The Vertex Sign Balance of (Hyper)graphs

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July 18, 2017

Positive subsums of a positive sum

Theorem (Manickam-Miklós, Manickam-Singhi)

For $x_1, x_2, \ldots, x_n$ given, with $\sum_{i=1}^{n} x_i > 0$, the minimum number of positive k-subsums of them is $\binom{n-1}{k-1}$ if $n > n_1(k)$ or $k \mid n$.

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Example

Consider $\{n, -1, -1, \ldots, -1\}$. But, like in EKR, not always the best: $\{-n, 1 + \frac{1}{n-2}, \ldots, 1 + \frac{1}{n-2}\}$ gives $\binom{n}{k-1}$ positive subsums, $< \binom{n-1}{k-1}$ for $2k < n$.

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Remark (Miklós)

Even worse: $\{2\frac{1}{3} - 3k, 2\frac{1}{3} - 3k, 2\frac{1}{3} - 3k, 3, 3, \ldots, 3\}$ gives $\binom{3k-2}{k}$ positive subsums, $< \binom{3k}{k-1}$.

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Conjecture ((Manickam-)Miklós-(Singhi))

$n_1(k) \leq 4k$

Earlier results

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Theorem

$n_1(2) = 6$, *i.e, the MMS conjecture is true for graphs.*

The Manickam-Miklós-Singhi property

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Definition (Huang-Sudakov, Pokrovskiy, 2013)

A hypergraph H with minimum degree $\delta(H)$ has the MMS (Manickam-Miklós-Singhi) property if for every weighting $w : V(H) \to \mathbb{R}$ satisfying $\sum_{x \in V(H)} w(x) \geq 0$, the number of nonnegative edges is at least $\delta(H)$.

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Theorem (Huang-Sudakov, 2013)

Let H be an k-uniform n-vertex hypergraph with $n > 10k^3$ and all codegrees equal to λ. Then for every weighting $w : V(H) \to \mathbb{R}$ satisfying $\sum_{x \in V(H)} w(x) \geq 0$, the number of nonnegative edges is at least $\delta(H)$, i.e., $V(K_n^{(k)})$ has the MMS property, $n_1(k) \leq 10k^3$ (in case of equality all nonnegative edges form a star).

Toward the exact bound

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Lemma (Pokrovskiy, 2013)

Let H be a d-regular k-uniform hypergraph on n vertices which has the MMS property. Then for every $w : V(K_n^{(k)}) \to \mathbb{R}$ satisfying $\sum_{x \in V(K_n^{(k)})} w(x) \geq 0$, the number of nonnegative edges is at least $\binom{n-1}{k-1}$, i.e., $V(K_n^{(k)})$ has the MMS property for this particular n and k.

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Corollary

$n_1(k) \leq 10^{46} k$.

The vertex sign balance

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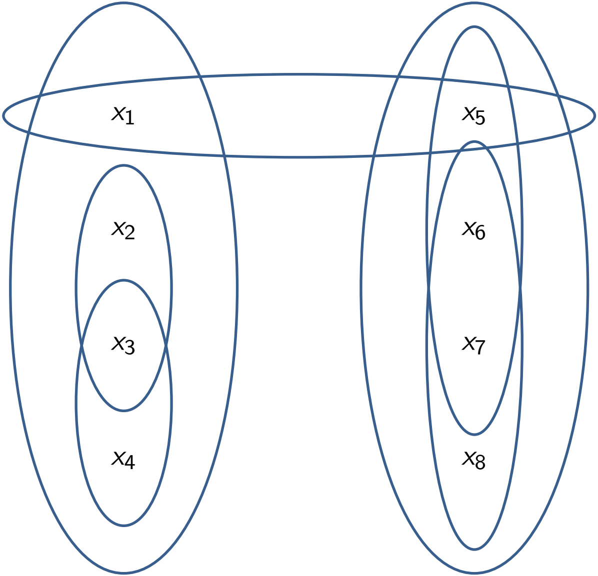
Remark

