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The house of an algebraic integer all of whose conjugates lie in a sector

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Abstract: Let α be a nonzero algebraic integer of degree d, all of whose conjugates α_i lie in a sector $|\arg z| \leq \theta,\ 0 \leq \theta < \pi$. The *house* of α is the largest modulus of its conjugates. We treat here the notion of house using the method of explicit auxiliary functions. This work seems to be the first of this kind. For $0 < \theta < \pi$, we compute the greatest lower bound $h(\theta)$ of the house of all such α, for θ belonging to nine subintervals of $[0, \pi)$. Moreover, among these subintervals, six are consecutive and complete. The polynomials involved in the auxiliary functions are found by our recursive algorithm.

Keywords: algebraic integer, house, explicit auxiliary functions, integer transfinite diameter, LLL algorithm

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1. Introduction

Let α be a nonzero algebraic integer of degree d, with conjugates $\alpha_1 = \alpha, \ldots, \alpha_d$ and minimal polynomial P. The *house* of α (and of P) is defined by:

$$\overline{|\alpha|} = \max_{1 \leq i \leq d} |\alpha_i|.$$

The Mahler measure of α is $M(\alpha) = \prod_{i=1}^{d} \max(1, |\alpha_i|)$ and its absolute Mahler measure is $\Omega(\alpha) = M(\alpha)^{1/d}$. We have the inequality: $\overline{|\alpha|} \geq \Omega(\alpha)$. It is clear that

$\lceil\overline{\alpha}\rceil \geq 1$ and, from a classical theorem of L. Kronecker [6], it follows that $\lceil\overline{\alpha}\rceil = 1$ if and only if α is a root of unity. In 1965, A. Schinzel and H. Zassenhaus [16] conjectured that there exists a constant $c > 0$ such that if α is not a root of unity then $\lceil\overline{\alpha}\rceil \geq 1 + c/d$. In 1985, a result of C. J. Smyth [17] led D. Boyd [3] to conjecture that c should be equal to $3/2 \log \theta_0$ where $\theta_0 = 1.324717\ldots$ is the smallest Pisot number, the real root of the polynomial $x^3 - x - 1$. P. Voutier [19] proved that, if α is an algebraic integer of degree $d \geq 3$, not a root of unity, then

$$\lceil\overline{\alpha}\rceil \geq 1 + \frac{1}{2d}(\log\log d/\log d)^3.$$

In 1991, E. M. Matveev [9] proved that, if α is an algebraic integer of degree $d \geq 2$, not a root of unity, then $\lceil\overline{\alpha}\rceil \geq \exp(\log(d + 0.5)/d^2)$. The best-known asymptotic result was given by A. Dubickas [2]:

$$\lceil\overline{\alpha}\rceil > 1 + \frac{1}{d}(64/\pi^2 - \epsilon)(\log\log d/\log d)^3) \quad \text{for } d > d_0(\epsilon).$$

More recently, in 2007, G. Rhin and Q. Wu [14] verified the conjecture of Schinzel and Zassenhaus with the constant of Boyd up to degree 28. They also established that, if α is an algebraic integer of degree $d \geq 4$, not a root of unity, then, if $d \leq 12$, $\lceil\overline{\alpha}\rceil \geq \exp(3\log(d/3)/d^2)$ and if $d \geq 13$, $\lceil\overline{\alpha}\rceil \geq \exp(3\log(d/2)/d^2)$. It appears that the result of [14] improves Matveev for $d \geq 6$.
Let $0 \leq \theta < \pi$ and S_θ be the sector of the complex plane such that $|\arg z| \leq \theta$. Let α be an algebraic integer, not a root of 1 and having all its conjugates in S_θ. The spectrum of the house of totally positive algebraic integers i.e., the $\theta = 0$ case, is well known. A result of L. Kronecker [6] tells us that all totally positive algebraic integers with house less than 4 have house of the form $2 + 2\cos(2\pi/n)$ for some positive integer n. Moreover, from a result of R. M. Robinson [15], this spectrum is dense in the interval $[4, \infty)$. Thus, we consider the case $0 < \theta < \pi$. We follow here the work of M. Langevin [7] on the absolute Mahler measure of algebraic integers α having all their conjugates in a sector. He proved that there exists a function $c(\theta)$ on $[0, \pi)$, always > 1, which is the greatest lower bound of the absolute Mahler measure of $\alpha \neq 0$, not a root of unity, all of whose conjugates lie in S_θ, i.e., $\Omega(\alpha) \geq c(\theta)$. G. Rhin and C. Smyth [12] succeeded in finding the exact value of $c(\theta)$ for θ in nine subintervals of $[0, 2\pi/3]$ and conjectured that $c(\theta)$ is a «staircase» function of θ, which is constant except for finitely many left discontinuities in any

