Reprint from

ISSN 2220-5438

Moscow Journal of Combinatorics and Number Theory

Moscow Journal of Combinatorics and Number Theory Volume 6 • Issue 2–3 2016

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Volume 6 • Issue 2–3 2016

Moscow Journal
of Combinatorics
and Number Theory
2016, vol. 6, iss. 2–3,
pp. 208–232, [pp. 324–348]

Linear independence of polylogarithms at algebraic points

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Abstract: We prove new qualitative and quantitative results on the linear independence measure of logarithms, dilogarithms and trilogarithms of algebraic numbers. For example, our method yields the linear independence over the rationals of $1, \mathrm{Li}_1(1/q), \mathrm{Li}_2(1/q), \mathrm{Li}_3(1/q)$ for any integer q with $q \geqslant 486$ or $q \leqslant -471$. We briefly indicate how to obtain linear independence measure over $\mathbb{Q}(i)$ of $1, \mathrm{Li}_1(i/\sqrt{q}), \mathrm{Li}_2(i/\sqrt{q})$ for $q \geqslant 28$, or improvement of some irrationality measures of $\mathrm{Li}_2(1/q)$ previously obtained by Rhin and Viola, for $q \geqslant 6$, and by Hata, for $q \leqslant -5$.

Keywords: irrationality measure, linear independence measures, polylogarithmic functions.

AMS Subject Classification: 11J82, 11J72, 11J91, 33C60

Received: 10.09.2015; **revised:** 07.01.2016

1. Introduction

Let $\mathrm{Li}_s(x)$ be the s-th polylogarithm defined by

$$\mathrm{Li}_s(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^s}, \qquad |x| < 1, \ s = 1, 2, 3, \dots;$$

in this note we adapt the method developed by Zudilin in [12, Section 1] and in [13] to obtain new qualitative and quantitative results for the linear independence over $\mathbb{Q}$ of $1, \mathrm{Li}_1(1/z), \mathrm{Li}_2(1/z)$ and $\mathrm{Li}_3(1/z)$ when z takes suitable rational values. In a simpler form, our method was introduced by Gutnik [1]. Later on, Hata [2] introduced simultaneous approximations to the Lerch functions, and in particular obtained a new proof of Gutnik's result.

A linear independence measure of a finite set of real numbers is a lower bound of the absolute value of a linear combination over $\mathbb{Z}$ of these numbers, in terms of the size of the coefficients of the combination itself (see e. g. Theorem 7.1 below). In the special case of the polylogarithms, Hata's constructions of simultaneous approximations to $\mathrm{Li}_1(1/z), \mathrm{Li}_2(1/z), \ldots, \mathrm{Li}_s(1/z)$ coincide with Gutnik's. Indeed, the simplest expression of the coefficient the polylogarithms in the s linear forms (i. e. the analogue of the polynomial $W_n(z)$ introduced below) is the same in the two constructions, and this implies that the s linear forms also coincide, by well known facts on Padé approximations of the second type. In contrast, the analytic expression of the s linear form is different in the two papers. We prove that $1, \mathrm{Li}_1(1/z), \mathrm{Li}_2(1/z), \mathrm{Li}_3(1/z)$ are linearly independent over $\mathbb{Q}$, when z belongs to a set of values wider than in Hata's paper [2]. We also obtain a linear independence measure that in particular improves the quantitative results of Gutnik and Hata.

It seems plausible to obtain the same results as in the present paper by writing the aforementioned polynomials $W_n(z)$ as "multiple Legendre polynomials", similarly to [6], and then by constructing the linear forms as integrals of these polynomials multiplied by suitable weights, i. e. by adapting Hata's method [3], developed there for the dilogarithm. Moreover, as a special case of a theorem of Nesterenko [8], the Barnes's type integral $I_n^{(3)}(z)$ below equals a suitable triple Euler-type integral, i. e. an integral in the unit cube of a product of powers of $x_1, 1-x_1, x_2, 1-x_2, x_3, 1-x_3$ divided by a power of a suitable polynomial of x_1, x_2, x_3, z. Therefore one could also adapt the Rhin—Viola method [10] (see section 8 below for a related remark on the double integrals in [10]) and seemingly obtain our new results along those lines. However, this could entail the application of a $\mathbb{C}^3$-saddle point method, or perhaps a restriction of the parameters similar to that operated in [10] (see section 8 below for more details on this point). It is worth noticing that in [11] Viola made an interesting tentative of adapting the method of [10] to polylogarithms, and proposed an explicit construction for the trilogarithms, based on suitable triple Euler-type integrals which are different from those in [8].