The vertex sign balance is always between 0 and δ, the minimum degree of the vertices.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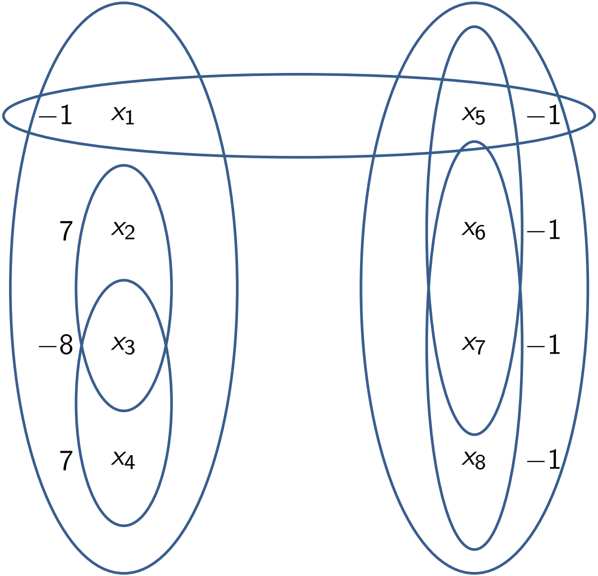
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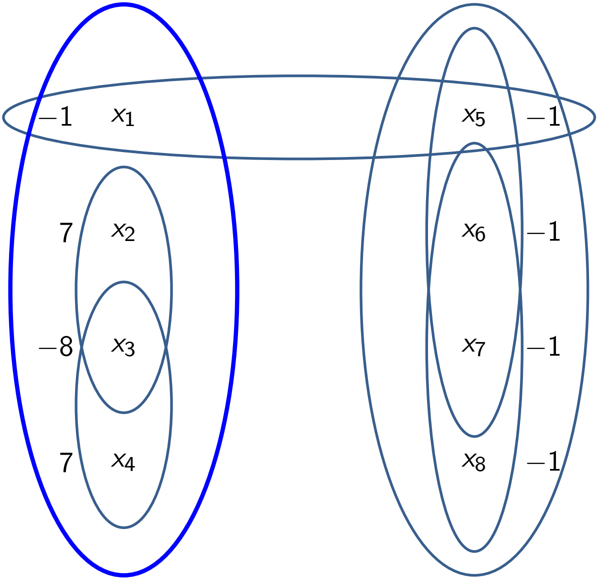
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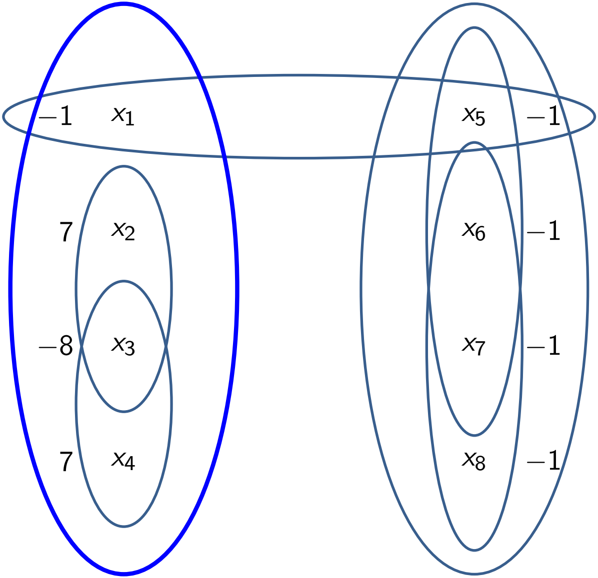
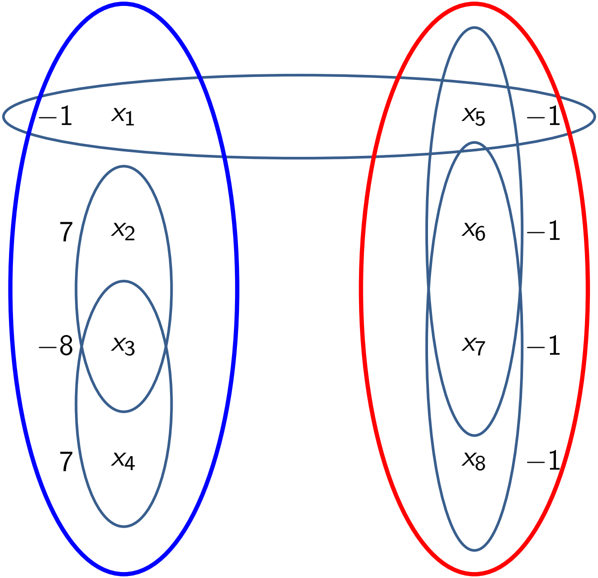
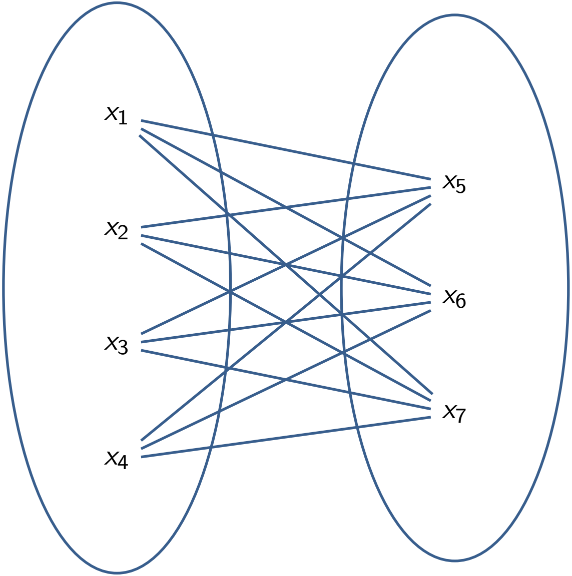


