# Sums of Multiplicative Characters with Additive Convolutions

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# Notations

Recall the following definitions and notations:

- The sumset of two sets $A$ and $B$ from the field $\mathbb{F}_p$ is the set

$$A + B = \{a + b \,:\, a \in A, b \in B\}\ .$$

- Here the variable $p$ stands for an arbitrary prime number and $\chi$ stands for an arbitrary multiplicative character modulo $p$.
- The notion of the convolution of two functions $f, g : \mathbb{F}_p \to \mathbb{C}$ is

$$(f * g)(x) = \sum_y f(y) g(x - y)\,.$$

# Paley Graph Conjecture

Consider a problem of obtaining upper bounds for the exponential sum

$$S_\chi(A, B) = \sum_{a \in A,\, b \in B} \chi(a + b)\,,$$

where $A, B$ are arbitrary subsets of the field $\mathbb{F}_p$.
There is a well-known conjecture on such sums.

**Paley Graph Conjecture**

Let $\delta > 0$ be a real number, $A, B \subset \mathbb{F}_p$ be arbitrary sets with $|A| > p^\delta$ and $|B| > p^\delta$. Then there exists a number $\tau = \tau(\delta)$ such that for any sufficiently large prime number $p$ the following holds

$$|S_\chi(A, B)| < p^{-\tau}\, |A|\, |B|\ .$$

# The Paley Graph

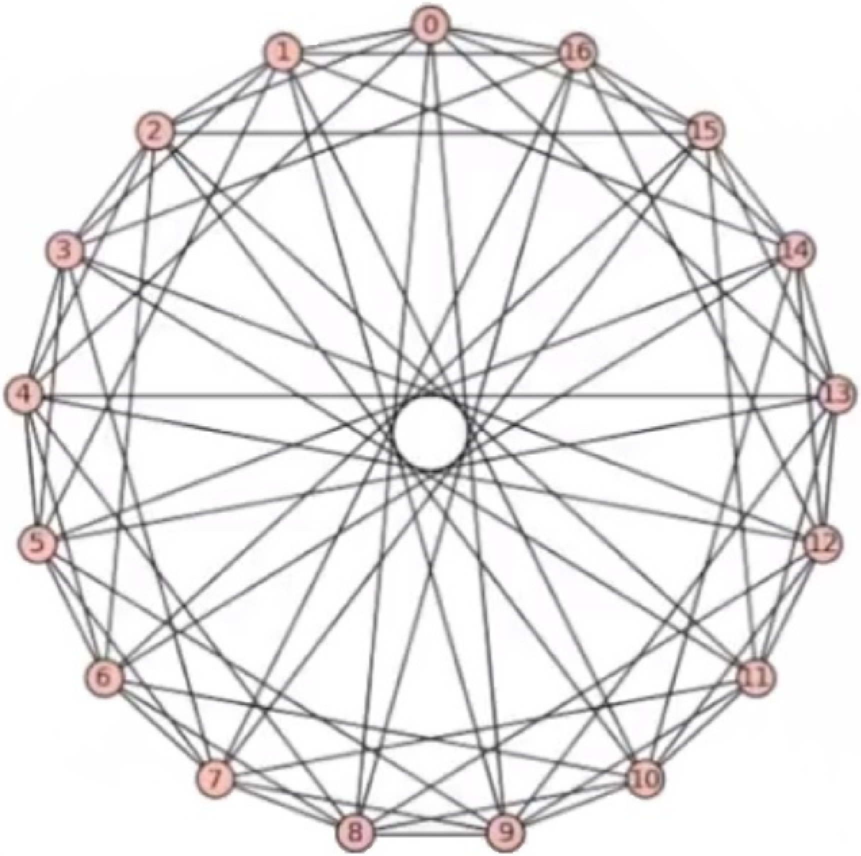
The Paley graph is the graph G(V,E) with

$$V = \mathbb{F}_p \,,$$

$$E = \{(a,b) \,:\, a-b \text{ is a quadratic residue}\} \,.$$

Paley Graph Conjecture implies that the size of the maximal clique in the Paley graph (as well as its independent number) grows slowly than $p^{\delta}$ for any positive $\delta$.

# The Paley Graph: example



# Some results

Unfortunately, at the moment we know few facts about the hypothesis. An affirmative answer was obtained just in the case of $|A| > p^{\frac{1}{2}+\delta}$, $|B| > p^{\delta}$. Even in the case $|A| \sim |B| \sim p^{\frac{1}{2}}$ is unknown. However, nontrivial bounds of sum can be obtained for structural sets $A$ and $B$ with weaker restrictions for the sizes of the sets. Thus, Mei–Chu Chang proved such an estimate provided one of the sets $A$ or $B$ has small sumset.

