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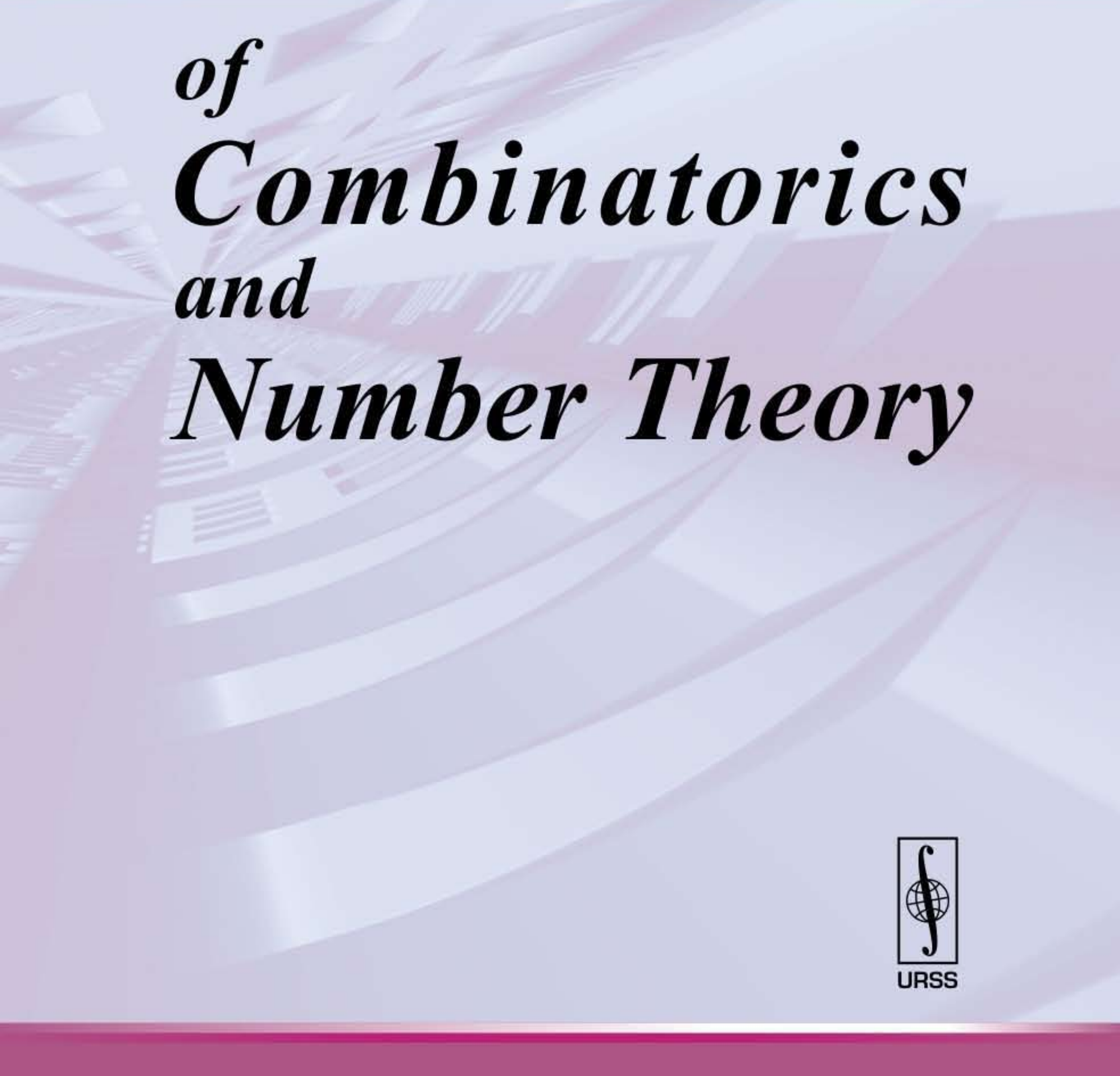

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On the value-distribution of the Riemann zeta-function on the critical line

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Abstract: We investigate the intersections of the curve $\mathbb{R} \ni t \mapsto \zeta\left(\frac{1}{2}+it\right)$ with the real axis. We show that if the Riemann hypothesis is true, the mean-value of those real values exists and is equal to 1. Moreover, we show that the zeta-function unconditionally assumes arbitrarily large real values on the critical line.

Keywords: Riemann zeta-function, value-distribution, critical line

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Dedicated to the memory
of Prof. Dr. K. Ramachandra

1. Introduction and statement of the main results

It is conjectured that the set of values of the Riemann zeta-function $\zeta(s)$ on the critical line $s = \frac{1}{2} + i\mathbb{R}$ is dense in the complex plane. However, not even a non-empty open subset of $\mathbb{C}$ is known such that $\zeta\left(\frac{1}{2}+i\mathbb{R}\right)$ is dense in this set. Some results give evidence in this direction. Bohr & Courant [1] have shown the denseness of $\zeta(\sigma+i\mathbb{R})$ in the complex plane for any fixed $\sigma \in \left(\frac{1}{2}, 1\right]$ and Selberg has proved (unpublished) that the values taken by an appropriate normalization of the Riemann zeta-function on the critical line are Gaussian normally distributed (the first published proof is due to Joyner [13]). However, Garunkštis & Steuding [8] showed recently that the curve $\mathbb{R} \ni t \mapsto \left(\zeta\left(\frac{1}{2}+it\right), \zeta'\left(\frac{1}{2}+it\right)\right)$ is not dense in $\mathbb{C}^2$. In this article we investigate, in particular, the real points of $\zeta\left(\frac{1}{2}+it\right)$ for real t. We show that the mean-value of those real values exists and is equal to 1 provided the Riemann hypothesis is true, and the zeta-function assumes arbitrarily large real values on the critical line. In his monograph [6], Edwards wrote *"... the real part of ζ has a strong tendency to be positive"* (page 121). We shall explain this phenomenon. For this purpose, we estimate the number of intersections $\zeta\left(\frac{1}{2}+i\mathbb{R}\right)$ with any given straight line $e^{i\phi}\mathbb{R}$ and prove asymptotic formulae for the first and second associated discrete moment of those values. This approach yields new information on the value-distribution of the zeta-function on the critical line.

