The Erdős-Hajnal problem on colorings of hypergraphs, its on-line generalizations and related questions

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Definitions

• A hypergraph \( H = (V, E) \) is a vertex set \( V \) and a family of subsets \( E \subseteq 2^V \) whose elements are called the edges of the hypergraph.

• A hypergraph \( H = (V, E) \) is said to be \( k \)-uniform if every edge consists of exactly \( k \) vertices.

• So, \( 2 \)-uniform hypergraph is just a usual graph.

• Example, a hypergraph of arithmetic progressions:
  • \( V = \{1, \ldots, n\} \),
  • \( E \) is a set of all AP-\( k \), all arithmetic progressions of length \( k \) in \( V \), i.e.
    \[ E = \{(1,2, \ldots, k), (2,3, \ldots, k + 1), \ldots\}. \]
Definitions

• Let $H = (V, E)$ be a hypergraph. A vertex coloring is called proper for $H$ if there is no monochromatic edges in this coloring.

• A hypergraph is said to be $r$-colorable if there is a proper coloring with $r$ colors for it.

• The chromatic number of the hypergraph $H$, denoted by $\chi(H)$, is the minimum number of colors required for a proper coloring.

• Example, Petersen graph

  It has 10 vertices, 15 edges,
  it is triangle-free,
  chromatic number equals 3.
Extremal problem

**Property B problem** (P. Erdős, A. Hajnal, 1961)

What is the minimum possible number of edges in a $k$-uniform hypergraph with chromatic number greater than two?

$$m(k) = \min \{ |E(H)|: \chi(H) > 2, H \text{ is } k-\text{uniform} \}.$$

Natural generalization of the problem:

$$m(k, r) = \min \{ |E(H)|: \chi(H) > r, H \text{ is } k-\text{uniform} \}.$$
Exact results

- $m(2) = 3$ (triangle).
- $m(2, r) = \binom{r+1}{2}$ (a complete graph on $r + 1$ vertices).
- $m(3) = 7$
- **Fano plane**
  - It is a 3-uniform 3-regular hypergraph on 7 vertices.
  - It has 7 edges and chromatic number equal to 3.
- $m(4) = 23$ (computer search)

What about asymptotic behavior?
Erdős-Hajnal problem

Simple bound

\[ m(k) \leq \binom{2k - 1}{k} \sim \frac{4^k}{2\sqrt{\pi k}}. \]

Best upper bound (P. Erdős, 1964): take a random hypergraph

\[ m(k) \leq \frac{e \ln 2}{4} k^2 2^k (1 + o(1)). \]

Best lower bound (J. Radhakrishnan, A. Srinivasan, 2000):

\[ m(k) \geq (\sqrt{3} - 1) \left( \frac{k}{\ln k} \right)^{1/2} 2^{k-1}. \]

Conjecture (P. Erdős, L. Lovász, 1973)

\[ m(k) = \Theta(k 2^k). \]
Erdős-Hajnal problem

What about \( r > 2 \)?

The random hypergraph gives the following upper bound: for any \( k, r \),

\[
m(k, r) \leq \frac{e}{2} k^2 r^k \ln r \left( 1 + O \left( \frac{1}{k} \right) \right).
\]

Lower bound (D. Cherkashin, J. Kozik 2014): for any \( k, r \),

\[
m(k, r) \geq c \left( \frac{k}{\ln k} \right)^{r - 1} r^{k - 1}.
\]

These bounds are the best for any fixed \( r \) and growing \( k \).
Case of growing number of colors

What is happening for \( r \gg k \)?

Note that for growing \( r \) and fixed \( k \), the simple bound

\[
m(k, r) \leq \binom{(r - 1)k + 1}{k} = \Theta_k(r^k)
\]

is better than the bound provided by random hypergraph. What is the right order on \( r \)?

