Small sumsets in $\mathbb{R}$

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Introduction

Let A, B be subsets in $\mathbb{R}$ or $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.
Define $A + B = \{a + b \,:\, a \in A,\ b \in B\}$.

Questions:

- How small can $A + B$ be compared to A and B ?
- If $A + B$ is close to the smallest possible, what can be said about the structures of A, B and $A + B$?

If $A \subset \mathbb{R}$, we write $\lambda(A)$ for the inner Lebesgue measure of A.
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Lower bounds in $\mathbb{R}$ and $\mathbb{T}$

If $A, B \subset \mathbb{R}$ are bounded, then

$$\lambda(A+B) \geq \lambda(A) + \lambda(B)$$

and equality holds if and only if A and B each have full measure in an interval.

Thm (Raikov, 1939)

If $A, B \subset \mathbb{T}$, then $\mu(A+B) \geq \min(\mu(A)+\mu(B), 1)$.

Thm (Ruzsa, 1991)

Let $A, B \subset \mathbb{R}$ bounded. If $\mathrm{diam}(A) = \sup A - \inf A$ and $\lambda(B) \leq \lambda(A)$ then $\lambda(A+B) \geq \min(\lambda(A) + 2\lambda(B), \mathrm{diam}(A) + \lambda(B))$.

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Proof of Ruzsa's thm in case $A = B$.

Assume A closed, $\min A = 0$ and $\max(A) = 1$ and define

$$S_1 = \{x \in [0,1] \ : \ x \in A + A \ \text{or} \ x + 1 \in A + A\};$$
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We have $\lambda(A + A) = \mu(S_1) + \mu(S_2)$.

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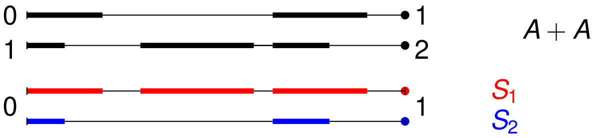

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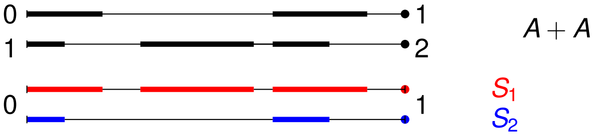

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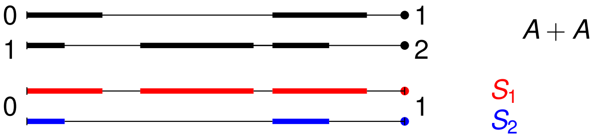

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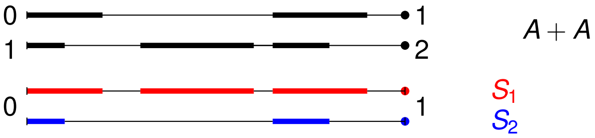

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A continuous Freiman $3k-4$ theorem.

Thm (dR, 2016)

Let $A, B \subset \mathbb{R}$ be bounded. If either

i) $\lambda(A+B) < \lambda(A) + \lambda(B) + \min(\lambda(A), \lambda(B))$;

ii) *or* $\mathrm{diam}(B) \leq \mathrm{diam}(A)$ *and* $\lambda(A+B) < \lambda(A) + 2\lambda(B)$;

then

1. $\mathrm{diam}(A) \leq \lambda(A+B) - \lambda(B)$,
2. $\mathrm{diam}(B) \leq \lambda(A+B) - \lambda(A)$,
3. *and there exists an interval $I \subset A+B$ of length $\ell(I) \geq \lambda(A) + \lambda(B)$.*

Discrete analogues by Freiman, Lev-Smeliansky, Stanchescu, Bardaji-Grynkiewicz.

Proof in the case $A = B$:
Assume $\lambda(A+A) < 3\lambda(A)$, A closed, bounded and $\min A = 0$, $\max A = D$.
By Ruzsa's inequality, $D \leq \lambda(A+A) - \lambda(A)$ and $\Delta = 2\lambda(A) - D > 0$.

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The method of switches for large sets : a density lemma

Assume A bounded and $\inf A = 0$.

1. If $x \geq 0$ and $x \notin A + A$ then $2\lambda(A \cap [0, x]) \leq x$.
2. If $x \leq 2D$ and $x \notin A + A$ then $2\lambda(A \cap [x - D, D]) \leq 2D - x$.