Let $\phi \in [0, \pi)$. The functional equation for $\zeta(s)$ in its asymmetrical form is given by

$$\zeta(s) = \Delta(s)\zeta(1-s), \quad \text{where} \quad \Delta(s) := 2^s\pi^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right). \tag{1}$$

That $\Delta(s)\Delta(1-s) = 1$ is essential for our approach (which follows directly from the functional equation and will be frequently used in the sequel). Consequently, $\Delta\left(\frac{1}{2}+it\right)$ is on the unit circle for any real t. Moreover,

$$\Phi(t) := \Phi(t;\phi) := \zeta\left(\frac{1}{2}+it\right) - e^{2i\phi}\zeta\left(\frac{1}{2}-it\right) = \zeta\left(\frac{1}{2}+it\right)\left(1-e^{2i\phi}\Delta\left(\frac{1}{2}-it\right)\right) \tag{2}$$

vanishes if and only if $\frac{1}{2}+it$ is a zero of the zeta-function or

$$1 = e^{2i\phi}\Delta\left(\frac{1}{2}-it\right) = e^{-2i\phi}\Delta\left(\frac{1}{2}+it\right),$$

where the last equality is derived by conjugation. Two values for ϕ are of special interest here: for $\phi = 0$ the roots of the equation $\Delta\left(\frac{1}{2}+it\right) = e^{2i\phi}$ correspond to real values of $\zeta\left(\frac{1}{2}+it\right)$, whereas $\phi = \frac{\pi}{2}$ yields the purely imaginary values; this follows immediately from the fact that for such values of t

$$\zeta\left(\frac{1}{2}+it\right) = \pm\zeta\left(\frac{1}{2}-it\right) = \pm\overline{\zeta\left(\frac{1}{2}+it\right)}.$$

The roots t of $\Delta\left(\frac{1}{2}+it\right)-1$ are called Gram points after Gram [10] who observed that the first of those roots separate consecutive zeta zeros on the critical line; we shall discuss our results concerning this aspect in a separate section below. For the roots of $\Phi(t)$ which are no ordinates of zeros of the zeta-function we obviously have

$$\arg\zeta\left(\frac{1}{2}+it\right)\equiv\phi \bmod \pi.$$

Hence, the roots of $\Phi(t)$ correspond to the intersection points of the curve $\zeta\left(\frac{1}{2}+i\mathbb{R}\right)$ with the straight line $e^{i\phi}\mathbb{R}$ through the origin. In this note we are mainly concerned with the non-zero intersection points. We expect that all but finitely many of the roots of the two factors of $\Phi(t)$ in (2) are distinct (for each fixed value of ϕ). Since the set of zeros of zeta is countable, there have to exist values of ϕ in the uncountable set $[0,\pi)$ for which $\Delta\left(\frac{1}{2}+i\gamma\right)\neq e^{2i\phi}$ for all ordinates γ of nontrivial zeros; however, we cannot even exclude the possibility of $\Delta\left(\frac{1}{2}+i\gamma\right)=1$ for a single ordinate γ.

Denote by $N_\phi(T)$ the number of zeros of the function $\Phi(t)$ with $t\in(0,T]$, then

$$N_\phi(T)=N_0(T)+N_\phi^\Delta(T),$$

where $N_0(T)$ counts the nontrivial zeros of $\zeta(s)$ on the critical line with imaginary part in $(0,T]$ and $N_\phi^\Delta(T)$ is the number of roots of the equation $\Delta\left(\frac{1}{2}+it\right)=e^{i\phi}$ with $t\in(0,T]$. All these functions are counting zeros with respect to their multiplicities; all roots counted by N_ϕ^Δ are simple, as is easily seen by (8) below. By a deep result of Conrey [2] (built upon a work of Levinson [15]), more than two fifths of the zeros of $\zeta(s)$ lie on the critical line,

$$N_0(T)\geqslant\frac{2}{5}N(T)=\frac{T}{5\pi}\log T+O(T), \tag{3}$$

where $N(T)=\frac{T}{2\pi}\log\frac{T}{2\pi e}+O(\log T)$ is the Riemann—von Mangoldt counting function for nontrivial zeros without specifying the real part (for this and for other basics from zeta-function theory we refer to Ivić [12]). First of all, we shall prove an asymptotic formula for the counting function $N_\phi^\Delta(T)$:

THEOREM 1. *For any* $\phi\in[0,\pi)$, *as* $T\to\infty$,

$$N_\phi^\Delta(T)=\frac{T}{2\pi}\log\frac{T}{2\pi e}+O(\log T);$$

in particular, the set $\zeta\left(\frac{1}{2}+i\mathbb{R}\right)\cap e^{i\phi}\mathbb{R}$ *is countable.*

If the Riemann hypothesis is true (or if almost all zeta zeros lie on the critical line $N(T)\sim N_0(T)$), then Theorem 1 implies

$$N_\phi^\Delta(T)\sim N_0(T)\quad\text{and}\quad N_\phi(T)\sim\frac{T}{\pi}\log T; \tag{4}$$

consequently, $T\log T$ is, unconditionally, the correct order of growth for $N_\phi(T)$.

Moreover, we shall prove asymptotic formulae for the associated first and second discrete moment:

THEOREM 2. *For any* $\phi \in [0,\pi)$, *as* $T \to \infty$,

$$\sum_{\substack{0<t\leqslant T\\ \zeta(\frac{1}{2}+it)\in e^{i\phi}\mathbb{R}}} \zeta\left(\frac{1}{2}+it\right) = 2e^{i\phi}\cos\phi\,\frac{T}{2\pi}\log\frac{T}{2\pi e} + O\big(T^{\frac{1}{2}+\epsilon}\big), \tag{5}$$

and

$$\sum_{\substack{0<t\leqslant T\\ \zeta(\frac{1}{2}+it)\in e^{i\phi}\mathbb{R}}} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 = \frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)^2 + \tag{6}$$

$$+ (2c + 2\cos(2\phi))\frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{T}{2\pi} + O\big(T^{\frac{1}{2}+\epsilon}\big), \tag{7}$$

where $c := \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n} - \log N\right) = 0.577\ldots$ *is the Euler—Mascheroni constant.*