$\nu \leq 1$



$\nu \leq 1 \quad \nu \geq 1 \quad \nu = 1$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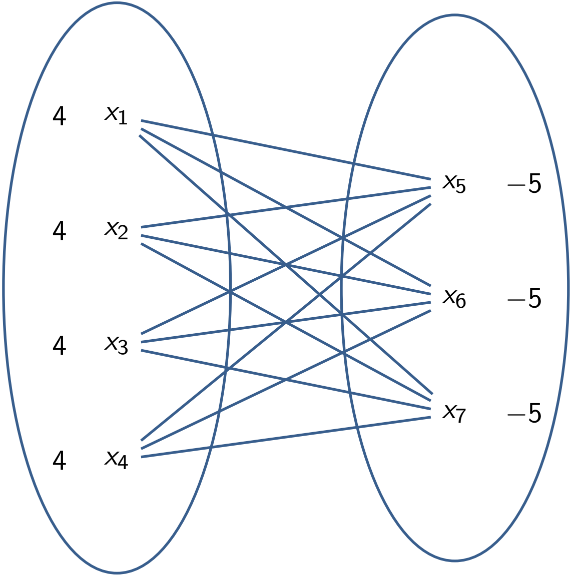
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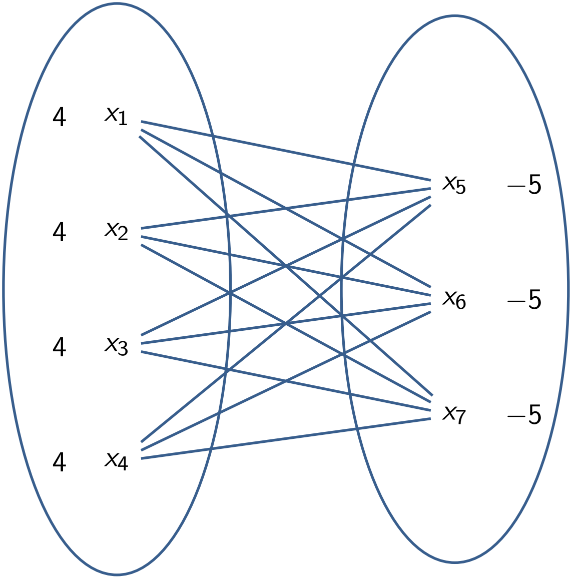
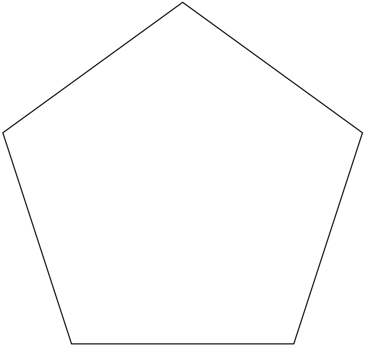
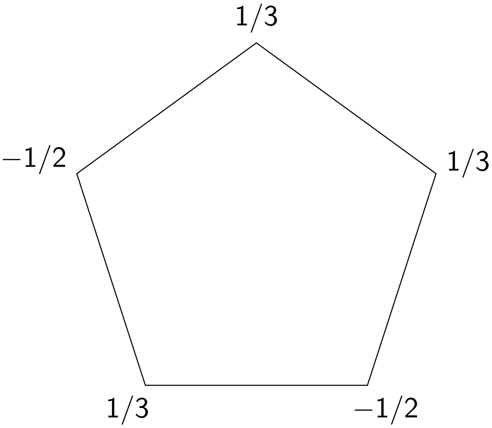


