On some open problems in Diophantine Approximation

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Some history: Luminy 1982, Problems and Conjectures

Schmidt, W.M.
Open problems in Diophantine approximation.
In: Diophantine Approximations and Transcendental Numbers (Luminy, 1982), pp. 271-287. Progr. Math. 31, Birkhäuser, Boston (1983)

Schmidt, W.M.: *Open problems in Diophantine approximation* (1982)

- Simultaneous behavior of $(k-1)$-th and $(k+1)$-th successive minima of a parametric family of lattices (Solved by Moshchevitin. The development of this problem lead to the concept of "PARAMETRIC GEOMETRY OF NUMBERS": Schmidt-Summerer, D. Roy, A. Keita etc...)
- "BAD-conjectute"(Solved by Badziahin, Pollington, Velani. Extensions by Beresnevich, J. An, Simmons, Nesharim, etc...)
- Small fractional parts of polynomials. ("Ingenious"result by Zaharescu $||n^2\alpha|| \leq n^{-2/3+\varepsilon}$. Lower bounds by Peres-Schlag method.)

Five new short stories

- Around Zaremba conjecture
- On numbers with restriction on digits
- Normal numbers and low discrepancy sequences
- Around Peres-Schlag method
- Multidimensional Diophantine Approximation

Zaremba's conjecture in continued fractions theory (1971)

$$\frac{a}{q} = [b_0(a); b_1(a), \dots, b_s(a)] =$$

$$= b_0(a) + \cfrac{1}{b_1(a) + \cfrac{1}{b_2(a) + \cfrac{1}{b_3(a) + \dots + \cfrac{1}{b_s(a)}}}}, \quad b_j(a) \in \mathbb{Z}_+$$

Conjecture.
$\forall\, q \in \mathbb{Z}_+ \quad \exists\, a: \quad (a, q) = 1 \;\; \text{s.t.} \;\; b_j(a) \le 5, \; 1 \le j \le s.$

Theorem. (Bourgain, Kontorovich, 2014) Conjecture is true for *almost all* q with $b_j(a) \le 50$.

$b_j(a) \le 4$: Frolenkov, Kan, Huang, Magee, Oh, Winter etc...

Zaremba's numbers: upper bound

$N_k(q) = \sharp\{a \pmod q : \ (a,q)=1, \text{ partial quotients of } a/q \text{ are } \leq k\}$

(weak) Zaremba conjecture $\iff \exists\, k : \ N_k(q) \geq 1, \ \forall q$

• O. David and U. Shapira, arXiv:1707.00427 (July 2017):
$\exists \varkappa \in (0,1) : \ N_k(q) \ll q^{\varkappa}$

• Moshchevitin, Sbornik Mathematics 198:4 (2007): $N_k(q) \ll q^{\varkappa_k}$
where $\varkappa_k = \text{Hausdorff dimension of } \mathcal{F}_k$,
$\mathcal{F}_k = \{x \in [0,1] : \text{ partial quotients of } x \text{ are } \leq k\}$
The result is a simple corollary of D. Hensley's bound

$$\sum_{q \leq Q} N_k(q) \asymp Q^{2\kappa_k}$$

Conjecture. $N_k(q) \ll q^{2\varkappa_k - 1 + \varepsilon}$

Numbers with missing digits

s, k – positive integers,

$$D = \{d_0, ..., d_k\},\ 0 = d_0 < d_1 < ... < d_k < s,\ \ 1 \le k \le s-2$$

$$(d_1, ..., d_k) = 1.$$

$$K_s^D(N) = \{x \in \mathbb{Z}_+ :\ \ x < N,\ \ x = \sum_{j=0}^{h} \delta_j s^j,\ \ \delta_j \in D\}$$

We are interested in properties of elements of $K_s^D(N)$ modulo q or p.

Numbers with missing digits; Konyagin's problem

Theorem. *For* $N \gg e^{\gamma \log q \log\log q}$ *the set* $K_s^D(N)$ *contains all residues modulo* q.
Proved by Konyagin (a version of a large sieve); alternative proofs: elementary properties of congruences for polynomial coefficients; the simplest proof - application of Plünecke's inequality.

Conjecture. (Konyagin) *For* $N \gg q^\gamma$ *the set* $K_s^D(N)$ *contains all residues modulo* q.

Theorem. (Moshchevitin) *For* $N \gg q^\gamma$ *the congruence*

$$x_1 x_2 \equiv \lambda \pmod{q}$$

with $q = p$ *prime is solvable in* $x_1, x_2 \in K_s^D(N)$

Normal numbers

$$\xi_1, \xi_2, \ldots, \xi_N, \xi_{N+1}, \ldots, \quad \xi_j \in [0,1)$$

Discrepancy

$$D_N = \sup_{x \in [0,1)} |\#\{n : 1 \le n \le N, \xi_n \in [0,x)\} - Nx|$$

Theorem. (Roth, Schmidt, 1972) For any sequence

$$\limsup_{N \to \infty} \frac{D_N}{\log N} > 0.$$

Optimality: e.g. for van der Corput sequence $D_N \ll \log N$

Theorem. (Levin, 1999) One can find α such that for the sequence $\xi_n = \{\alpha \cdot 2^n\}$ one has $D_N \ll (\log N)^2$

Normal numbers: what is the smallest order of discrepancy?