Note that the leading main term of the second moment is independent of ϕ whereas the first one vanishes for $\phi = \pi/2$. These asymptotic formulae provide interesting information about the value-distribution of the zeta-function on the critical line. As an immediate consequence of (4) in combination with (5) we note

COROLLARY 1. *For* $\phi \neq \frac{\pi}{2}$,

$$1 \leqslant \liminf_{T\to\infty} \frac{(e^{i\phi}\cos\phi)^{-1}}{N_\phi(T)} \sum_{\substack{0<t\leqslant T\\ \zeta(\frac{1}{2}+it)\in e^{i\phi}\mathbb{R}}} \zeta\left(\frac{1}{2}+it\right) \leqslant$$

$$\leqslant \limsup_{T\to\infty} \frac{(e^{i\phi}\cos\phi)^{-1}}{N_\phi(T)} \sum_{\substack{0<t\leqslant T\\ \zeta(\frac{1}{2}+it)\in e^{i\phi}\mathbb{R}}} \zeta\left(\frac{1}{2}+it\right) \leqslant \frac{10}{7}.$$

If the Riemann hypothesis is true (*or if almost all zeta zeros lie on the critical line*), *then the mean-value of the points in* $\zeta\left(\frac{1}{2}+i\mathbb{R}\right) \cap e^{i\phi}\mathbb{R}$ *exists and is equal to*

$$\lim_{T\to\infty}\frac{1}{N_\phi(T)} \sum_{\substack{0<t\leqslant T\\ \zeta(\frac{1}{2}+it)\in e^{i\phi}\mathbb{R}}} \zeta\left(\frac{1}{2}+it\right) = e^{i\phi}\cos\phi;$$

for purely imaginary values ($\phi = \pi/2$) *the statement is unconditionally true.*

The case of purely imaginary values is special because formula (5) gives for $\phi = \pi/2$ just an upper bound for the first moment which leads to mean-value zero. For

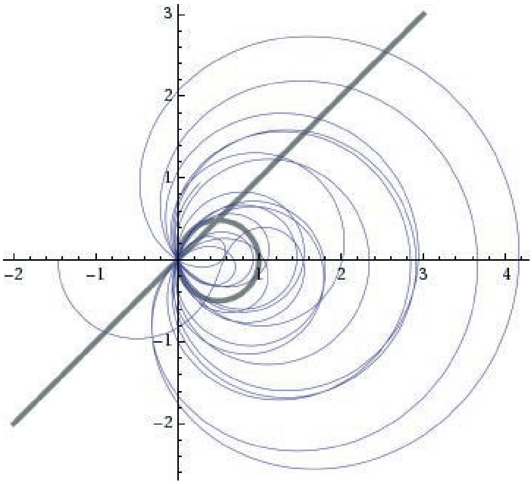

Fig. 1. The curve $t \mapsto \zeta\left(\frac{1}{2}+it\right)$ for $t \in [0,70]$ in addition with the circle of the mean-values and the bisecting line of the first and third quadrant ($\phi = \pi/4$); here the mean-value is $\frac{1+i}{2}$ which is the intersection point of the grey circle with the bisecting line

the real values of the zeta-function on the critical line the mean-value equals one provided the Riemann hypothesis is true. For any value of ϕ, the real part of the corresponding mean-value is non-negative, explaining Edwards' observation from above. These different mean-values according to ϕ are well reflected in the graph of the curve $t \mapsto \zeta\left(\frac{1}{2}+it\right)$, see Figure 1 below: the grey circle stands for the means $e^{i\phi}\cos\phi$ of the values $\zeta\left(\frac{1}{2}+i\mathbb{R}\right)$ on the straight line $e^{i\phi}\mathbb{R}$ as ϕ varies; the almost symmetry with respect to the real axis and the tendency for $\operatorname{Re}\zeta\left(\frac{1}{2}+it\right)$ to be positive are nicely explained by (5) and (6).

Our next application shows that the zeta-function assumes arbitrarily large values on any half-line $e^{i\phi}\mathbb{R}$ through the origin in the right half-plane.

Corollary 2. *For any $\phi \in [0,\pi)$ there exist infinitely many real values $t \to \infty$ such that*

$$e^{-i\phi}\zeta\left(\frac{1}{2}+it\right) \geqslant (\log t)^{\frac{1}{2}};$$

more precisely, any interval $(T, 2T]$ with sufficiently large T contains such a value t. In the purely imaginary case ($\phi = \frac{\pi}{2}$) there exist also values $t \to +\infty$ such that

$$i\zeta\left(\frac{1}{2}+it\right) \geqslant (\log t)^{\frac{1}{2}}.$$

In particular, $\zeta\left(\frac{1}{2}+it\right)$ assumes arbitrarily large real values. So far, to authors best knowledge, the various Ω-results for the zeta-function in the literature (e.g. Soundararajan [16]) do not imply this corollary.

Finally, we give a lower bound for the number of roots of $\Delta\left(\frac{1}{2}+it\right) - e^{2i\phi}$ distinct from ordinates of zeta zeros:

COROLLARY 3. *For* $\phi \neq \frac{\pi}{2}$, *as* $T \to \infty$,

$$\sum_{\substack{0<t\leqslant T\\ \Delta(\frac{1}{2}+it)=e^{2i\phi},\\ \zeta(\frac{1}{2}+it)\neq 0}} 1 \geqslant \frac{2(\cos\phi)^2}{\pi}\big(1+o(1)\big)T.$$

For $\phi = \frac{\pi}{2}$, that is, the case of purely imaginary values, we do not have any non-trivial lower bound. It has been conjectured by Fujii [7] that the values $\frac{1}{2\pi i}\log\Delta\left(\frac{1}{2}+i\gamma_n\right)$ are uniformly distributed modulo one (in the sense of Weyl), where γ_n denotes the n-th ordinate of the nontrivial zeros of $\zeta(s)$ in ascending order. A proof of this conjecture would certainly yield more information about the quantity estimated in the last corollary.

