On some open problems in Diophantine Approximation

by Nikolay Moshchevitin

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Some history: Luminy 1982, Problems and Conjectures

Schmidt, W.M.

*Open problems in Diophantine approximation.*

- Simultaneous behavior of \((k - 1)\)-th and \((k + 1)\)-th successive minima of a parametric family of lattices (Solved by Moshchevitin. The development of this problem lead to the concept of "PARAMETRIC GEOMETRY OF NUMBERS": Schmidt-Summerer, D. Roy, A. Keita etc...)

- "BAD-conjecture" (Solved by Badziahin, Pollington, Velani. Extensions by Beresnevich, J. An, Simmons, Nesharim, etc...)

- Small fractional parts of polynomials. ("Ingenious" result by Zaharescu \(\| n^2 \alpha \| \leq n^{-2/3+\varepsilon}\). Lower bounds by Peres-Schlag method.)
Five new short stories

▶ Around Zaremba conjecture
▶ On numbers with restriction on digits
▶ Normal numbers and low discrepancy sequences
▶ Around Peres-Schlag method
▶ Multidimensional Diophantine Approximation
Zaremba’s conjecture in continued fractions theory (1971)

\[
\frac{a}{q} = [b_0(a); b_1(a), \ldots, b_s(a)] = \\
= b_0(a) + \frac{1}{b_1(a) + \frac{1}{b_2(a) + \frac{1}{b_3(a) + \cdots + \frac{1}{b_s(a)}}}}, \quad b_j(a) \in \mathbb{Z}_+,
\]

**Conjecture.**
\[
\forall q \in \mathbb{Z}_+ \exists a : (a, q) = 1 \text{ s.t. } b_j(a) \leq 5, 1 \leq j \leq s.
\]

**Theorem.** (Bourgain, Kontorovich, 2014) Conjecture is true for almost all \( q \) with \( b_j(a) \leq 50 \).

\( b_j(a) \leq 4 : \) Frolenkov, Kan, Huang, Magee, Oh, Winter etc...
Zaremba’s numbers: upper bound

\[ N_k(q) = \#\{ a \pmod{q} : (a, q) = 1, \text{ partial quotients of } a/q \text{ are } \leq k \} \]

(weak) Zaremba conjecture \iff \exists k : N_k(q) \geq 1, \forall q

  \( \exists \kappa \in (0, 1) : N_k(q) \ll q^{\kappa} \)

- Moshchevitin, Sbornik Mathematics 198:4 (2007): \( N_k(q) \ll q^{\kappa_k} \)
  where \( \kappa_k = \text{Hausdorff dimension of } \mathcal{F}_k \),
  \( \mathcal{F}_k = \{ x \in [0, 1] : \text{partial quotients of } x \text{ are } \leq k \} \)

The result is a simple corollary of D. Hensley’s bound

\[ \sum_{q \leq Q} N_k(q) \asymp Q^{2\kappa_k} \]

Conjecture. \( N_k(q) \ll q^{2\kappa_k - 1 + \varepsilon} \)
Numbers with missing digits

$s, k$ – positive integers,

\[ D = \{d_0, \ldots, d_k\}, \quad 0 = d_0 < d_1 < \ldots < d_k < s, \quad 1 \leq k \leq s - 2 \]

\[ (d_1, \ldots, d_k) = 1. \]

\[ K_s^D(N) = \{x \in \mathbb{Z}_+ : x < N, \quad x = \sum_{j=0}^{h} \delta_j s^j, \quad \delta_j \in D\} \]

We are interested in properties of elements of $K_s^D(N)$ modulo $q$ or $p$. 
Numbers with missing digits; Konyagin’s problem

**Theorem.** For $N \gg e^{\gamma \log q \log \log q}$ the set $K_s^D(N)$ contains all residues modulo $q$.

Proved by Konyagin (a version of a large sieve); alternative proofs: elementary properties of congruences for polynomial coefficients; the simplest proof - application of Plünecke’s inequality.

**Conjecture.** (Konyagin) For $N \gg q^\gamma$ the set $K_s^D(N)$ contains all residues modulo $q$.

**Theorem.** (Moshchevitin) For $N \gg q^\gamma$ the congruence

$$x_1x_2 \equiv \lambda \pmod{q}$$

with $q = p$ prime is solvable in $x_1, x_2 \in K_s^D(N)$.
Normal numbers

\[ \xi_1, \xi_2, \ldots, \xi_N, \xi_{N+1}, \ldots, \quad \xi_j \in [0, 1) \]

Discrepancy

\[ D_N = \sup_{x \in [0,1)} |\#\{n : 1 \leq n \leq N, \xi_n \in [0, x)\} - Nx| \]

**Theorem.** (Roth, Schmidt, 1972) For any sequence

\[ \limsup_{N \to \infty} \frac{D_N}{\log N} > 0. \]

Optimality: e.g. for van der Corput sequence \( D_N \ll \log N \)

**Theorem.** (Levin, 1999) One can find \( \alpha \) such that for the sequence \( \xi_n = \{\alpha \cdot 2^n\} \) one has \( D_N \ll (\log N)^2 \)
Question. Can one construct a number $\alpha$ such that the discrepancy of $\xi_n = \{\alpha \cdot 2^n\}$ is $o(\log^2 N)$ or even $O(\log N)$?