closed subinterval of $[0, \pi)$. The polynomials involved in their auxiliary functions were found by heuristic methods. In 2004, thanks to Wu's algorithm [20], G. Rhin and Q. Wu [13] gave the exact value of $c(\theta)$ for four new subintervals of $[0, \pi)$ and extended four existing subintervals. In 2013, the author and G. Rhin [5] found for the first time a complete subinterval and a fourteenth subinterval. A complete interval is an interval on which the function $c(\theta)$ is constant, with jump discontinuities at each end. These improvements are due to our recursive algorithm.

Definition 1 *Let us define* $h(\theta) = \inf_{\alpha} \overline{|\alpha|}$ *where the infimum is taken over all nonzero algebraic integers* α *that are not* $= 2$ *or roots of unity and having all conjugates, including* α *itself, in the sector* S_θ.

Using the polynomials $x^{2n+1} - 2$, when $n \to \infty$ it is clear that $c(\theta)$ and $h(\theta) \to 1$ when $\theta \to \pi$. We define the spectrum $Spec(\theta) = \{ \overline{|\alpha|} : \alpha$ has all its conjugates in the sector $S_\theta\}$. Then, as a consequence of a result of Mignotte [10], for $\delta > 0$ the smallest limit point of the set $Spec(\pi - \delta)$ is at least $1 + c\delta^3$, for an effective positive constant c.

Remark 1.1. Now, we write the angles in degrees.

We give in Table 4 a list of 20 polynomials Q_i with $\theta_i = \varphi(Q_i)$. Now we define two functions f and g on $[0, 180)$. The function $g(\theta)$ is the decreasing staircase function having left discontinuities at the angles θ_i given in Table 4 and such that $g(\theta_i) = \overline{|Q_i|}$. It gives the smallest known value of $\overline{|\alpha|}$ for $\alpha \in S_\theta$ then $h(\theta) \leq g(\theta)$. For $1 \leq i \leq 9$, we define 9 non-increasing functions f_i for $\theta \in [\theta_i, \theta_{i+1}]$ as follows:

$$f_i(\theta) = \min_{z \in S_\theta} \left(\log \max(B_i, |z|) - \sum_{1 \leq j \leq J} c_{ij} \log |Q_{ij}(z)| \right),$$

where $B_i = \overline{|Q_i|}$ can be read off from Table 4. The polynomials Q_{ij} and the coefficients c_{ij} can be read off from Table 2 and Table 3. The function f is such that $f(\theta) = f_i(\theta)$ when $\theta \in [\theta_i, \theta_{i+1})$ for $1 \leq i \leq 9$. Since the functions f_i are continuous we have $f(\theta) \to f_i(\theta_{i+1})$ when $\theta \to \theta_{i+1}^-$. We do not find any function f_i such that $f_i(\theta_i) > g(\theta_i)$ for the other intervals $[\theta_i, \theta_{i+1})$, by Kronecker's theorem we may define $f(\theta) = 1$ for $\theta \geq \theta_{10}$. Then the function f is non-increasing on $[0, 180)$.

Table 1

The 9 intervals where $h(\theta)$ is known. The polynomials in the last column are the minimal polynomial of an algebraic integer belonging to S_{θ_i} and they are also listed in Table 4

i	$h(\theta)$	θ_i	θ'_i	Q
1	2.618033	0	14.066992	$z^2 - 3z + 1$
2	2.494446	14.066992	19.542882	$z^3 - 5z^2 + 7z - 1$
3	2.369276	19.542882	21.640384	$z^3 - 5z^2 + 8z - 3$
4	2.019800	21.640384	30	$z^3 - 4z^2 + 5z - 1$
5	1.732050	30	45	$z^2 - 3z + 3$
6	1.414213	45	71.65	$z^2 - 2z + 2$
7	1.363626	75.179481	79.75	$z^4 - 3z^3 + 5z^2 - 5z + 3$
8	1.324717	80.656153	82.97	$z^3 - z^2 + 2z - 1$
9	1.227949	87.978495	92.18	$z^4 - 2z^3 + 3z^2 - 3z + 2$

Theorem 1 *The non-increasing functions f, g, h satisfy the following inequalities*:

$$\min(f(\theta), g(\theta)) \le h(\theta) \le g(\theta) \qquad (0 \le \theta < 180).$$

Moreover, the exact value of $h(\theta)$ is known on nine subintervals of $[0, 180)$.