The remaining parts of this article are organized as follows. The next two sections contain the proofs of Theorem 1 and Theorem 2. For the sake of completeness we give the proofs of the corollaries in the fourth section. In Section 5 we discuss the special case $\phi = 0$ and how our results provide information about Gram points. Finally, we conclude with some remarks on the shape of the curve $t \mapsto \zeta\left(\frac{1}{2}+it\right)$ and further applications of our method.

2. Proof of Theorem 1

Recall that $\Delta\left(\frac{1}{2}+it\right)$ is a complex number from the unit circle whenever $t \in \mathbb{R}$. Moreover, $\Delta'\left(\frac{1}{2}+it\right)$ is non-vanishing by the asymptotic formula

$$\frac{\Delta'}{\Delta}(\sigma+it) = -\log\frac{|t|}{2\pi} + O(|t|^{-1}) \qquad \text{for} \quad |t| \geqslant 1. \tag{8}$$

Consequently, $\Delta\left(\frac{1}{2}+it\right)$ is spinning clockwise on the unit circle around the origin with increasing speed as $t \to \infty$. Moreover, there exists no proper real interval $\mathcal{I}$ such that $\zeta\left(\frac{1}{2}+it\right)$ lies on a straight line $e^{i\phi}\mathbb{R}$ for all $t \in \mathcal{I}$. Firstly, let us assume that

$$\Delta\left(\frac{1}{2}+iT\right) = \Delta\left(\frac{1}{2}\right) = 1. \tag{9}$$

Then the number of roots of the equation $\Delta\left(\frac{1}{2}+it\right) = e^{2i\phi}$ with $0 \leqslant t \leqslant T$ is up to a sign equal to the winding number of the curve

$$\eta\,:\,[0,1]\to\mathbb{C},\quad \lambda\mapsto\eta(\lambda):=\Delta\left(\frac{1}{2}+i\lambda T\right).$$

This yields

$$-N_\phi^\Delta(T)=\frac{1}{2\pi i}\int_\eta\frac{\mathrm{d}s}{s}=\frac{T}{2\pi}\int_0^1\frac{\Delta'}{\Delta}\left(\frac{1}{2}+i\lambda T\right)\mathrm{d}\lambda.$$

In order to use (8) we divide the integration interval into two subintervals. Noting that there are only finitely many roots of $\Delta\left(\frac{1}{2}+it\right)-e^{2i\phi}$ for $0<t\leqslant 1$, we find for the term with the integral on the right-hand side above

$$\frac{T}{2\pi}\left\{\int_0^{1/T}+\int_{1/T}^1\right\}\frac{\Delta'}{\Delta}\left(\frac{1}{2}+i\lambda T\right)\mathrm{d}\lambda=O(1)+\frac{T}{2\pi}\int_{1/T}^1\left(-\log\frac{\lambda T}{2\pi}+O\big((\lambda T)^{-1}\big)\right)\mathrm{d}\lambda.$$

Hence, the asymptotic formula for $N_\phi^\Delta(T)$ follows by integration; however, to get rid of our assumption (9) on T, by (8) we may substitute this by any T at the expense of an error $O(\log T)$. This proves Theorem 1.

3. Proof of Theorem 2

The proof relies on a new variation of the method of Conrey, Ghosh & Gonek (see, e.g., [3]); for our purpose we shall work with the logarithmic derivative of $\Delta(s)-e^{2i\phi}$. Previously, only logarithmic derivatives of Dirichlet series were studied.

3.1. Proof for the first moment

Since the nontrivial zeros of the zeta-function do not contribute to the sum in question, we only have to take the roots of $\Delta\left(\frac{1}{2}+it\right)-e^{2i\phi}$ into account. Denoting those roots by t_n^ϕ in ascending order $0<t_1^\phi\leqslant t_2^\phi\leqslant\ldots$, we thus have

$$\sum_{\substack{0<t\leqslant T\\ \zeta(\frac{1}{2}+it)\in e^{i\phi}\mathbb{R}}}\zeta\left(\frac{1}{2}+it\right)=\sum_{0<t_n^\phi\leqslant T}\zeta\left(\frac{1}{2}+it\right).$$

In view of Theorem 1 the sequence of these t_n^ϕ cannot lie too densely. As a matter of fact, for any given T_0 there exists $T\in[T_0,T_0+1)$ such that

$$\min_{t_n^\phi}|T-t_n^\phi|\geqslant\frac{1}{\log T},\tag{10}$$

where the minimum is being taken over all t_n^ϕ. For $|t| \geqslant 10$, one has $|\Delta(s)| = 1$ if and only if $\mathrm{Re}\, s = \frac{1}{2}$ (see Spira [17], resp. Dixon & Schoenfeld [5] for a slight improvement). For smaller values of t there are roots off the critical line, however, they form a sparse set. Hence, we may assume that there are no roots of $\Delta(s) - e^{2i\phi}$ on the rectangular contour $\mathcal{C}$ with corners $2+i, 2+iT, 1-a+iT, 1-a+i$ in counterclockwise direction, where $a = 1+\frac{1}{\log T}$ (if there are roots on the contour we make small indentions at the expense of a bounded error). By the calculus of residues,

$$\sum_{0<t_n^\phi\leqslant T} \zeta\left(\frac{1}{2}+it\right) = \frac{1}{2\pi i}\left\{\int_{2+i}^{2+iT} + \int_{2+iT}^{1-a+iT} + \int_{1-a+iT}^{1-a+i} + \int_{1-a+i}^{2+i}\right\}\zeta(s)\frac{\Delta'(s)}{\Delta(s)-e^{2i\phi}}\,\mathrm{d}s + O(1) =$$

$$= \sum_{i=1}^{4} \mathcal{I}_i + O(1).$$

The bounded error term $O(1)$ results from at most finitely many possible residues outside $\mathcal{C}$ and contributions $\zeta\left(\frac{1}{2}+it_n^\phi\right)$ with $0 < t_n \leqslant 1$.