$\nu = 0$

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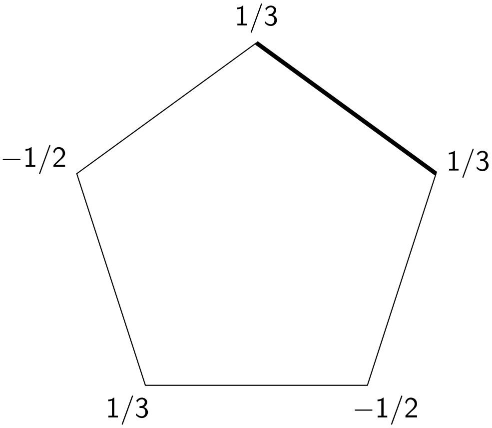
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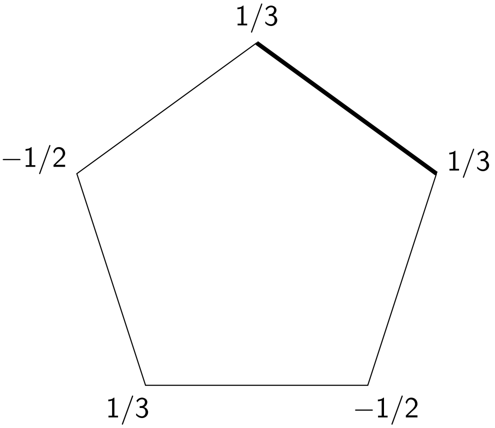
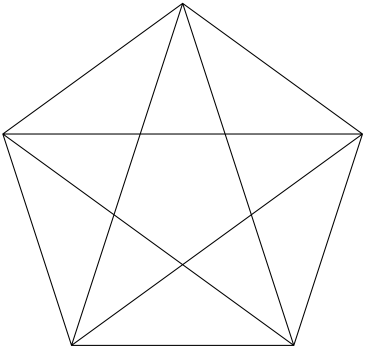
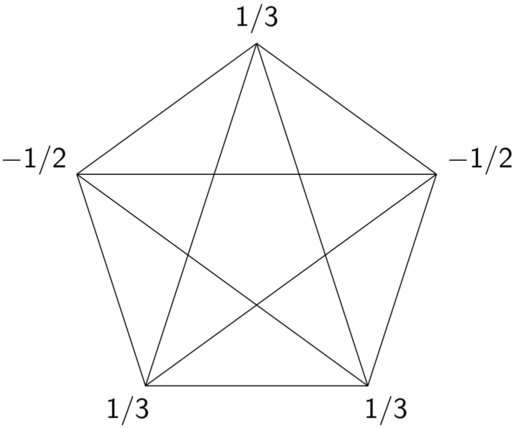


$\nu = 1$

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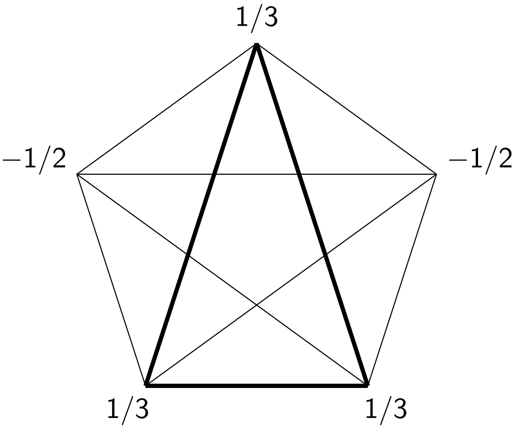
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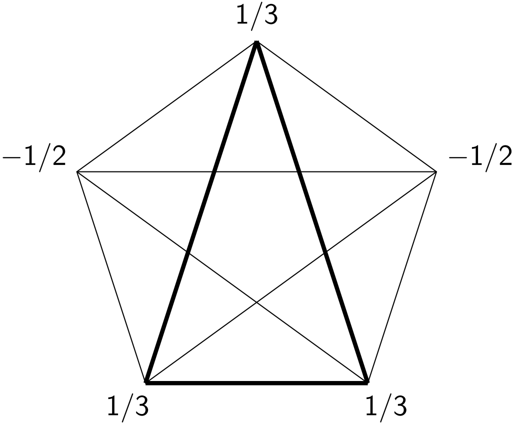
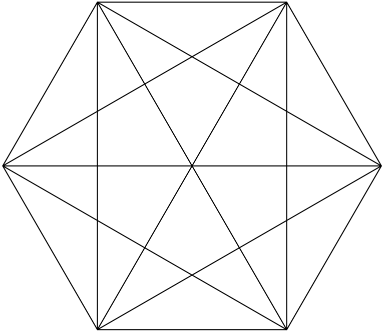
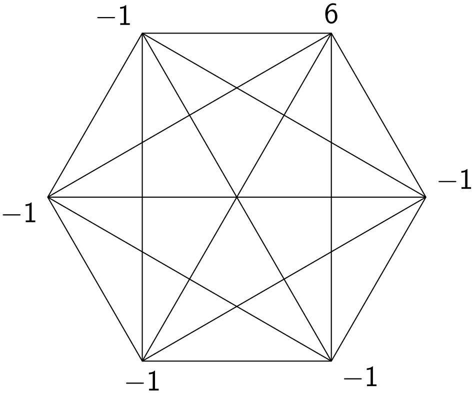