These intervals are given in Table 1. One can read off the five intervals $[\theta_i, \theta'_i)$ for $1 \le i \le 5$ and the four intervals $[\theta_i, \theta'_i]$ for $6 \le i \le 9$ where $h(\theta)$ is known exactly. For θ in each of these intervals, we have $f(\theta) > g(\theta)$ so that $h(\theta) = h(\theta_i)$. Outside these intervals we have $h(\theta) \le g(\theta)$.

Remark 1.2. Since Langevin proved the existence of functions with the same properties as $c(\theta)$ for house, absolute trace and absolute length in the sector $[0, 90)$, we may extend the conjecture of G. Rhin and C. J. Smyth [12] on the nature of the function $c(\theta)$ to all these functions. Finding consecutive and complete subintervals appears here for the first time.

In Section 2 we describe the method of explicit auxiliary functions. In Section 3, we link these functions with the classical integer transfinite diameter. In Section 4, we detail how our recursive algorithm [4] enables us to prove Theorem 1. All the computations were done on a MacBookPro with the languages Pascal and Pari [11].

2. The explicit auxiliary functions

In this section we assume that α is an algebraic integer in S_θ with minimal polynomial P of degree d. We let $\alpha_1 = \alpha, \ldots, \alpha_d$ denote the conjugates of α. The auxiliary functions f_i, $1 \leq i \leq 9$, are of the following type:

$$\forall\, z \in S_\theta,\, f(z) = \log\max(B, |z|) - \sum_{1\leq j\leq J} c_j \log|Q_j(z)|, \qquad (2.1)$$

where B and the coefficients c_j are positive real numbers and the polynomials Q_j are nonzero in $\mathbb{Z}$, not necessarily irreducible, but not divisible by P. The main point is to choose the numbers c_j and the polynomials Q_j in order to maximize the minimum m of f on S_θ.

If we have $\sum_{i=1}^{d} f(\alpha_i) \geq md$ and $B \leq \lceil\overline{\alpha}\rceil$ then

$$\log\lceil\overline{\alpha}\rceil^d \geq \sum_{i=1}^{d} \log\max(B, |\alpha_i|) \geq md + \sum_{j=1}^{J} c_j \log|\prod_{i=1}^{d} Q_j(\alpha_i)|.$$

Since P does not divide any Q_j, $\prod_{i=1}^{d} Q_j(\alpha_i)$ is a nonzero integer because it is the resultant of P and Q_j. Therefore, we have

$$\lceil\overline{\alpha}\rceil \geq e^m.$$

The main difficulty in this procedure is to find a good list of polynomials Q_j which gives a value of m as large as possible. For this purpose, we link the auxiliary function to the integer transfinite diameter in order to find our polynomials by the recursive algorithm.

3. Auxiliary functions and integer transfinite diameter

In this section, we will need the following definition:
Let K be a compact subset of $\mathbb{C}$. We define *the integer transfinite diameter of* K by

$$t_{\mathbb{Z}}(K) = \liminf_{\substack{n\geq 1\\ n\to\infty}} \inf_{\substack{P\in\mathbb{Z}[z]\\ \deg(P)=n}} |P|_{\infty,K}^{\frac{1}{n}},$$

where $|P|_{\infty,K} = \sup_{z\in K}|P(z)|$. If φ is a positive function defined on K, the φ*-integer transfinite diameter of* K is defined as

$$t_{\mathbb{Z},\varphi}(K) = \liminf_{\substack{n\geq 1\\ n\to\infty}} \inf_{\substack{P\in\mathbb{Z}[z]\\ \deg(P)=n}} \sup_{z\in K}\left(|P(z)|^{\frac{1}{n}}\ \varphi(z)\right).$$

