $o(n)$ distinct distances. Joint work with Frank de Zeeuw.

In forthcoming paper, joint with Konrad Swanepoel, we answer some questions of Martini and Soltan, somewhat related to the first problem of Erdős, mentioned above. Two elements, p and q, of a finite point set P in Euclidean d-space form a *double-normal pair* if P is contained in the closed strip between the hyperplanes through p and q orthogonal to the segment pq. Obviously, any *diameter*, that is, any pair of points in P at maximum distance form a double-normal pair. Due to work of Erdős, Grünbaum, Heppes, and others, it is well understood at most how many diameters an n-element set can determine? We study the analogous problem for double-normal pairs.