$\nu \leq 3$, so K_5 does not have the MMS property

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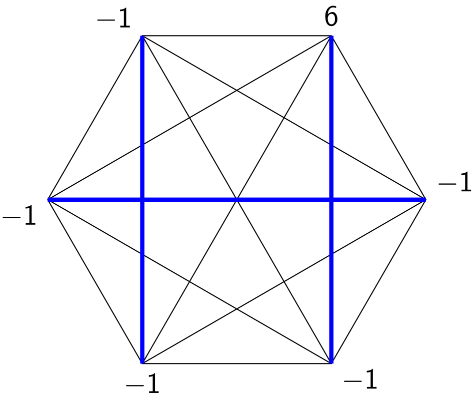
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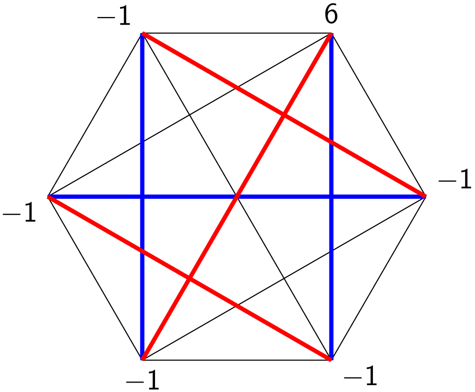
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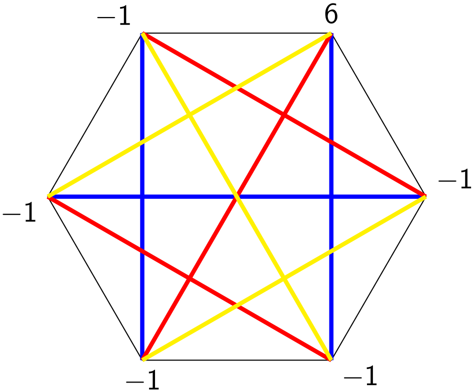
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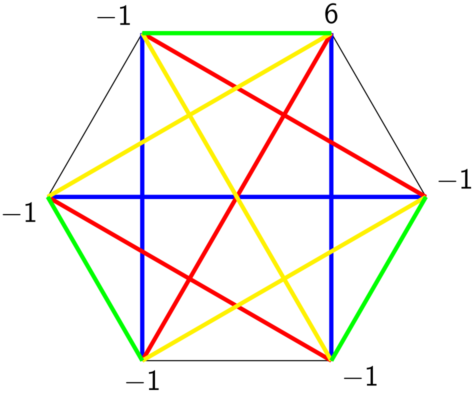
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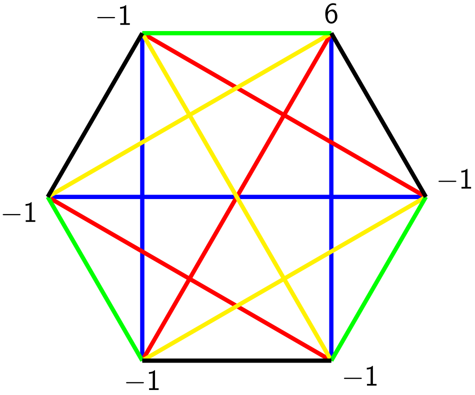
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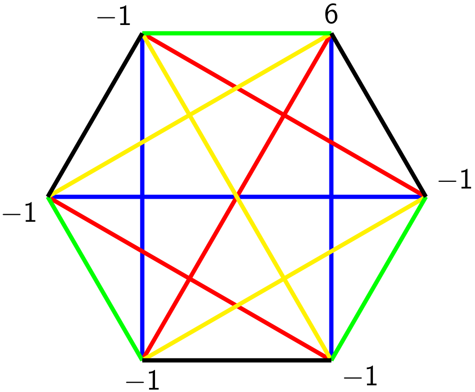