Our method could be extended to polylogaritms $\mathrm{Li}_s(1/z)$, but we confine our discussions to trilogarithms for the sake of simplicity. In the last section, as a byproduct of the present work, we briefly indicate how to adapt our method to the linear independence of $1, \mathrm{Li}_1(1/z), \mathrm{Li}_2(1/z)$, where z takes rational or imaginary quadratic values, by using definitions and results from the paper [7] by Viola and the present author. The linear independence criterion we borrow from [7] could be extended to higher dimensions, therefore it would be possible to obtain linear independence measures of $1, \mathrm{Li}_1(1/z), \mathrm{Li}_2(1/z), \mathrm{Li}_3(1/z)$ for suitable algebraic values of z: for ease of presentation, we omit the details and the numerical examples.

The paper is organized as follows. In section 2 we introduce our simultaneous approximations in the form of suitable hypergeometric series, inspired by [13]. In section 3 we study the arithmetical properties of our approximations. In section 4 we refine the results of section 3 by a standard comparison between companion constructions. In section 5 we apply the saddle point method and obtain the asymptotic behavior of our linear forms, and of the common coefficient of $\mathrm{Li}_s(1/z)$ $(s = 1, 2, 3)$ in the three linear forms. In section 6 we compute a standard but crucial determinant which guarantees that a certain degenerescence does not occur. In section 7 we state our main theorem and give some numerical applications. In section 8 we indicate how to adapt the argument when we deal with $\mathrm{Li}_1(1/z)$ and $\mathrm{Li}_2(1/z)$ only, and z is algebraic. For $z \in \mathbb{Q}$ this leads to an improvement of the results in [3], for z negative, and in [10], for z positive.

2. Linear forms in trilogarithms

Let a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 be integer numbers such that

$$b_1, b_2, b_3 \leqslant a_1, a_2, a_3, a_4 < b_4$$

as in [13]. In the numerical examples we presente in section 7, the quantity

$$d = (a_1 + a_2 + a_3 + a_4) - (b_1 + b_2 + b_3 + b_4) + 1$$

will be zero; however, for methodological reasons, we do not make any assumption on the sign of d (we also remark that the related quantity in the application we present in section 8 is positive in several numerical examples). Let $\widehat{R}(t)$ be the product of four rational functions as follows:

$$\widehat{R}(t) = \prod_{i=1}^{4} R(a_i, b_i; t),$$

where

$$R(a, b; t) = \Gamma(b - a)\frac{\Gamma(a + t)}{\Gamma(b + t)} \quad \text{if } a < b$$

and

$$R(a,b;t)=\frac{1}{\Gamma(1+a-b)}\frac{\Gamma(a+t)}{\Gamma(b+t)}\qquad \text{if } a\geq b,$$

and Γ is Euler's Gamma function. In order to fix the ideas, we take the integers a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4-1 all multiple of an integer $n=0,1,2,\ldots$, and introduce the fixed nonnegative integers $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma$ as follows:

$$\alpha_1 n=a_4-b_1,\quad \alpha_2 n=a_4-b_2,\quad \alpha_3 n=a_4-b_3,$$
$$\beta_1 n=a_1-b_1,\quad \beta_2 n=a_2-b_2,\quad \beta_3 n=a_3-b_3$$

and

$$\gamma n+1=b_4-a_4.$$

We consider the following three series defined for any $z\in\mathbb{C}$ with $|z|>1$:

$$S_n^{(j)}(z)=z^{\delta n}(1-z)^{\max\{0,(\beta_1+\beta_2+\beta_3-\gamma)n\}}$$
$$\times\sum_{t\geqslant 1+\delta n}\frac{1}{(j-1)!}\frac{\mathrm{d}^{j-1}}{\mathrm{d}t^{j-1}}\left(\frac{(t-\alpha_1 n)_{\beta_1 n}}{(\beta_1 n)!}\frac{(t-\alpha_2 n)_{\beta_2 n}}{(\beta_2 n)!}\frac{(t-\alpha_3 n)_{\beta_3 n}}{(\beta_3 n)!}\frac{(\gamma n)!}{(t)_{\gamma n+1}}\right)z^{-t},$$
$$(j=1,2,3),$$