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**Theorem (N. Alon, 1985)**

\[
m(k, r) > (k - 1) \left\lfloor \frac{r}{k} \right\rfloor \left\lfloor \frac{k - 1}{k} \frac{r}{k} \right\rfloor^{k-1} r^{k-1},
\]

i.e. for \( r > k, m(k, r) > \Omega(r^k) \).
Case of growing number of colors

Theorem (N. Alon, 1985)

For $r > k$,

$$m(k, r) = O\left(\frac{5}{k^2} (\ln k) \left(\frac{3}{4}\right)^k \binom{(r - 1)k + 1}{k}\right).$$

Theorem (D. Shabanov, 2010)

For $r > k$,

$$m(k, r) = \Omega(k^{1/2}r^k).$$
Recent bounds

Theorem (I. Akolzin, D. Shabanov, 2016)

There exist absolute constants $c_1, c_2 > 0$ such that for any $r > k$,

$$c_1 \frac{k}{\ln k} r^k \leq m(k, r) \leq c_2 k^3 (\ln k) r^k.$$ 

• The lower bound improves the previous result of 2010 and generalizes the result of Cherkashin and Kozik.

• The upper bound improves the Alon’s bound which has the order $k^2 (\ln k) \left(\frac{3e}{4}\right)^k r^k$.

• It also improves the bound given by the random hypergraph for $k(\ln k) < \ln r$. 
Alon's conjecture

Conjecture (N. Alon, 1985)

For every \( k \geq 2 \), there exists a limit

\[
\lim_{r \to \infty} \frac{m(k, r)}{r^k}.
\]

- We do not prove the existence of a limit, but give reasonable (polynomial as functions of \( k \)) bounds for \( \liminf_{r \to \infty} \frac{m(k, r)}{r^k} \) and \( \limsup_{r \to \infty} \frac{m(k, r)}{r^k} \).
Ideas of the proofs

Let $H = (V, E)$ be a hypergraph and let $\sigma$ be an ordering of its vertex set. The ordered subset of edges $(A_1, \ldots, A_r)$ forms an ordered $r$-chain with respect to $\sigma$ if for any $i < j$, $v \in A_i, u \in A_j$ it holds that $\sigma(v) \leq \sigma(u)$ and $|A_i \cap A_{i+1}| = 1, |A_i \cap A_j| = 0$ when $j > i + 1$.

Theorem (A. Pluhár, 2009)

$\chi(H) \leq r$ iff there is an ordering of the vertices of $H$ without ordered $r$-chains.
Ideas of the proofs

1) Let $H = (V,E)$ be a $k$-uniform hypergraph with small number of edges.

2) Set $a = \left\lfloor \frac{k-1}{k} r \right\rfloor$, $b = \left\lfloor r/k \right\rfloor$ and take for any vertex $v \in V$ independent random variable $X_v$ with uniform distribution on $[0,1]$.

3) For any edge $A \in E$, define
   \[ f(A) = \min_{v \in A} X_v, \quad l(A) = \max_{v \in A} X_v. \]

The edge $A$ is said to be bad if $l(A) - f(A) \leq \frac{1-p}{r}$. Otherwise it is called good.

4) Denote $H'$ for the hypergraph of good edges. We show that with positive probability $\chi(H') \leq a$ and the number of bad edges is less than $(k-1)b$ and they can be made non-monochromatic by using the remained colors.
List chromatic number

- Let $H = (V, E)$ be a hypergraph and let $L = (L(v), v \in V)$ be a set of color lists. It is called a list assignment.
- A list assignment $L$ is said to be $r$-uniform if $|L(v)| = r$ for any $v \in V$.
- A vertex coloring $f$ corresponds to the list assignment $L$ if $f(v) \in L(v)$ for every $v \in V$ ($f$ is a coloring from the lists).
- A hypergraph $H$ is called $r$-choosable if for every $r$-uniform list assignment, there is a proper coloring from the lists.
- The list chromatic number of the hypergraph $H$, denoted by $\chi_l(H)$, is the minimum $r$ such that $H$ is $r$-choosable.
- Clearly, $\chi(H) \leq \chi_l(H)$.
List chromatic number

• Example, bipartite graph $K_{3,3}$. Its chromatic number equals 2.