This weighted version of the integer transfinite diameter was introduced by F. Amoroso [1]. It is an important tool in the study of rational approximations of logarithms of rational numbers.
Inside the auxiliary function (2.1), we replace the numbers c_j by rational numbers a_j/q where q is a common denominator of the c_j for $1 \leq j \leq J$. Then we can write:

$$\text{for } z \in S_\theta,\ \ f(z) = \log\max(B,|z|) - \frac{t}{r}\log|Q(z)| \geq m, \tag{3.1}$$

where $Q = \prod_{j=1}^{J} Q_j^{a_j} \in \mathbb{Z}[z]$ is of degree $r = \sum_{i=1}^{J} a_j \deg Q_j$ and $t = \sum_{j=1}^{J} c_j \deg Q_j$.
We want to get a function whose minimum m in the sector S_θ is as large as possible. Thus we search for a polynomial $Q \in \mathbb{Z}[z]$ such that

$$\sup_{z\in S_\theta}|Q(z)|^{t/r}(\max(B,|z|))^{-1} \leq e^{-m}.$$

If we suppose that t is fixed, it is clear that we need an effective upper bound for the quantity

$$t_{\mathbb{Z},\varphi}(S_\theta) = \liminf_{\substack{r\geq 1\\ r\to+\infty}} \inf_{\substack{P\in\mathbb{Z}[z]\\ \deg(P)=r}} \sup_{z\in S_\theta}\left(|P(z)|^{\frac{t}{r}}\ \varphi(z)\right).$$

where we use the weight $\varphi(z) = (\max(B,|z|))^{-1}$.

4. Construction of the auxiliary functions

The polynomials involved in the auxiliary function are found by our recursive algorithm developed in [4] from Wu's algorithm [20]. It replaces a heuristic search for suitable polynomials by a systematic inductive search. Suppose that we have already found a list Q_1, Q_2, ..., Q_J of suitable polynomials. Then we use semi-infinite linear programming (introduced into number theory by C. J. Smyth [18]) to optimize f for this set of polynomials (i.e., to get the greatest possible m). We obtain the real positive numbers c_1, c_2, ..., c_J and then f in the form (3.1) as above. The function f is invariant under complex conjugation so we can limit ourselves to the sector $S'_\theta = \{z \in \mathbb{C}$ such that $0 \leq \arg z \leq \theta\}$. Since the function f is harmonic in S'_θ outside the union of arbitrarily small disks around the roots of the polynomials Q_j, the minimum is taken on the boundary of S'_θ. Thus, it is sufficient to consider the minimum of f on the arc $A_\theta = \{z = Be^{ix}, 0 \leq x \leq \theta\}$ and on the half line $R_\theta = \{z = se^{i\theta}, s \geq 0\}$.

The auxiliary function on the arc is:

$$f(z) = \log B - \sum_{1 \leq j \leq J} c_j \log |Q_j(z)| \geq m_1,$$

while we have on the half line:

$$f(z) = \log \max(B, s) - \sum_{1 \leq j \leq J} c_j \log |Q_j(z)| \geq m_2.$$

Thus, by our algorithm, we seek a polynomial $R(z) = \sum_{l=0}^{k} a_l z^l \in \mathbb{Z}[z]$, where k is varying from 4 to 15 successively, such that

$$\sup_{0 \leq x \leq \theta} |Q(Be^{ix})R(Be^{ix})|^{\frac{t}{r+k}} B^{-1} \leq e^{-m_1}$$

and

$$\sup_{s \geq 0} |Q(se^{i\theta})R(se^{i\theta})|^{\frac{t}{r+k}} \max(B, s)^{-1} \leq e^{-m_2},$$

i.e., such that

$$\sup_{0\le x\le\theta} |Q(Be^{ix})R(Be^{ix})|B^{-\frac{r+k}{t}} \quad\text{and}\quad \sup_{s\ge 0} |Q(se^{i\theta})R(se^{i\theta})| \max(B,s)^{-\frac{r+k}{t}}$$

are as small as possible.
But, here, $R(Be^{ix})$ and $R(se^{i\theta})$ are now linear forms in the unknown coefficients a_l of R. We replace them by their real parts and their imaginary parts. Then, we get the following real linear forms

$$|Q(Be^{ix_n})|.\mathrm{Re}(R(Be^{ix_n}))B^{-\frac{r+k}{t}} \quad\text{and}\quad |Q(Be^{ix_n})|.\ \mathrm{Im}(R(Be^{ix_n}))B^{-\frac{r+k}{t}},$$

$$|Q(s_n e^{i\theta})|.\mathrm{Re}(R(s_n e^{i\theta}))\max(B,s_n)^{-\frac{r+k}{t}}$$

and

$$|Q(s_n e^{i\theta})|.\mathrm{Im}(R(s_n e^{i\theta}))\max(B,s_n)^{-\frac{r+k}{t}}.$$

The x_n are suitable points in $[0,\theta]$, including the points where f_1 has its least local minima. The s_n are suitable points in $(0,\infty)$, including the points where f_2 has its least local minima. All these linear forms define a real lattice on $\mathbb{Z}^{k+1}$. We use algorithm LLL to obtain a small vector in this lattice.

