Schur conjecture in $\mathbb{R}^d$

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In this paper we present the proof of Schur's conjecture concerning diameter graphs in $\mathbb{R}^d$. First let us remind the definition of a diameter graph.

Definition 1. *A graph $G = (V, E)$ is a diameter graph in $\mathbb{R}^d$ if $V \subset \mathbb{R}^d$, V is finite,* $\operatorname{diam} V = 1$ *and $E \subseteq \{(x, y), x, y \in \mathbb{R}^d, |x - y| = 1\}$, where $|x - y|$ denotes the Euclidean distance between x and y.*

We also consider diameter graphs on the sphere S_r^d of radius r. The definition is almost word for word repeats the one given above. Note only that we think of the sphere being embedded in $\mathbb{R}^{d+1}$, and the (unit) distance is induced from the ambient space.

Different properties of diameter graphs arise naturally in the study of the famous Borsuk's conjecture. During the 80 years since the Borsuk's conjecture was formulated, different properties of diameter graphs were examined. The main direction in the research are, however, extremal properties of diameter graphs, that is, the maximum chromatic number, or the maximum number of cliques of a given size (depending on the number of vertices) a diameter graph in $\mathbb{R}^d$ may have. For additional information about diameter graphs and Borsuk's conjecture we refer the reader to [1, 6, 7].

Denote by $D_d(l, n)$, which is the maximum number of cliques of size l in a diameter graph on n vertices in $\mathbb{R}^d$. We focus on one conjecture, posed by Schur et. al. in [8]:

Conjecture 1 (Schur et. al., [8])**.** *We have $D_d(d, n) = n$ for $n \geqslant d + 1$.*

This was proved by Hopf and Pannwitz for $d = 2$ in [3] and for $d = 3$ by Schur et. al. in [8]. A general case of this problem (as well as several related problems) was mentioned by J. Pach on Erdős centennial conference in Budapest. Pach and Morić proved the following result concerning Schur's conjecture:

Theorem 1 (Morić, Pach, [5])**.** *The number of d-cliques in a graph of diameters on n vertices in $\mathbb{R}^d$ is at most n, provided that any two d-cliques share at least $d - 2$ vertices.*

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Recently, Kupavskii [4] managed to prove Schur's conjecture for $d = 4$, and also studied $D_4(l, n)$ for other values of l. Another proof of the conjecture for $d = 4$ was given by Bulankina, Kupavskii and Polyanskiy in [2]. The main result of this paper is the proof of Schur's conjecture in the following general setting:

Theorem 2. *Let $d \in \mathbb{N}$. Schur's conjecture holds*
1. In the space $\mathbb{R}^d$,
2. On the sphere S_r^d of radius $r > 1/\sqrt{2}$.

Remark Note that Schur's conjecture on the sphere formulates exactly like in the space, with the exception that instead of diameter graphs in $\mathbb{R}^d$ we consider diameter graphs on S_r^d (and not on S_r^{d-1}).

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