First we consider $\mathcal{I}_1$. The well-known formula

$$\Delta(\sigma+it) = \left(\frac{|t|}{2\pi}\right)^{\frac{1}{2}-\sigma-it} \exp\left(i\left(t+\frac{\pi}{4}\right)\right)\left(1+O(|t|^{-1})\right) \quad \text{for} \quad |t| \geqslant 1 \tag{11}$$

holds uniformly for any σ from a bounded interval. Hence,

$$\frac{1}{\Delta(s)-e^{2i\phi}} = \frac{-e^{-2i\phi}}{1-e^{-2i\phi}\Delta(s)} = -e^{-2i\phi}\left(1+\sum_{k=1}^{\infty} e^{-2ki\phi}\Delta(s)^k\right); \tag{12}$$

in view of (11) the infinite geometric series on the right-hand side is converging very quickly for $s \in 2+i\mathbb{R}$. Writing $\Delta'(s) = \Delta(s)\frac{\Delta'}{\Delta}(s)$ and using (11) in combination with (8), we easily find that

$$\frac{1}{2\pi i}\int_{2+i}^{2+iT} \zeta(s)\Delta'(s)\sum_{k=1}^{\infty} e^{-2ki\phi}\Delta(s)^k\,\mathrm{d}s \ll 1.$$

The same trick leads to

$$\frac{1}{2\pi i}\int_{2+i}^{2+iT} \zeta(s)\Delta'(s)\,\mathrm{d}s \ll 1,$$

which implies $\mathcal{I}_1 \ll 1$. The horizontal integral $\mathcal{I}_4$ is independent of T, so $\mathcal{I}_4 \ll 1$. For $\mathcal{I}_2$ we notice that the denominator $\Delta(s) - e^{i\phi}$ of the integrand being off the critical line is either dominated by $\Delta(s)$ for $\mathrm{Re}\, s < \frac{1}{2}$ or by $e^{i\phi}$ if $\mathrm{Re}\, s > \frac{1}{2}$. On the critical line we have

$$\Delta\left(\frac{1}{2}+iT\right)-e^{i\phi}=\Delta\left(\frac{1}{2}+iT\right)-\Delta\left(\frac{1}{2}+it_m^{\phi}\right)$$

for some integer m. In view of (11) and (10) we find that

$$\Delta\left(\frac{1}{2}+iT\right)-e^{i\phi}\gg|T-t_m^{\phi}|\log T\gg 1.$$

Now using (8) in combination with

$$\zeta(\sigma+it)\ll 1+|t|^{\frac{1}{2}(1-\sigma)+\epsilon}\qquad\text{for}\quad 1-a\leqslant\sigma\leqslant 2,\ |t|\geqslant 1, \tag{13}$$

we may deduce that $\mathcal{I}_2\ll T^{\frac{1}{2}+\epsilon}$.

The main term of the asymptotic formula comes from the integral $\mathcal{I}_3$ which we have to evaluate now. Via $s\mapsto 1-\overline{s}$ we find that

$$\mathcal{I}_3=-\frac{1}{2\pi i}\int\limits_{a+i}^{a+iT}\zeta(1-\overline{s})\frac{\Delta'(1-\overline{s})}{\Delta(1-\overline{s})-e^{2i\phi}}\,\mathrm{d}s.$$

By the reflection principle,

$$\overline{\mathcal{I}_3}=-\frac{1}{2\pi i}\int\limits_{a+i}^{a+iT}\zeta(1-s)\frac{\Delta'(1-s)}{\Delta(1-s)-e^{-2i\phi}}\,\mathrm{d}s.$$

In view of the functional equation (1) the integrand can be rewritten as

$$\zeta(1-s)\frac{\Delta'(1-s)}{\Delta(1-s)-e^{-2i\phi}}=\zeta(s)\Delta'(1-s)\frac{1}{1-e^{-2i\phi}\Delta(1-s)^{-1}}.$$

Note that $\Delta(1-s)^{-1}=\Delta(s)$ (again by (1)). The function given in (11) is small on $s=a+i\mathbb{R}$, and we may thus use the geometric series expansion to find

$$\begin{aligned}\overline{\mathcal{I}_3}&=-\frac{1}{2\pi i}\int\limits_{a+i}^{a+iT}\zeta(s)\Delta'(1-s)\left(1+\frac{e^{-2i\phi}}{\Delta(1-s)}+\sum_{k=2}^{\infty}e^{-2ki\phi}\Delta(s)^k\right)\mathrm{d}s\\&=\mathcal{J}_1+\mathcal{J}_2+O\left(T^{\frac{1}{2}+\epsilon}\right).\end{aligned}\tag{14}$$

We start with $\mathcal{J}_1$. According to (8) we have

$$\mathcal{J}_1=\int\limits_1^T\left(\log\frac{\tau}{2\pi}+O\left(\frac{1}{\tau}\right)\right)\mathrm{d}\left(\frac{1}{2\pi i}\int\limits_{a+i}^{a+i\tau}\zeta(s)\Delta(1-s)\,\mathrm{d}s\right).$$

For the inner integral we use Gonek's lemma:

LEMMA 1. *Suppose that* $\sum_{n=1}^{\infty} a(n)n^{-s}$ *converges for* $\sigma > 1$ *where* $a(n) \ll n^{\epsilon}$ *for any* $\epsilon > 0$. *Then we have, uniformly for* $1 < a \leqslant 2$,

$$\frac{1}{2\pi i} \int\limits_{a+i}^{a+iT} \left(\frac{m}{2\pi}\right)^{s} \Gamma(s) \exp\left(\delta \frac{\pi i s}{2}\right) \sum_{n=1}^{\infty} \frac{a(n)}{n^{s}} \, \mathrm{d}s =$$

$$= \begin{cases} \sum\limits_{n \leqslant \frac{Tm}{2\pi}} a(n) \exp\left(-2\pi i \dfrac{n}{m}\right) + O\left(m^{a} T^{a-\frac{1}{2}+\epsilon}\right) & \textit{if} \quad \delta = -1, \\ O(m^{a}) & \textit{if} \quad \delta = +1. \end{cases}$$

PROOF. Because of the absolute convergence we may interchange the order of summation and integration. For the integral we use Lemma 1 from [4] and for the sum Lemma 4 from [9]. □