Then, we get a polynomial R whose factors R_j are good candidates to enlarge the set of polynomials $(Q_1, Q_2, \ldots, Q_J)$. We only keep the polynomials R_j which have a nonzero coefficient c_j in the newly optimized auxiliary function f. After optimization, some previous polynomials Q_j may have a zero coefficient c_j and so are removed. The polynomials in Table 4 are found during all these computations.

4.1. The Computations.

We give here some explanations about the computations of the functions f_i.

A complete interval.

For the first interval, we start with the four polynomials z, $z-1$, $z-2$ and z^2-3z+1. Then we introduce the unknown polynomials R of degree growing from 10 to 20. The final function $f_1(\theta)$ decreases from 2.623674 at $\theta = 0$ to 2.618837 at $\theta = 14.066992$. Then the function $f_1(\theta)$ is greater than $g(\theta)$ in the whole interval $[\theta_1, \theta_2)$ so the value of $h(\theta)$ in this range is $\overline{|Q_1|} = 2.61803399$.

A non-complete interval.

For the sixth interval, we start with the polynomials z, $z-1$, z^2-z+1 (which is cyclotomic) and z^2-2z+2 which has a root on the half-line $\theta = 45$. We proceed as above and obtain the function $f_6(\theta)$. For $45 \le \theta \le 71.65$, we have $f_6(\theta) \ge 1.414338 \ge 1.414213 = g(\theta)$ so that $h(\theta) = g(\theta) = 1.414213$ and we get the non-complete interval $[45, 71.65]$.

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Table 2

The auxiliary functions f_i, $1 \le i \le 9$, involved in Theorem 1

i	Q_j	c_j
1	Q_1 Q_2 Q_3 Q_4 Q_5	0.018506 0.034524 0.097738 0.003427 0.035709
	Q_6 Q_{13}	0.001583 0.014710
2	Q_1 Q_2 Q_3 Q_5 Q_7	0.020139 0.043439 0.086792 0.017384 0.024384
	Q_8 Q_9 Q_{10} Q_{16}	0.003915 0.000425 0.006998 0.008772
3	Q_1 Q_2 Q_3 Q_7 Q_9	0.025456 0.042125 0.086111 0.033529 0.019139
	Q_{11} Q_{12} Q_{15} Q_{18} Q_{34}	0.017640 0.000161 0.000490 0.000859 0.003452
4	Q_1 Q_2 Q_3 Q_9 Q_{14}	0.017777 0.082418 0.029814 0.016027 0.039697
	Q_{17} Q_{19} Q_{23}	0.010587 0.023249 0.001665
5	Q_1 Q_2 Q_{14} Q_{20} Q_{21} Q_{22}	0.044136 0.184437 0.023350 0.027677 0.041457 0.042631
6	Q_1 Q_2 Q_{21} Q_{24} Q_{25} Q_{26}	0.024507 0.175057 0.032525 0.004751 0.079304 0.050438
7	Q_1 Q_2 Q_{25} Q_{26} Q_{28}	0.008444 0.0575607 0.065330 0.0004682 0.016565
	Q_{29} Q_{30} Q_{32} Q_{33} Q_{34}	0.001730 0.000014 0.000423 0.004915 0.001592
	Q_{35} Q_{36} Q_{37} Q_{45} Q_{47}	0.015610 0.001888 0.001414 0.002149 0.000687
8	Q_1 Q_2 Q_{25} Q_{28} Q_{30} Q_{31}	0.006904 0.0975367 0.117138 0.007135 0.042038 0.015259
9	Q_1 Q_2 Q_{25} Q_{38} Q_{39}	0.019950 0.074188 0.055458 0.019050 0.049961
	Q_{40} Q_{41} Q_{42}	0.001051 0.011188 0.006327

