Zero free regions of the derivative of the Lerch zeta-function

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Preliminaries

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- Hurwitz zeta-function

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s},$$

$0 < \alpha \leq 1$. Note that $\zeta(s) = \zeta(s, 1)$.

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$$L(s, \alpha, \lambda) = \sum_{n=0}^{\infty} \frac{e^{2\pi i\lambda n}}{(n+\alpha)^s},$$

where $0 < \alpha \leq 1$ and $0 < \lambda \leq 1$. Here $\sigma > 1$. As to the rest of the complex plane, the Lerch zeta-function is defined as the meromorphic continuation.

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- The following equations hold

$$L(s, 1, 1) = \zeta(s)$$

and

$$L(s, \alpha, 1) = \zeta(s, \alpha).$$

History

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- The latter result was instrumental in Levinson's (1974) proof that at least one-third of the non-trivial zeros of the Riemann zeta-function lie on the critical line.
- Garunkštis and R. Š. (2015) extended Speiser's result to the extended Selberg class. This class does contain functions which are known to have non-trivial zeros left of the critical line.

Functional equation

Lerch zeta-function satisfies the following functional equation

$$
\begin{aligned}
L(1-s,\alpha,\lambda) =&(2\pi)^{-s}\Gamma(s)\left(\exp\left(\frac{\pi is}{2}-2\pi i\alpha\lambda\right)L(s,\lambda,-\alpha)\right. \\
&\left.\exp\left(-\frac{\pi is}{2}+2\pi i\alpha(1-\lambda)\right)L(s,1-\lambda,\alpha)\right).
\end{aligned}
$$

Zero free regions of L'

Theorem

If

$$\sigma > \max\left\{2, \left(\log\log\alpha^{-1} - \log\left(\frac{1}{2e} + \frac{3\log 2 + 2}{4}\right)\right)(\log\alpha)^{-1}\right\}$$

and $t \in \mathbb{R}$, *then* $L'(\sigma + it, \alpha, \lambda) \neq 0$.

Zero free regions of L'

Let

$$l : \sigma = 1 - \pi t \left(\log \left(\frac{\lambda}{1-\lambda} \right) \right)^{-1}, \ \lambda \neq \frac{1}{2}.$$

Let $d(s, l)$ be the distance of s from l.

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Let $\lambda \neq 1/2$. For any $\epsilon > 0$ there is $\sigma_0 < 0$ such that $L'(s, \alpha, \lambda) \neq 0$ if $\sigma < \sigma_0$ and $d(s, l) > \epsilon$.

Theorem

Let $\lambda = 1/2$ For $|t| \geq 1$, we can choose such $\sigma' < 0$ that for all $s = \sigma + it$, $\sigma \leq \sigma'$, we have $L'(s, \alpha, 1/2) \neq 0$.

Ideas of proof

- Let $\sigma > 1$. Then

$$
\begin{aligned}
-\frac{\alpha^s}{\log \alpha} L'(s, \alpha, \lambda) = & \sum_{n=0}^{\infty} e^{2\pi i \lambda n} \frac{\log(n+\alpha)}{\log \alpha} \left(\frac{\alpha}{n+\alpha} \right)^s \\
= & 1 + \frac{\alpha^s}{\log \alpha} \sum_{n=1}^{\infty} e^{2\pi i \lambda n} \frac{\log(n+\alpha)}{(n+\alpha)^s} \\
:= & 1 + D(s, \alpha, \lambda).
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$$:=1+D(s,\alpha,\lambda).$$

- For $x > e^{1/\sigma} - \alpha$,

$$\left(\frac{\log(x+\alpha)}{(x+\alpha)^\sigma}\right)'_x = \frac{1-\sigma\log(x+\alpha)}{(x+\alpha)^{\sigma+1}} < 0.$$

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- We get

$$\sum_{n=3}^{\infty} \frac{\log(n+\alpha)}{(n+\alpha)^{\sigma}} \leq \int_{2}^{\infty} \frac{\log(x+\alpha)}{(x+\alpha)^{\sigma}}\, dx = \frac{1+(\sigma-1)\log(\alpha+1)}{(\sigma-1)^2(\alpha+1)^{\sigma-1}}.$$

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- So

$$\begin{aligned}|D(s,\alpha,\lambda)| \le & \frac{\alpha^\sigma}{\log \alpha^{-1}} \left(\frac{\log(1+\alpha)}{(1+\alpha)^\sigma} + \frac{\log(2+\alpha)}{(2+\alpha)^\sigma} \right. \\ & \left. + \frac{1+(\sigma-1)\log(2+\alpha)}{(\sigma-1)^2(2+\alpha)^{\sigma-1}} \right).\end{aligned}$$

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- Let $\sigma \geq 2$. We have the following bounds

$$\frac{\log(1+\alpha)}{(1+\alpha)^\sigma} \leq \frac{1}{2e}, \quad \frac{\log(2+\alpha)}{(2+\alpha)^\sigma} \leq \frac{\log 2}{4},$$
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- Finally,

$$|D(\sigma+it,\alpha,\lambda)| \leq \frac{\alpha^{\sigma}}{\log \alpha^{-1}} \left(\frac{1}{2e} + \frac{3\log 2 + 2}{4} \right).$$

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- Choosing the right σ gives $|D(\sigma+it,\alpha,\lambda)| < 1$, which proves the theorem.

Thank you for your attention!