This yields

$$\frac{1}{2\pi i} \int\limits_{a+i}^{a+i\tau} \zeta(s)\Delta(1-s) \, \mathrm{d}s = \frac{\tau}{2\pi} + O(\tau^{\frac{1}{2}+\epsilon}),$$

and

$$\mathcal{J}_1 = \int\limits_{1}^{T} \left(\log \frac{\tau}{2\pi} + O\left(\frac{1}{\tau}\right) \right) \mathrm{d}\left(\frac{\tau}{2\pi} + O\left(\tau^{\frac{1}{2}+\epsilon}\right) \right) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O\left(T^{\frac{1}{2}+\epsilon}\right).$$

In order to estimate $\mathcal{J}_2$ we use (8) again and obtain

$$\mathcal{J}_2 = e^{-2i\phi} \int\limits_{1}^{T} \left(\log \frac{\tau}{2\pi} + O\left(\frac{1}{\tau}\right) \right) \mathrm{d}\left(\frac{1}{2\pi i} \int\limits_{a+i}^{a+i\tau} \zeta(s) \, \mathrm{d}s \right).$$

Since $a = 1 + \dfrac{1}{\log T}$ we have

$$\frac{1}{2\pi i} \int\limits_{a+i}^{a+i\tau} \zeta(s) \, \mathrm{d}s = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n^{a}} \int\limits_{1}^{\tau} \frac{1}{n^{it}} \, \mathrm{d}t = \frac{\tau}{2\pi} + O(\log T),$$

which leads to

$$\mathcal{J}_2 = e^{-2i\phi} \frac{T}{2\pi} \log \frac{T}{2\pi e} + O\left((\log T)^{2}\right).$$

Hence, after conjugation (because of (14)) we arrive at

$$\sum_{0 < t_n^{\phi} \leqslant T} \zeta\left(\frac{1}{2} + it\right) = (1 + e^{2i\phi}) \frac{T}{2\pi} \log \frac{T}{2\pi e} + O\left(T^{\frac{1}{2}+\epsilon}\right).$$

Note that

$$1+e^{2i\phi}=2e^{i\phi}\cos\phi, \tag{15}$$

and so we have proved the first asymptotic formula (5) for any value T, satisfying (10). To get this uniformly in T we allow an arbitrary T at the expense of an error

$$\zeta\left(\frac{1}{2}+it\right)\ll T^{\frac{1}{4}+\epsilon};$$

the latter estimate is a trivial consequence of the Phragmén—Lindelöf principle applied to the functional equation. This proves formula (5).

3.2. Proof for the second moment

We procced in a similar way as in the previous proof. Assuming the same as for the first moment, we have

$$\sum_{0<t_n^\phi\leqslant T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2=\frac{1}{2\pi i}\left\{\int\limits_{2+i}^{2+iT}+\int\limits_{2+iT}^{1-a+iT}+\int\limits_{1-a+iT}^{1-a+i}+\int\limits_{1-a+i}^{2+i}\right\}\times$$

$$\times\,\zeta(s)\zeta(1-s)\frac{\Delta'(s)}{\Delta(s)-e^{2i\phi}}\,\mathrm{d}s+O(1)=\sum_{i=1}^{4}\mathcal{I}_i+O(1).$$

We begin with $\mathcal{I}_1$. According to (8) and the geometric series expansion we find that

$$\mathcal{I}_1=-\,e^{-2i\phi}\frac{1}{2\pi i}\int\limits_{2+i}^{2+iT}\zeta(s)^2\frac{\Delta'}{\Delta}(s)\Big(1+\sum_{k=1}^{\infty}e^{-2ki\phi}\Delta(s)^k\Big)\,\mathrm{d}s=$$

$$=e^{-2i\phi}\int\limits_{1}^{T}\left(\log\frac{\tau}{2\pi}+O\left(\frac{1}{\tau}\right)\right)\mathrm{d}\left(\frac{1}{2\pi i}\int\limits_{2+i}^{2+i\tau}\zeta(s)^2\,\mathrm{d}s\right)+O(1).$$

Since

$$\frac{1}{2\pi i}\int\limits_{2+i}^{2+i\tau}\zeta(s)^2\,\mathrm{d}s=\frac{1}{2\pi}\sum_{m,n=1}^{\infty}\frac{1}{(mn)^2}\int\limits_{1}^{\tau}\frac{1}{(mn)^{it}}\,\mathrm{d}t=\frac{\tau}{2\pi}+O(1),$$

we get

$$\mathcal{I}_1=e^{-2i\phi}\frac{T}{2\pi}\log\frac{T}{2\pi e}+O(1).$$

By the same argument as for the first moment we find that $\mathcal{I}_2\ll T^{\frac{1}{2}+\epsilon},\mathcal{I}_4\ll 1$. Again, the integral $\mathcal{I}_3$ gives the main contribution. Via $s\mapsto 1-\overline{s}$ we find that

$$\overline{\mathcal{I}_3}=-\frac{1}{2\pi i}\int\limits_{a+i}^{a+iT}\zeta(1-s)\zeta(s)\frac{\Delta'(1-s)}{\Delta(1-s)-e^{-2i\phi}}\,\mathrm{d}s.$$

By the functional equation (1) and the geometric series expansion,

$$\overline{\mathcal{I}_3} = -\frac{1}{2\pi i}\int_{a+i}^{a+iT} \zeta(s)^2\Delta'(1-s)\left(1+\frac{e^{-2i\phi}}{\Delta(1-s)}+\sum_{k=2}^{\infty} e^{-2ki\phi}\Delta(s)^k\right)\mathrm{d}s =$$

$$= \mathcal{J}_1 + \mathcal{J}_2 + O\left(T^{\frac{1}{2}+\epsilon}\right).$$

We start with $\mathcal{J}_1$. In view of (8) we have

$$\mathcal{J}_1 = \int_1^T \left(\log\frac{\tau}{2\pi}+O\left(\frac{1}{\tau}\right)\right)\mathrm{d}\left(\frac{1}{2\pi i}\int_{a+i}^{a+i\tau}\zeta(s)^2\Delta(1-s)\,\mathrm{d}s\right).$$