Question. Can one construct a number α such that the discrepancy of $\xi_n = \{\alpha \cdot 2^n\}$ is $o(\log^2 N)$ or even $O(\log N)$?

Remark. Schmidt's theorem for the sequences of the type $\xi_n = \{\alpha \cdot 2^n\}$ is trivial: Suppose that discrepancy $D_N \leq \varepsilon \log N$; then in the dyadic expansion of α successive $k = O(\log N)$ zeros occurs infinitely often, and we immediately have a contradiction if we consider the local discrepancy for the segment $(0, 1/2)$ (the events $\xi_{n+j} \in (0, 1/2)$ are NOT independent!)

Lacunary sequences and Peres-Schlag Method

Lacunary sequence $\{t_n\}_{n\in\mathbb{Z}_+}$:

$$t_n \geq 1, \quad \frac{t_{n+1}}{t_n} \geq 1 + \frac{1}{M}, \; M > 1, \quad \forall\, n$$

- A. Khintchine (1926): there exists $\xi \in \mathbb{R}$ s.t.

$$||t_n\xi|| \geq \frac{\gamma}{(M \log M)^2}, \quad \gamma > 0, \quad \forall\, n$$

- P. Erdős (1974) formulated a conjecture that for lacunary $\{t_n\}$ there should exist ξ such that $\{t_n\xi\}$ is not dense $\pmod 1$.
- Solutions and improvements - de Mathan, Pollington, Katznelson, etc...
- Y. Peres and W. Schlag (2007?): there exists $\xi \in \mathbb{R}$ s.t.

$$||t_n\xi|| \geq \frac{\gamma}{M \log M}, \quad \gamma > 0, \quad \forall\, n$$

(they apply Lovasz local lemma).

Optimality of Peres-Schlag's Bound

- Y. Peres and W. Schlag (2007?):

$t_n \geq 1, \quad \frac{t_{n+1}}{t_n} \geq 1 + \frac{1}{M}, \; M > 1, \quad \forall\, n$ there exists $\xi \in \mathbb{R}$ s.t.

$$||t_n \xi|| \geq \frac{\gamma}{M \log M}, \quad \gamma > 0, \quad \forall\, n$$

- There exists a sequence t_n of integers with lacunarity condition under the consideration such that for every $\xi \in \mathbb{R}$

$$||t_n \xi|| \ll \frac{\gamma_1}{M}$$

happens infinitely often.

Question. What is the optimal bound here?

Peres-Schlag method has a lot of applications in Diophantine approximation!

Just one example:

$\exists\, \alpha : \; \inf_n ||\alpha n^2|| \cdot n \log n > 0$ (weak Schmidt's conjecture);

Strong Schmidt's conjecture ($\exists\, \alpha : \; \inf_n ||\alpha n^2|| \cdot n > 0$) is open!

Approximation to two-dimensional subspaces in $\mathbb{R}^4$

$L \subset \mathbb{R}^4$ - two-dimensional subspace, $\mathbb{Z}^4$ - lattice.

$$L: \begin{cases} \theta_{1,1}x_1 + \theta_{1,2}x_2 - y_1 = 0 \\ \theta_{2,1}x_1 + \theta_{2,2}x_2 - y_2 = 0 \end{cases}$$

best approximation vectors $\boldsymbol{z}_\nu = (x_{1,\nu}, x_{2,\nu}, y_{1,\nu}, y_{2,\nu}) \in \mathbb{Z}^4$:

$$\max_{j=1,2} ||\theta_{j,1}x_{1,\nu} + \theta_{j,2}x_{2,\nu}|| = \max_{j=1,2} |\theta_{j,1}x_{1,\nu} + \theta_{j,2}x_{2,\nu} - y_{j,\nu}| <$$

$$\max_{j=1,2} ||\theta_{j,1}x_1 + \theta_{j,2}x_2||$$

for all $(x_1, x_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ s.t.

$$\max(|x_1|, |x_2|) < \max(|x_{1,\nu}|, |x_{2,\nu}|)$$

Approximation to two-dimensional subspaces in $\mathbb{R}^4$: strange problem about asymptotic dimension

$R(\Theta) = \min\{\dim L : \ L - \text{linear subspace, such that } \ \boldsymbol{z}_\nu \in L \ \forall \nu \geq \nu_0\}$

Suppose that the vectors $\boldsymbol{z}_\nu \in \mathbb{Z}^4$ of the best approximations for the 2×2 matrix Θ are uniquely defined up to sign $\pm$ (the matrix Θ is "good").

From Jarnik's result it follows that $R(\Theta) \neq 3$.

Question. Is it possible for a "good" 2×2 matrix Θ that $R(\Theta) = 2$?

For 2×3 matrices it is possible!

THANK YOU!