$\nu(K_6) = 5 = \delta(K_6)$, so K_6 has the MMS property

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4. The fractional matching number of G is $\frac{n}{2}$ (where $n = |V|$).

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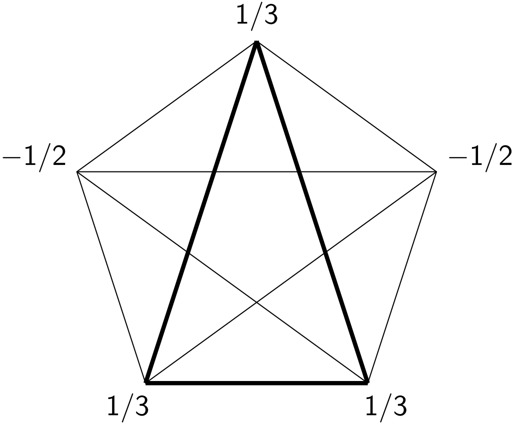
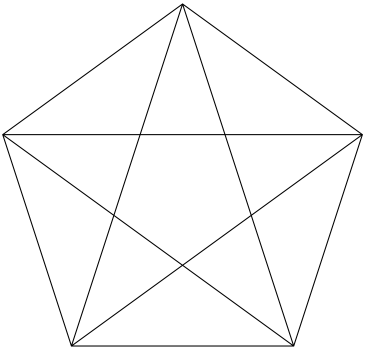
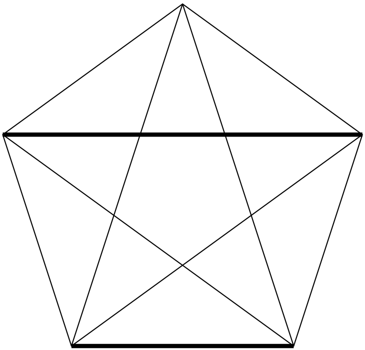
Corollary

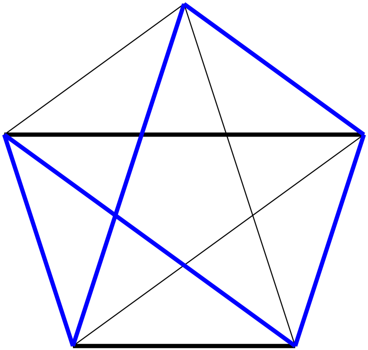
The existence of $\delta(G)$ edge-independent perfect 2-matchings implies the MMS-property.

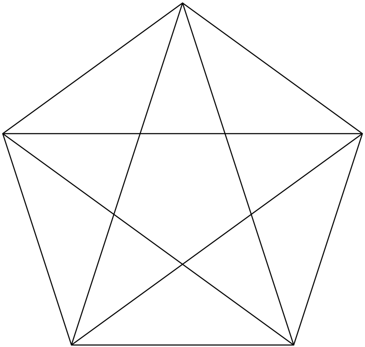


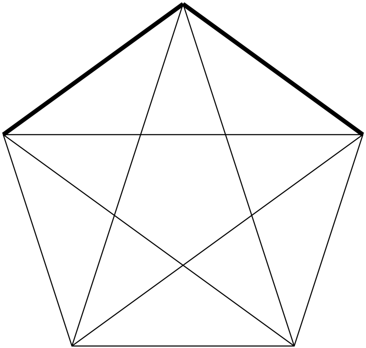
$\nu \leq 3$, so K_5 does not have the MMS property