Using Gonek's Lemma 1 for the inner integral and a standard bound for the Dirichlet divisor problem (see [12]), we find that

$$\frac{1}{2\pi i}\int_{a+i}^{a+i\tau}\zeta(s)^2\Delta(1-s)\,\mathrm{d}s = \sum_{mn\leqslant\frac{\tau}{2\pi}} 1 + O\left(\tau^{\frac{1}{2}+\epsilon}\right)$$

$$= \frac{\tau}{2\pi}\log\frac{\tau}{2\pi} + (2c-1)\frac{\tau}{2\pi} + O\left(\tau^{\frac{1}{2}+\epsilon}\right).$$

This yields the asymptotic formula

$$\mathcal{J}_1 = \int_1^T \left(\log\frac{\tau}{2\pi}+O\left(\frac{1}{\tau}\right)\right)\mathrm{d}\left(\frac{\tau}{2\pi}\log\frac{\tau}{2\pi}+(2c-1)\frac{\tau}{2\pi}+O\left(\tau^{\frac{1}{2}+\epsilon}\right)\right) =$$

$$=\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^2 + (2c-2)\frac{T}{2\pi}\log\frac{T}{2\pi e} + O\left(T^{\frac{1}{2}+\epsilon}\right).$$

Next we consider $\mathcal{J}_2$. In view of (8) we have

$$\mathcal{J}_2 = e^{-2i\phi}\int_1^T \left(\log\frac{\tau}{2\pi}+O\left(\frac{1}{\tau}\right)\right)\mathrm{d}\left(\frac{1}{2\pi i}\int_{a+i}^{a+i\tau}\zeta(s)^2\,\mathrm{d}s\right).$$

Since $a = 1 + \dfrac{1}{\log T}$,

$$\frac{1}{2\pi i}\int_{a+i}^{a+i\tau}\zeta(s)^2\,\mathrm{d}s = \frac{1}{2\pi}\sum_{m,n=1}^{\infty}\frac{1}{(mn)^a}\int_1^\tau\frac{1}{(mn)^{it}}\,\mathrm{d}t = \frac{\tau}{2\pi}+O\left((\log T)^2\right),$$

which leads to

$$\mathcal{J}_2 = e^{-2i\phi}\frac{T}{2\pi}\log\frac{T}{2\pi e} + O\left((\log T)^3\right).$$

Collecting all estimates together, formula (6) of the theorem follows. This finishes the proof of Theorem 2.

4. Proof of the corollaries

4.1. Proof of Corollary 1

If the Riemann hypothesis is true, then $N_\phi(T) \sim \frac{T}{\pi}\log T$ by (4). Hence, the existence of the mean-value and the formula for the mean-value follow immediately from (5). In the case $\phi = \frac{\pi}{2}$ the main term of (5) vanishes and, using the unconditional bound $N_\phi(T) \geqslant N_\phi^\Delta(T)$ in place of the unproved Riemann hypothesis, we obtain

$$\frac{1}{N_\phi(T)} \sum_{\substack{0<t\leqslant T\\ 0\neq\zeta(\frac12+it)\in i\mathbb{R}}} \zeta\left(\frac12+it\right) \ll T^{-\frac12+\epsilon}; \tag{16}$$

consequently, the mean-value is zero for $\phi = \frac{\pi}{2}$. If we do not assume the Riemann hypothesis, then we may use (3) to derive the bounds for lim inf and lim sup as in the corollary.

4.2. Proof of Corollary 2

By Theorem 1 and 2 we find that

$$N := N_\phi^\Delta(2T) - N_\phi^\Delta(T) \sim \frac{T}{2\pi}\log T$$

and

$$\Sigma := \sum_{\substack{T<t\leqslant 2T\\ \zeta(\frac12+it)\in e^{i\phi}\mathbb{R}}} \left|\zeta\left(\frac12+it\right)\right|^2 \sim \frac{T}{2\pi}(\log T)^2. \tag{17}$$

Now suppose that $|\zeta\left(\frac12+it\right)| < c(\log T)^\alpha$ with some positive constant c for all terms in the latter sum. Then we obtain the trivial bound

$$\Sigma < c^2(\log T)^{2\alpha}N \sim c^2\frac{T}{2\pi}(\log T)^{1+2\alpha}.$$

In order to have no contradiction with respect to the right-hand side of (17) we deduce $\alpha \leqslant \frac12$. Consequently, for any sufficiently large T there exists a value t in the interval $(T,2T]$ such that $\zeta\left(\frac12+it\right) \in e^{i\phi}\mathbb{R}$ and $\left|\zeta\left(\frac12+it\right)\right| \geqslant (\log t)^{\frac12}$. If all values $e^{-i\phi}\zeta\left(\frac12+it\right)$ were negative, then we would get a contradiction to (5). If ϕ is purely imaginary, then we can deduce from (16) the existence of large values with both signs. This proves the corollary.

4.3. Proof of Corollary 3

Applying the Cauchy—Schwarz inequality, we find that

$$\Bigg| \sum_{\substack{0<t\leqslant T\\ \zeta(\frac{1}{2}+it)\in e^{i\phi}\mathbb{R}}} \zeta\left(\frac{1}{2}+it\right)\Bigg|^2 \leqslant \Bigg(\sum_{\substack{0<t\leqslant T\\ \Delta(\frac{1}{2}+it)=e^{2i\phi},\\ \zeta(\frac{1}{2}+it)\neq 0}} 1\Bigg) \times \Bigg(\sum_{\substack{0<t\leqslant T\\ \zeta(\frac{1}{2}+it)\in e^{i\phi}\mathbb{R}}} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2\Bigg).$$

Inserting the asymptotic formulae of Theorem 2 yields the assertion of the corollary.