• But its list chromatic number is equal to 3. Indeed, there is a bad 2-uniform list assignment.
Asymptotics for graphs

Theorem (P. Erdős, A. Rubin, H. Taylor, 1980)

1. If $2m < m(k)$ then $\chi_l(K_{m,m}) \leq k$.
2. If $m \geq m(k)$ then $\chi_l(K_{m,m}) > k$.
3. Together with the known bounds for $m(k)$, this implies that

$$\chi_l(K_{m,m}) = \log_2 m - O(\log_2 \log_2 m).$$

Theorem (D. Saxton, A. Thomason, 2012)

Let $G$ be a graph with average degree $d$. Then

$$\chi_l(G) \geq (1 + o(1))\log_2 d.$$
List on-line chromatic number

• Let $G = (V, E)$ be a graph and let $r \geq 2$ be an integer.

$(G, r)$ - GAME 1.

• Suppose there are two players Lister and Painter. Set $X_0 = \emptyset$.
• In round $i$ Lister picks a vertex subset $V_i \subset V \setminus (X_1 \cup \cdots \cup X_{i-1})$.
• Painter chooses an independent subset $X_i \subset V_i$ (i.e. the vertices of $X_i$ are not adjacent in $G$) and colors all its vertices with color $i$.
• After round $i$ the vertices in $X_1 \cup \cdots \cup X_i$ are colored.
• If a vertex $v$ belongs to $l$ sets among $V_1, \ldots, V_i$ then it is said to have $l$ permissible colors after round $i$. 
List on-line chromatic number

• The winning rule is the following:
  – Lister wins if after some round there exists a non-colored vertex with $r$ permissible colors.
  – Painter wins if after some round all the vertices are colored.

• Graph $G$ is said to be list on-line $r$-colorable if Painter has a winning strategy in $(G, r)$-game 1.

• The list on-line chromatic number of a graph $G$, $\chi_{ol}(G)$, is the minimum $r$ such that $G$ is list on-line $r$-colorable.

• Since during the game Lister constructs the color lists on-line,

\[ \chi_l(G) \leq \chi_{ol}(G).\]
Asymptotics for bipartite graphs

Theorem (L. Duraj, G. Gebowski, J. Kozik, 2015)

\[ \chi_{ol}(K_{m,m}) = \log_2 m - O(1). \]

- Recall that \( \chi_l(K_{m,m}) = \log_2 m - O(\log_2 \log_2 m). \)

- So, the difference between the list on-line chromatic number and the list chromatic number can be arbitrarily large.

- The key ingredient of the proof is the connection with the on-line version of the Erdős-Hajnal problem.
On-line colorings of hypergraphs

• Let \( n \geq 1, k \geq 2 \) and \( r \geq 2 \) be integers.

\( (n, k, r) \) – GAME 2.

• Suppose there are two players Lister and Painter.
• In round \( i \) Lister presents a new vertex \( v_i \) and declares in which edges it is contained (i.e. gives some numbers among \{1, ..., n\}).
• Lister cannot add vertices to the edges that already have \( k \) vertices.
• Painter must immediately assign one of the \( r \) colors to the vertex \( v_i \).
On-line colorings of hypergraphs

- The game ends when all $n$ edges contain exactly $k$ vertices.
  - Painter wins if the obtained $r$-coloring is proper for the constructed hypergraph.
  - Otherwise Lister wins.

- By analogy with the classical Erdős-Hajnal problem we introduce the value $m_{ol}(k, r)$ which is equal to the minimum $n$ such that Lister has a winning strategy in $(n, k, r)$ – GAME 2.

- Clearly,
  $$m_{ol}(k, r) \leq m(k, r),$$
  because for $n = m(k, r)$ Lister can always construct a non-$r$-colorable hypergraph.
## Known results

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<tr>
<th>Theorem (J. Aslam, A. Dhagat, 1993)</th>
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<td>$m_{ol}(k, r) \geq r^{k-1}$.</td>
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<th>Theorem (L. Duraj, G. Gebowski, J. Kozik, 2015)</th>
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<td>$m_{ol}(k, 2) \leq 8 \cdot 2^{k-1}$.</td>
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- So, in the case of two colors $m_{ol}(k, 2) = \Theta(2^k)$.
- Recall that $m(k, 2) = m(k) = \Omega\left(\frac{1}{2^{o(1)}} 2^k\right)$. 
New results

Recall that for fixed $k \geq 2$, $m(k, r) = \Theta_k(r^k)$.