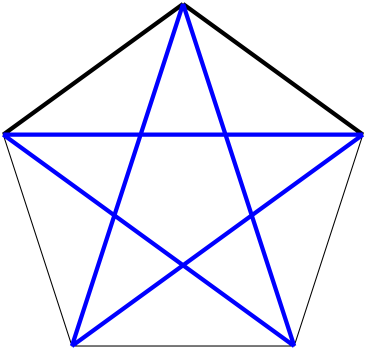


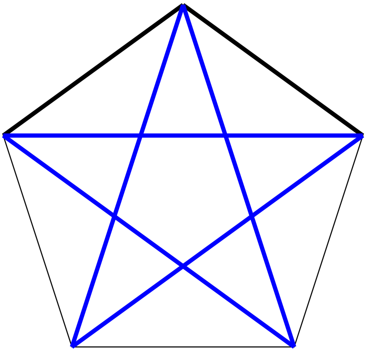












$\nu \geq 3$, so $\nu = 3$

Theorem

If G is a k-regular graph with n vertices, then
1. $\nu(G) \geq k/2$ with equality iff G has an independent vertex subset S such that $|S| = (n-1)/2$.
2. If $k \geq (n^2-1)/(2n-6)$, then $\nu(G) = k$, that is, G has the MMS property.

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Theorem

Let G and H be graphs with respective minimal degrees d and e. If $\nu(H) = e > 0$, then $\nu(G \times H) = d + e$, where $G \times H$ is the Cartesian product of G and H.

The complexity of vertex sign balance [Y. G., Gy. Y. K.,D.M.]

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Theorem (Zenkluse-Ries-Picouleau-Werra-Bentz)

Given an undirected (bipartite) graph $G = (V, E)$ and a positive integer $0 \leq k \leq |E|$, the question whether there exists a set $T \subseteq E$ with $|T| \leq k$ such that for each maximum matching M in G, $|M \cap T| \geq 1$ is NP-complete.

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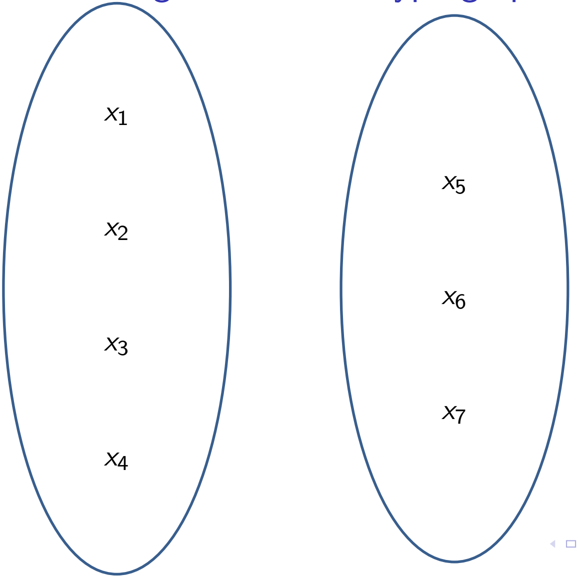
Theorem (Y. Guo - Gy. Y. Katona - DM)

The questions whether $\nu(G) \leq k$ (for a given k) and whether $\nu(G) \leq \delta(G)$ (the min degree of G) are both NP-complete.

The vertex sign balance of hypergraphs



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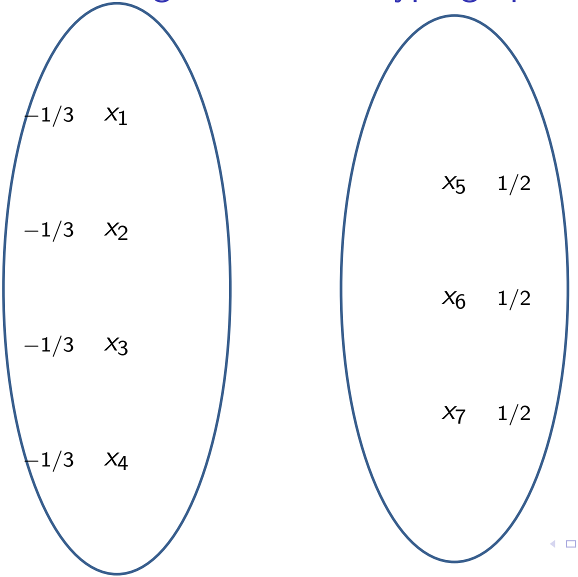