5. Gram points

The roots of the equation $\Delta\left(\frac{1}{2}+it\right)=1$ constitute the so-called Gram points t_n which are usually defined by the roots of the equations $\vartheta(t)=\pi n$ with $n\in\mathbb{N}$, where $\exp(i\vartheta(t))=\Delta\left(\frac{1}{2}+it\right)^{-\frac{1}{2}}$. The function

$$Z(t) := \exp\big(i\vartheta(t)\big)\zeta\left(\frac{1}{2}+it\right)$$

is real-valued for real t and Gram's law states that

$$(-1)^n Z(t_n) > 0.$$

In fact, each sign change of the real-valued function $Z(t)$ corresponds to a zero and the number of Gram points matches the number of nontrivial zeros. By Theorem 1 we have that $N_\phi^\Delta(T)-N(T)\ll\log T$. As Gram observed in 1903 for the first zeta zeros, consecutive ordinates of nontrivial zeros are separated by a Gram point and vice versa; it should be mentioned that Gram himself doubted that this separation would persist indefinitely (cf. [6]; page 127). The first failure of Gram's law appears for $n=126$, as discovered by Hutchinson [11]. It was shown by Titchmarsh [18] that Gram's law is violated infinitely often, and, recently, Trudgian [19] has proved that it fails for a postive proportion of values of n as $n\to\infty$. Since $\zeta\left(\frac{1}{2}+it\right)<0$ if and only if t is a Gram point for which Gram's law is not true, there exist infinitely many intersections of the curve $t\mapsto\zeta\left(\frac{1}{2}+it\right)$ with the negative real axis and we may ask whether those points all lie in a bounded interval or not. This seems to be a difficult question which is of particular interest with respect to the conjectured denseness of $\zeta\left(\frac{1}{2}+i\mathbb{R}\right)$ in the complex plane.

Our results have an impact on Gram points too. Using

$$n=\frac{1}{\pi}\vartheta(t_n)\sim\frac{1}{2\pi}t_n\log t_n,$$

the asymptotic formulae from Theorem 2 translate to

$$\sum_{n\leqslant N} Z(t_n) \sim 2N \quad \text{and} \quad \sum_{n\leqslant N} Z(t_n)^2 \sim N\log N; \tag{18}$$

the first formula above was already found by Titchmarsh [18] (by a different method). By the Cauchy—Schwarz inequality, we find that

$$\Big(\sum_{n\leqslant N} Z(t_n)^2\Big)^2 \leqslant \Big(\sum_{n\leqslant N} Z(t_n)^4\Big)\Big(\sum_{n\leqslant N} 1\Big).$$

Substituting (18) implies the inequality

$$\sum_{n\leqslant N} Z(t_n)^4 \geqslant (1+o(1))N(\log N)^2,$$

as $N \to \infty$. However, Lavrik [14] obtained by a different method the stronger asymptotic formula

$$\sum_{n\leqslant N} Z(t_n)^4 = \frac{1}{2\pi^2}N(\log N)^4 + O(N(\log N)^{\frac{7}{2}}).$$

6. Concluding Remarks

We conclude with a few heuristic remarks. In Corollary 1 we have proved the conditional existence and attained explicit values for the mean of $\zeta\left(\frac{1}{2}+it\right)$ on straight lines $e^{i\phi}\mathbb{R}$ passing through the origin. Their positivity has confirmed an observation made by Edwards in [6] (as mentioned in the introduction). The almost symmetry of the curve generated by the values $\zeta\left(\frac{1}{2}+it\right)$ with respect to the real line is well reflected in the vanishing main term of the first moment (5) for $\phi = \frac{\pi}{2}$ whereas the non-vanishing of the second moment (6) implies the existence of infinitely many non-zero values $\zeta\left(\frac{1}{2}+it\right)$ on the imaginary axis. Assuming the Riemann hypothesis and taking Corollary 1 into account, we compute the mean of all individual mean-values by integration with respect to ϕ as

$$\frac{1}{\pi}\int_0^{\pi} \lim_{T\to\infty}\frac{1}{N_\phi(T)} \sum_{\substack{0<t\leqslant T\\ \zeta(\frac{1}{2}+it)\in e^{i\phi}\mathbb{R}}} \zeta\left(\frac{1}{2}+it\right)\,\mathrm{d}\phi = \frac{1}{\pi}\int_0^{\pi}\frac{1}{2}(1+e^{2i\phi})\,\mathrm{d}\phi = \frac{1}{2};$$

here we have used (15) to express the individual mean-values from Corollary 1 in a more convenient expression for the integration. The latter formula is of special interest with respect to another discrete moment recently considered by Garunkštis

& Steuding. In [8], they have proved that, for any fixed complex number a, as $T \to \infty$,

$$\sum_{0<\gamma_a\leqslant T} \zeta'(\rho_a) = \left(\frac{1}{2} - a\right)\frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right)^2 + 2(c_0 - 1 + a)\frac{T}{2\pi}\log\frac{T}{2\pi} + \\ + 2(c_1 - c_0 - a)\frac{T}{2\pi} + E(T),$$

where the summation is over nontrivial roots $\rho_a = \beta_a + i\gamma_a$ of the equation $\zeta(s) = a$, the numbers c_n are the Stieltjes constants (defined by the Laurent expansion of $\zeta(s)$ around $s = 1$), and the error term is $E(T) \ll T\exp(-C(\log T)^{\frac{1}{3}})$ with some absolute positive constant C; if the Riemann hypothesis is true, then $E(T) \ll T^{\frac{1}{2}+\epsilon}$. For $a = \frac{1}{2}$ the main term is of lower order which we may interpret in the following way: the curve $t \mapsto \zeta\left(\frac{1}{2} + it\right)$ avoids passing through a small neighbourhood of the point $\frac{1}{2}$. It seems that for generic ϕ the set $\zeta\left(\frac{1}{2} + i\mathbb{R}\right) \cap e^{i\phi}\mathbb{R}$ splits up into two subsets of approximately equal size each of which consisting of elements having either very small or large absolute value; the mean $e^{i\phi}\cos\phi$ separates these sets and the quantity $\frac{1}{2}$ is the average over all these mean-values.

Our results are easily extendable to Dirichlet L-functions; however, the case of higher degree L-functions seems to be rather difficult. This difficulty is already visible in the case of Dedekind zeta-functions $\zeta_{\mathbb{K}}$ to quadratic number fields for which Conrey, Ghosh & Gonek [3] succeeded to prove the existence of infinitely many simple zeros by factoring $\zeta_{\mathbb{K}} = \zeta L$, where L is the Dirichlet L-function to the character associated with $\mathbb{K}$.

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