**Theorem (Mei–Chu Chang, 2008)**

Let $A$, $B \subset \mathbb{F}_p$ be arbitrary sets and $K, \delta$ be positive numbers with

$$|A| > p^{\frac{4}{9}+\delta}, \quad |B| > p^{\frac{4}{9}+\delta}, \quad |B+B| < K|B|\,.$$

Then there exists $\tau = \tau(\delta, K) > 0$ such that the inequality

$$|S_\chi(A,B)| < p^{-\tau}\,|A|\,|B|$$

holds for all $p > p(\delta, K)$.

# Some results

We refine Chang's assumption $|A| > p^{\frac{4}{9}+\delta}, |B| > p^{\frac{4}{9}+\delta}$.

**Theorem (Shkredov-Volostnov, 2016)**

Let $A, B \subset \mathbb{F}_p$ be sets and $K, L, \delta > 0$ be numbers with

$$|A| > p^{\frac{12}{31}+\delta}, \quad |B| > p^{\frac{12}{31}+\delta},$$

$$|A+A| < K|A|, \quad |A+B| < L|B|.$$

Then one has

$$|S_\chi(A,B)| \ll \sqrt{\frac{L \log 2K}{\delta \log p}} \cdot |A||B|$$

provided $p > p(\delta, K, L)$.

This result is not a direct improvement of Chang's theorem because of the additional assumption $|A+B| < L\,|B|$. However it is applicable in the case $B = -A$ and hence in terms of the Paley graph our result is better.

# Almost periodicity lemma

The proof relies on the Croot–Sisask lemma on almost periodicity of convolutions of the characteristic functions of sets.

**Lemma (Croot-Sisask, 2010)**

Let $\varepsilon \in (0,1)$, $K \geqslant 1$, $q \geqslant 2$ be real numbers, $A$ and $S$ be subsets of an abelian group $G$ such that $|A+S| \leqslant K|A|$ and let $f \in L_q(G)$ be an arbitrary function. Then there is $s \in S$ and a set $T \subset S - s$, $|T| \geqslant |S|(2K)^{-O(\varepsilon^{-2}q)}$ such that for all $t \in T$ the following holds

$$\|(f * A)(x+t) - (f * A)(x)\|_{L_q(G)} \leqslant \varepsilon |A| \, \|f\|_{L_q(G)}.$$

Thanks to the result we reduce the binary character sum to a sum with more variables.

# The sketch of the proof

Using Croot-Sisask lemma we find the «almost periodicity» set T.

$$|T|\,|S_\chi(A,B)| = \left| \sum_{t\in T,\, x\in \mathbb{F}_p} (A*B)(x)\chi(x) \right| =$$

$$= \left| \sum_{t\in T,\, x\in \mathbb{F}_p} (A*B)(x+t)\chi(x) + \sum_{t\in T,\, x\in \mathbb{F}_p} ((A*B)(x) - (A*B)(x+t))\chi(x) \right| \leqslant$$

$$\leqslant \left| \sum_{t\in T,\, x\in \mathbb{F}_p} (A*B)(x+t)\chi(x) \right| + \sum_{t\in T} \|(A*B)(x+t) - (A*B)(x)\|_1 \,.$$

# The sketch of the proof

By theCauchy-Schwarz inequality we have

$$\|(A*B)(x+t)-(A*B)(x)\|_1 \leqslant \|(A*B)(x+t)-(A*B)(x)\|_2\,(2\,|A+B|)^{\frac{1}{2}}\,,$$

which allows us to estimate the second sum.
Finally, we obtain need to obtain a good bound for a ternary sum

$$\sum_{t\in T,\, x\in\mathbb{F}_p}(A*B)(x+t)\chi(x)=\sum_{a\in A,\, b\in B,\, t\in T}\chi(a+b-t)=S_\chi(A,B,-T)\,.$$

# The bound for the ternary sum

Using the method of the proof of Chang's Theorem as well as some recent results from sum—product theory, we obtain an upper bound for the ternary sum in the case of sets with small additive doubling.

## Theorem

Suppose that $A$, $B$, $C \subset \mathbb{F}_p$ are arbitrary sets and $K, L, \delta > 0$ are real numbers such that

$$|A|, |B|, |C| > p^{\frac{12}{31}+\delta},$$
$$|A+A| < K|A|,$$
$$|B+C| < L|B|.$$

Then there exists $\tau = \tau(\delta, K) = \delta^2 (\log 2K)^{-3+o(1)}$ with the property

$$|S_\chi(A, B, C)| < p^{-\tau} |A| |B| |C|$$

for all $p > p(\delta, K, L)$.

Thank you for your attention!