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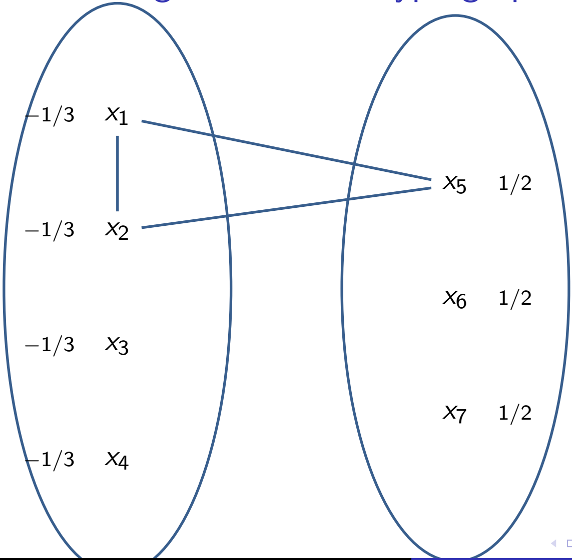

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4. The fractional matching number of G is $\frac{n}{2}$ (where $n = |V|$).

Theorem (Repeat)

For any graph $G = (V, E)$, the following statements are equivalent:
1. $\nu(G) \geq 1$.
2. There is a perfect 2-matching of G.
3. For every independent set $S \subset V$ of vertices, there are at least $|S|$ many vertices not in S, incident to a vertex in S.
4. The fractional matching number of G is $\frac{n}{2}$ (where $n = |V|$).

Theorem

For a t-uniform hypergraph H, $\nu(H) \geq 1$ iff the fractional matching number of $H = n/t$.

Lemma

The existence of $\delta(H)$ edge-independent perfect c-matchings (collections of edges - probably counted with multiplicity - covering each vertex c times) implies the MMS-property.

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Lemma

For a hypergraph H $\nu(H) \geq$ minimum number of edges covering all perfect c-matchings of H.

Proof: Take a weighting of the vertices with minimum number (i.e., $\nu(H)$) of positive edges. If these edges do not cover all perfect c-matchings of H,consider one, it should also contain a positive edge.

The edge sign balance

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Definition

The edge sign balance of a (hyper)graph G, denoted by $\varepsilon(G)$, is defined as the minimum number of vertices whose total weight is nonnegative, where the minimum is taken over all assignments of weights to the edges with nonnegative overall sum.

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Remark

The edge sign balance is always between 0 (for uniform hypergraphs 1) and the minimum size of the vertices.

Remark

The edge sign balance of a hypergraph is equal to the vertex sign balance of the dual of it.

The edge sign balance of graphs

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Theorem

The edge sign balance of a graph G is between 1 and 2 and is equal to 2 iff the graph is bipartite.

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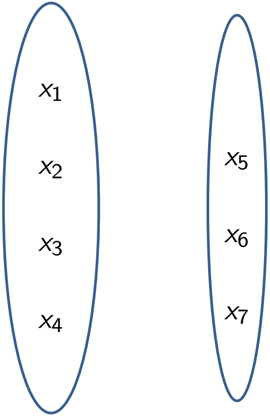

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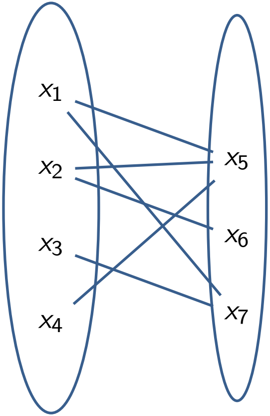

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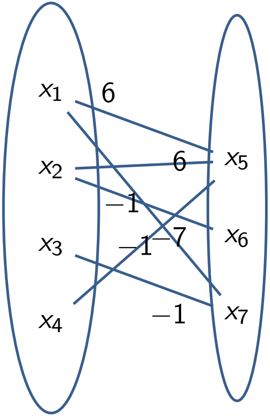

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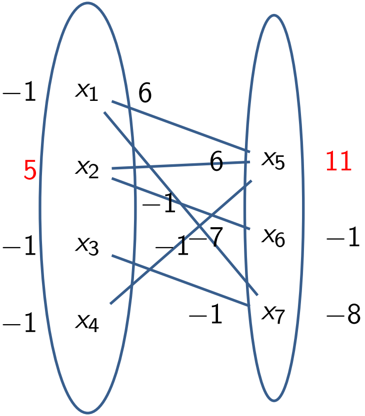

The edge sign balance of 3-uniform hypergraphs

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Conjecture

The edge sign balance of a 3-uniform hypergraph equals the maximum number of vertex-independent perfect 2-coverings.

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Question

For a 3-uniform hypergraph H what is the relation of 3-partitness and $\varepsilon(H) = 3$?