We define $g(x) = 2\lambda(A \cap [0, x])$.
We have

- $g(x) > x \Rightarrow x \in A + A$,
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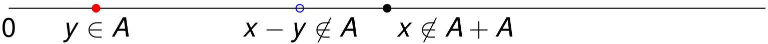

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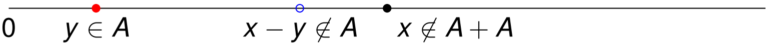

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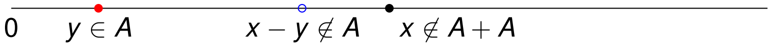

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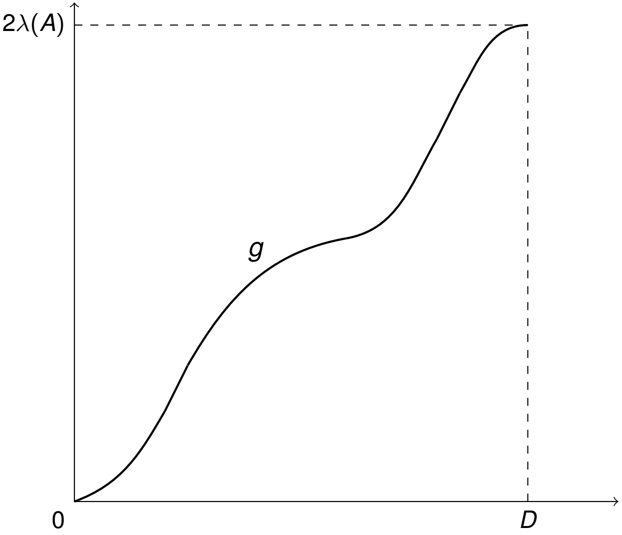

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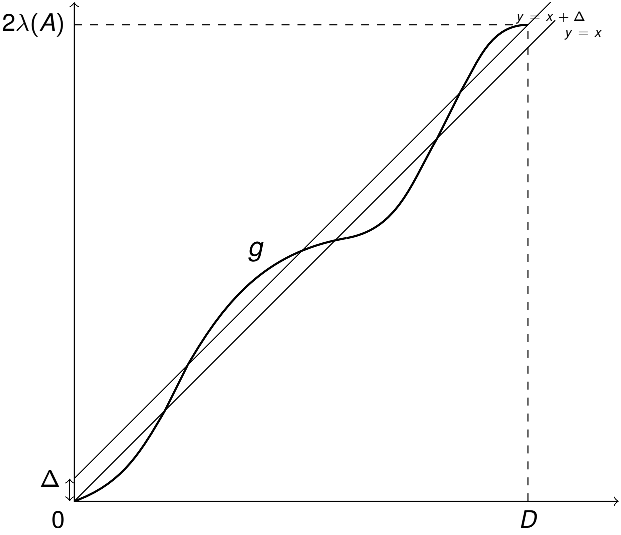

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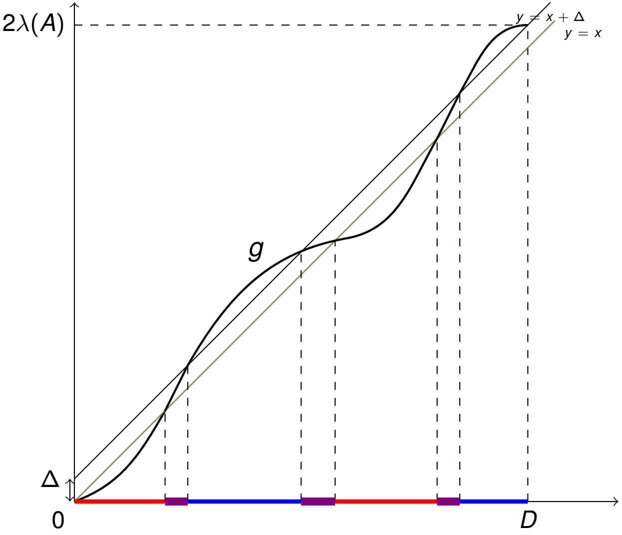

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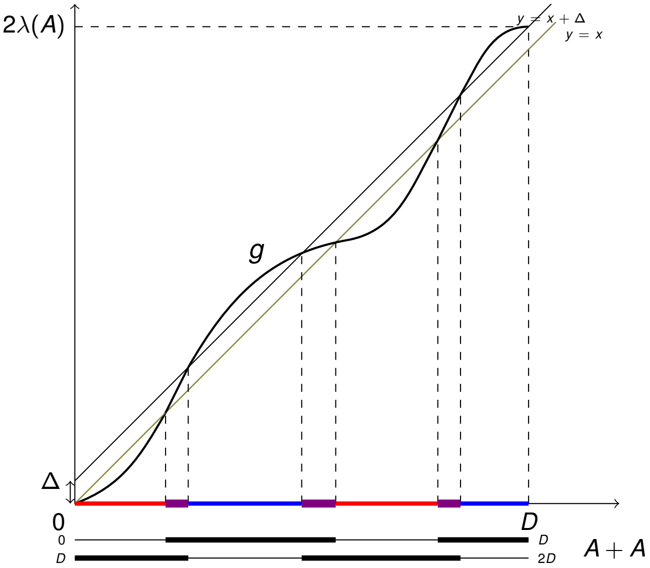