Table 3

Polynomials used in the auxiliary functions f_i, $1 \le i \le 9$,
where $\varphi(Q) = \max\{|\arg z| \text{ such that } Q(z) = 0\}$

j	$\boxed{Q_j}$	$\varphi(Q_j)$	Q_j
Q_1	0	0	z
Q_2	1	0	$z - 1$
Q_3	2	0	$z - 2$
Q_4	2.61803399	0	$z^2 - 3z + 1$
Q_5	2.49444634	14.0669922	$z^3 - 5z^2 + 7z - 1$
Q_6	2.69420745	18.5821464	$z^6 - 10z^5 + 40z^4 - 79z^3 + 76z^2 - 28z + 1$
Q_7	2.36927632	19.5428819	$z^3 - 5z^2 + 8z - 3$
Q_8	2.57530660	21.3762053	$z^5 - 8z^4 + 24z^3 - 31z^2 + 14z - 1$
Q_9	2.01980089	21.6403840	$z^3 - 4z^2 + 5z - 1$
Q_{10}	2.51388480	22.3336303	$z^5 - 8z^4 + 25z^3 - 36z^2 + 21z - 1$
Q_{11}	2.41421356	23.9057118	$z^4 - 6z^3 + 13z^2 - 10z + 1$
Q_{12}	2.61132069	24.1327514	$z^8 - 12z^7 + 62z^6 - 177z^5 + 299z^4 - 296z^3 + 158z^2 - 36z + 2$
Q_{13}	2.63594853	28.6295307	$z^4 - 6z^3 + 12z^2 - 8z + 2$
Q_{14}	1.73205081	30	$z^2 - 3z + 3$
Q_{15}	2.47024633	30.0052009	$z^8 - 11z^7 + 53z^6 - 142z^5 + 225z^4 - 207z^3 + 101z^2 - 20z + 1$
Q_{16}	2.56602444	30.6160763	$z^5 - 7z^4 + 18z^3 - 19z^2 + 7z - 1$
Q_{17}	2.09355577	32.1850043	$z^3 - 4z^2 + 6z - 2$
Q_{18}	2.34123073	34.9178743	$z^5 - 7z^4 + 20z^3 - 28z^2 + 19z - 4$
Q_{19}	1.99264807	36.8903690	$z^4 - 5z^3 + 10z^2 - 8z + 1$
Q_{20}	1.77423196	40.8948445	$z^3 - 3z^2 + 4z - 1$
Q_{21}	1.41421356	45	$z^2 - 2z + 2$
Q_{22}	1.78466730	47.0903760	$z^3 - 4z^2 + 7z - 5$
Q_{23}	2.01980089	58.1255013	$z^6 - 6z^5 + 16z^4 - 22z^3 + 16z^2 - 5z + 1$
Q_{24}	1.52470258	59.0157696	$z^3 - 2z^2 + 3z - 1$