Theorem (A. Khuzieva, D. Shabanov, P. Svyatokum, 2017+)

Suppose $r > k$ and let us denote $a = \left\lfloor \frac{k-1}{k} r \right\rfloor$, $b = r - a = \left\lfloor \frac{r}{n} \right\rfloor$. Then

$$m_{ol}(k, r) \geq ((k - 1)b + 1)a^{k-1} = \Omega(r^k).$$

• Thus, for fixed $k \geq 2$, $m_{ol}(k, r) = \Theta_k(r^k)$.

Theorem (A. Khuzieva, D. Shabanov, P. Svyatokum, 2017+)

$$m_{ol}(k, r) \leq k(r - 1)^2 r^k.$$
Ideas of the proof

We have to show that for

\[ n < ((k - 1)b + 1)a^{k-1}, \]

Painter has a winning strategy in \((n, k, r) - \text{GAME 2}\). It can be described as follows.

1. Let us split the set of colors into two groups: \([1, ..., a]\) and \([a + 1, ..., r]\).
2. Suppose after round \(i\) vertices \(v_1, ..., v_i\) are colored.
3. In round \(i + 1\) Lister presents a vertex \(v_{i+1}\) and names the edges that contain it.
4. Let random variable \(X\) denotes the number of monochromatic edges when Painter colors all the remaining uncolored vertices (including \(v_{i+1}\)) randomly with first \(a\) colors.
Ideas of the proof

5. Painter calculates $a$ numbers:

$$d_j = \mathbb{E}(X|v_{i+1} \text{ is colored with } j), j = 1, \ldots, a,$$

and chooses the smallest value $d_q$.

6. If $d_q < 1$ then Painter colors $v_{i+1}$ with color $q$.

7. If $d_q \geq 1$ then chooses a color $q'$ from $\{a + 1, \ldots, r\}$ that has not been used $k - 1$ times and Painter colors $v_{i+1}$ with color $q'$.

8. If $d_q \geq 1$ and every color in $\{a + 1, \ldots, r\}$ has already been used $k - 1$ times then Painter colors $v_{i+1}$ with color $q$.

It can be shown that Painter always wins by using the described strategy.
Multipartite graphs

- Let $K_{m^*r}$ denote a complete $r$-partite graph with $m$ vertices in every part.
- It is known (M. Krivelevich, N. Gazit, 2006) that for any fixed $r \geq 3$,
  $$\chi_l(K_{m^*r}) = (1 + o(1)) \log \frac{r}{r-1} m$$
as $m \to \infty$.
- We extends the above result.

**Theorem (A. Khuzieva, D. Shabanov, P. Svyatokum, 2017+)**

For any fixed $r \geq 3$,
$$\chi_{ol}(K_{m^*r}) = (1 + o(1)) \log \frac{r}{r-1} m$$
as $m \to \infty$. 
Panchromatic colorings

- Let $H = (V, E)$ be a hypergraph. A vertex coloring with $r$ colors is called *panchromatic* for $H$ if every edge meets every color under this coloring.

- It appears that panchromatic $r$-colorings play the same role for $K_{m^*r}$ as proper 2-colorings for $K_{m,m}$.

- Consider the panchromatic version of $(n, k, r)$ – GAME 2:
  - the game process is absolutely the same;
  - Painter wins if the obtained $r$-coloring is panchromatic for the constructed hypergraph.

- Let $p_{ol}(k, r)$ denote the minimum $n$ such that Lister has a winning strategy in the panchromatic version of $(n, k, r)$ – GAME 2.
Panchromatic on-line colorings

**Lemma 1**

1. If $rm < p_{ol}(k, r)$ then $\chi_{ol}(K_{m*r}) \leq k$.
2. If $m \geq p_{ol}(k, r)$ then $\chi_{ol}(K_{m*r}) > k$.

**Lemma 2**

$$p_{ol}(k, r) > \frac{1}{r} \left(\frac{r}{r-1}\right)^{k-1}.$$
Few open questions

➢ Is it true that for $r > k$,

$$m_{ol} = \Theta(r^k)?$$

➢ Is it true that for $r < k$,

$$m_{ol} = \Theta_r(r^k)?$$

➢ Is it true that for fixed $r > 2$ and growing $m$,

$$\chi_{ol}(K_{m^*r}) - \chi_l(K_{m^*r}) \to \infty?$$