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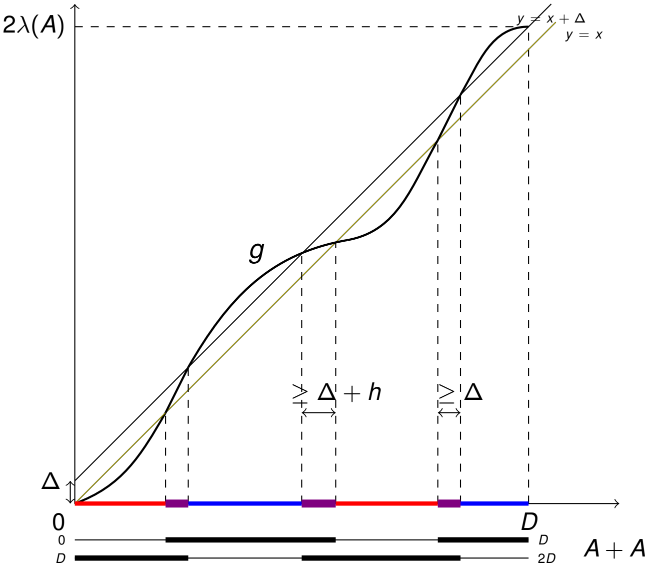

End of the proof : structure of $A+A$

$$\lambda(A+A) = D + \lambda(A) + \lambda(S_2 \cap A^c)$$

If any down crossing, we have $\lambda(S_2 \cap A^c) \geq \Delta$, thus

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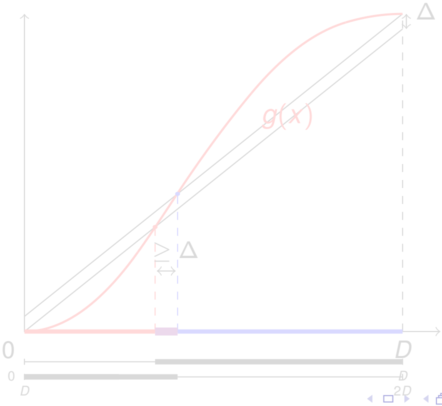

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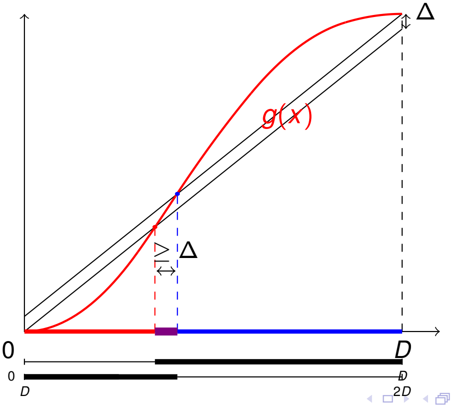

Small sumsets in $\mathbb{T}$

Thm (Candela, dR, 2017)

If $A \subset \mathbb{T}$ satisfy $\mu(A+A) < \min\left((2+10^{-4})\mu(A)\,;\,\frac{1}{2}+\mu(A)\right)$, then $\exists I, K \subset \mathbb{T}$ intervals, $\exists n \geq 1$ such that $n \cdot A \subset I$, $K \subset n \cdot (A+A)$ and $\mu(I) \leq \mu(A+A) - \mu(A)$, $\mu(K) \geq 2\mu(A)$.

Thm (Bilu, 1998)

There exists $c > 0$ such that for any $A \subset \mathbb{T}$ satisfying $\mu(A) \leq c$ and $\mu(A+A) < \min(1, 3\mu(A))$, there exist a positive integer n and an interval I of length $\ell(I) \leq \mu(A+A) - \mu(A)$ such that $n \cdot A \subset I$.

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Sets of doubling constant less than $3+\varepsilon$ in $\mathbb{R}$.

Thm (Eberhard, Green, Manners, 2014)

If $A \subset [0,1]$ is an open set with $\lambda(A+A) \le 4\lambda(A) - \delta$ then there is an interval I of length $\ell(I) \gg_\delta 1$ such that $\lambda(A \cap I) \ge (\frac{1}{2} + \frac{1}{7}\delta)\lambda(I)$.

Thm (Candela, dR, 2017)

Let $A \subset [0,1]$ satisfy $\lambda(A+A) \le (3+\varepsilon)\lambda(A)$ with $\varepsilon \le 10^{-4}$. Then $A \subset I$ with I an interval of $\mathbb{T}$ of length at most $(1+\varepsilon)\lambda(A)$.

Corollary

If $A \subset [0,1]$ is a closed set with $\lambda(A+A) \le \min\left(4\lambda(A) - \delta, \frac{1}{4} + \frac{\delta}{2}\right)$ and either $\lambda(A) < cD_A$ or $\delta > (1-10^{-4})\lambda(A)$, then there is an interval I of length $\ell(I) \ge \min(\delta^2, \delta/4)$ such that $\lambda(A \cap I) \ge (\frac{1}{2} + \frac{1}{4}\delta)\lambda(I)$.

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