Q_{25}	1	60	$z^2 - z + 1$
Q_{26}	1.41421356	69.2951889	$z^2 - z + 2$
Q_{27}	1.64015200	74.7163425	$z^5 - 3z^4 + 5z^3 - 6z^2 + 3z - 1$
Q_{28}	1.36362651	75.1794810	$z^4 - 3z^3 + 5z^2 - 5z + 3$
Q_{29}	1.46916905	75.8507515	$z^6 - 4z^5 + 9z^4 - 13z^3 + 12z^2 - 7z + 1$
Q_{30}	1.41372670	77.3596055	$z^8 - 5z^7 + 14z^6 - 26z^5 + 34z^4 - 31z^3 + 19z^2 - 6z + 1$
Q_{31}	1.73491605	78.3230933	$z^8 - 4z^7 + 12z^6 - 21z^5 + 30z^4 - 25z^3 + 18z^2 - 3z + 1$
Q_{32}	1.46807570	79.7313691	$z^{11} - 7z^{10} + 27z^9 - 71z^8 + 139z^7 - 210z^6 + 248z^5 - 227z^4 + 157z^3 - 76z^2 + 22z - 1$
Q_{33}	1.41425545	80.3585322	$z^{10} - 6z^9 + 20z^8 - 45z^7 + 74z^6 - 91z^5 + 83z^4 - 54z^3 + 23z^2 - 5z + 1$
Q_{34}	1.44038921	80.5418045	$z^{12} - 8z^{11} + 34z^{10} - 98z^9 + 210z^8 - 349z^7 + 458z^6 - 476z^5 + 388z^4 - 242z^3 +$ $+ 111z^2 - 35z + 7$
Q_{35}	1.32471796	80.6561536	$z^3 - z^2 + 2z - 1$
Q_{36}	1.41711530	81.7436262	$z^6 - 3z^5 + 6z^4 - 7z^3 + 6z^2 - 2z + 1$
Q_{37}	1.42267845	87.5431247	$z^{15} - 9z^{14} + 43z^{13} - 141z^{12} + 349z^{11} - 684z^{10} + 1089z^9 - 1427z^8 + 1546z^7 -$ $- 1380z^6 + 1003z^5 - 579z^4 + 253z^3 - 76z^2 + 12z - 1$
Q_{38}	1.22794984	87.9784953	$z^4 - 2z^3 + 3z^2 - 3z + 2$
Q_{39}	1	90	$z^2 + 1$
Q_{40}	1.27230966	92.1291431	$z^5 - 2z^4 + 4z^3 - 5z^2 + 4z - 3$
Q_{41}	1.35321020	94.0044886	$z^9 - 4z^8 + 10z^7 - 18z^6 + 25z^5 - 27z^4 + 22z^3 - 14z^2 + 5z - 1$
Q_{42}	1.27375339	96.0500257	$z^7 - 3z^6 + 6z^5 - 9z^4 + 10z^3 - 8z^2 + 5z - 1$
Q_{43}	1.41817747	180	$z^5 - 2z^4 + 4z^3 - 3z^2 + 2z + 1$
Q_{44}	2.29649166	180	$z^5 - 7z^4 + 19z^3 - 23z^2 + 10z + 1$
Q_{45}	1.37594383	180	$z^8 - 5z^7 + 14z^6 - 26z^5 + 34z^4 - 31z^3 + 18z^2 - 5z - 1$
Q_{46}	3.23814583	180	$z^9 - 2z^7 + 14z^6 - 28z^5 + 38z^4 - 35z^3 + 18z^2 - 4z + 1$
Q_{47}	1.40815538	180	$z^{17} - 10z^{16} + 52z^{15} - 183z^{14} + 481z^{13} - 990z^{12} + 1633z^{11} - 2175z^{10} + 2323z^9$ $-1935z^8 + 1165z^7 - 379z^6 - 109z^5 + 238z^4 - 164z^3 + 66z^2 - 17z + 4$

Table 4

Polynomials used in the function $g(\theta)$
where $\varphi(Q) = \max\{|\arg z| \text{ such that } Q(z) = 0\}$

j	$\lceil Q_j \rceil$	$\varphi(Q_j)$	Q_j
1	2.61803399	0	$z^2 - 3z + 1$
2	2.49444634	14.0669922	$z^3 - 5z^2 + 7z - 1$
3	2.36927632	19.5428819	$z^3 - 5z^2 + 8z - 3$
4	2.01980089	21.6403840	$z^3 - 4z^2 + 5z - 1$
5	1.73205081	30	$z^2 - 3z + 3$
6	1.41421356	45	$z^2 - 2z + 2$
7	1.3636265	75.1794810	$z^4 - 3z^3 + 5z^2 - 5z + 3$
8	1.32471796	80.6561536	$z^3 - z^2 + 2z - 1$
9	1.22794984	87.9784953	$z^4 - 2z^3 + 3z^2 - 3z + 2$
10	1.21060779	106.368385	$z^3 + z - 1$
11	1.18920712	112.500000	$z^4 + 2z^2 + 2$
12	1.15096393	116.481702	$z^6 - z^5 + 2z^4 - 2z^3 + 2z^2 - 2z + 1$
13	1.14150997	120.702429	$z^5 - z^4 + z^3 - z^2 + 2z - 1$
14	1.13925030	130.049673	$z^5 + z^3 + z - 1$
15	1.13635300	132.505907	$z^7 + z^5 - z^4 + z^3 - z^2 + z - 1$
16	1.12246205	135.000000	$z^6 - 2z^3 + 2$
17	1.10452431	141.700857	$z^7 + z^5 + z^3 + z - 1$
18	1.10027624	143.184193	$z^6 + z^2 + 1$
19	1.09373169	155.927080	$z^6 + z^5 - z^3 - z^2 + 1$
20	1.07282987	159.835962	$z^6 - z^4 + 1$

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