Remark. Schmidt’s theorem for the sequences of the type $\xi_n = \{\alpha \cdot 2^n\}$ is trivial: Suppose that discrepancy $D_N \leq \varepsilon \log N$; then in the dyadic expansion of $\alpha$ successive $k = O(\log N)$ zeros occurs infinitely often, and we immediately have a contradiction if we consider the local discrepancy for the segment $(0, 1/2)$ (the events $\xi_{n+j} \in (0, 1/2)$ are NOT independent!)
Lacunary sequences and Peres-Schlag Method

Lacunary sequence \( \{t_n\}_{n \in \mathbb{Z}_+} \):

\[
t_n \geq 1, \quad \frac{t_{n+1}}{t_n} \geq 1 + \frac{1}{M}, \quad M > 1, \quad \forall n
\]

- A. Khintchine (1926): there exists \( \xi \in \mathbb{R} \) s.t.

\[
\|t_n \xi\| \geq \frac{\gamma}{(M \log M)^2}, \quad \gamma > 0, \quad \forall n
\]

- P. Erdős (1974) formulated a conjecture that for lacunary \( \{t_n\} \) there should exist \( \xi \) such that \( \{t_n \xi\} \) is not dense (mod 1).

- Solutions and improvements - de Mathan, Pollington, Katznelson, etc...

- Y. Peres and W. Schlag (2007?): there exists \( \xi \in \mathbb{R} \) s.t.

\[
\|t_n \xi\| \geq \frac{\gamma}{M \log M}, \quad \gamma > 0, \quad \forall n
\]

(they apply Lovasz local lemma).
Optimality of Peres-Schlag’s Bound

- Y. Peres and W. Schlag (2007?):
  \( t_n \geq 1, \quad \frac{t_{n+1}}{t_n} \geq 1 + \frac{1}{M}, \quad M > 1, \quad \forall \ n \) there exists \( \xi \in \mathbb{R} \) s.t.
  \[
  \| t_n \xi \| \geq \frac{\gamma}{M \log M}, \quad \gamma > 0, \quad \forall \ n
  \]

- There exists a sequence \( t_n \) of integers with lacunarity condition under the consideration such that for every \( \xi \in \mathbb{R} \)
  \[
  \| t_n \xi \| \ll \frac{\gamma 1}{M}
  \]
  happens infinitely often.

**Question.** What is the optimal bound here?

Peres-Schlag method has a lot of applications in Diophantine approximation!

Just one example:
\( \exists \alpha : \inf_n \| \alpha n^2 \| \cdot n \log n > 0 \) (weak Schmidt’s conjecture);
Strong Schmidt’s conjecture (\( \exists \alpha : \inf_n \| \alpha n^2 \| \cdot n > 0 \)) is open!
Approximation to two-dimensional subspaces in $\mathbb{R}^4$

$L \subset \mathbb{R}^4$ - two-dimensional subspace, $\mathbb{Z}^4$ - lattice.

\[
L : \begin{cases} 
\theta_{1,1}x_1 + \theta_{1,2}x_2 - y_1 = 0 \\
\theta_{2,1}x_1 + \theta_{2,2}x_2 - y_2 = 0
\end{cases}
\]

best approximation vectors $z_\nu = (x_{1,\nu}, x_{2,\nu}, y_{1,\nu}, y_{2,\nu}) \in \mathbb{Z}^4$: 

\[
\max_{j=1,2} ||\theta_{j,1}x_{1,\nu} + \theta_{j,2}x_{2,\nu}|| = \max_{j=1,2} |\theta_{j,1}x_{1,\nu} + \theta_{j,2}x_{2,\nu} - y_{j,\nu}| < \\
\max_{j=1,2} ||\theta_{j,1}x_1 + \theta_{j,2}x_2||
\]

for all $(x_1, x_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ s.t.

\[
\max(|x_1|, |x_2|) < \max(|x_{1,\nu}|, |x_{2,\nu}|)
\]
Approximation to two-dimensional subspaces in $\mathbb{R}^4$: strange problem about asymptotic dimension

$$R(\Theta) = \min\{\dim L : L – \text{linear subspace, such that } z_\nu \in L \ \forall \nu \geq \nu_0\}$$

Suppose that the vectors $z_\nu \in \mathbb{Z}^4$ of the best approximations for the $2 \times 2$ matrix $\Theta$ are uniquely defined up to sign $\pm$ (the matrix $\Theta$ is "good").

From Jarnik’s result it follows that $R(\Theta) \neq 3$.

**Question.** Is it possible for a "good" $2 \times 2$ matrix $\Theta$ that $R(\Theta) = 2$?

For $2 \times 3$ matrices it is possible!
THANK YOU!