where

$$\delta=\max\{0,\alpha_1-\beta_1,\alpha_2-\beta_2,\alpha_3-\beta_3\}.$$

The rational function $\widehat{R}(t-a_4)$ has a unique decomposition in partial fractions:

$$\widehat{R}(t-a_4)=\frac{(t-\alpha_1 n)_{\beta_1 n}}{(\beta_1 n)!}\frac{(t-\alpha_2 n)_{\beta_2 n}}{(\beta_2 n)!}\frac{(t-\alpha_3 n)_{\beta_3 n}}{(\beta_3 n)!}\frac{(\gamma n)!}{(t)_{\gamma n+1}}=\sum_{l=0}^{\gamma n}\frac{A_l}{t+l}+B(t),$$

where

$$A_l=\left(\widehat{R}(t-a_4)(t+l)\right)_{t=-l}$$
$$=(-1)^{(\beta_1+\beta_2+\beta_3)n+l}\binom{\gamma n}{l}\binom{\alpha_1 n+l}{\beta_1 n}\binom{\alpha_2 n+l}{\beta_2 n}\binom{\alpha_3 n+l}{\beta_3 n}$$

and $B(t)$ is a polynomial in t of degree $< d+1 = (\beta_1+\beta_2+\beta_3-\gamma)n$ with rational coefficients (by definition $\deg 0 = -\infty$). Note also that $\widehat{R}(t-a_4)$ vanishes with its first and second derivatives for $t = \delta n+1, \dots, \min\{\alpha_1, \alpha_2, \alpha_3\}n$.

Therefore $S_n^{(1)}(z)$ splits into two parts:

$$\begin{aligned} S_n^{(1)}(z) &= z^{\delta n}(1-z)^{\max\{0,(\beta_1+\beta_2+\beta_3-\gamma)n\}} \sum_{t\geq 1+\delta n} \left(\sum_{l=0}^{\gamma n} \frac{A_l}{t+l} + B(t) \right) z^{-t} \\ &= (1-z)^{\max\{0,(\beta_1+\beta_2+\beta_3-\gamma)n\}} \sum_{l=0}^{\gamma n} A_l \sum_{t\geq 1+\delta n} \frac{z^{\delta n - t}}{t+l} \\ &\qquad + z^{\delta n}(1-z)^{\max\{0,(\beta_1+\beta_2+\beta_3-\gamma)n\}} \sum_{t\geq 1+\delta n} B(t) z^{-t}, \end{aligned}$$

and similarly for $S_n^{(2)}(z)$ and $S_n^{(3)}(z)$, as we see later on. The first part of $S_n^{(j)}(z)$ $(j=1,2,3)$ splits further, using

$$\sum_{t\geq 1+\delta n} \frac{z^{\delta n - t}}{(t+l)^j} = z^{\delta n + l}\mathrm{Li}_j(1/z) - \sum_{t=1}^{\delta n + l} \frac{z^{\delta n + l - t}}{t^j} \qquad (l = 0, \dots, \gamma n;\ j = 1, 2, \dots),$$

where the finite sum at the right-hand side is 0 if $\delta n + l = 0$. For the second part of $S_n^{(1)}(z)$ we have

$$-\sum_{t\geq 1+\delta n} B(t) z^{-t} = \frac{U_n^{(1)}(z)}{z^{\delta n}(1-z)^{(\beta_1+\beta_2+\beta_3-\gamma)n}}$$

for a suitable polynomial $U_n^{(1)}(z)$ of degree $< (\beta_1+\beta_2+\beta_3-\gamma)n$ (see e. g. [9, Lemma 1]). Hence $U_n^{(1)}(z)$ is nonzero if and only if $(\beta_1+\beta_2+\beta_3-\gamma)n > 0$.

For any positive integer m, we consider the least common multiple of $1, 2, \dots, m$, and denote it by D_m.

By [13, Lemma 5] we have that

$$D_{n\max\{\beta_1,\beta_2,\beta_3\}} B(t)$$

is an integer-valued polynomial (i. e. a polynomial that takes integer values over integers). Therefore, again by [9, Lemma 1]

$$D_{n\max\{\beta_1,\beta_2,\beta_3\}} U_n^{(1)}(z) \in \mathbb{Z}[z].$$

Hence

$$S_n^{(1)}(z) = W_n(z)\mathrm{Li}_1(1/z) - \big(U_n^{(1)}(z) + V_n^{(1)}(z)\big),$$

where

$$W_n(z) = z^{\delta n}(1-z)^{\max\{0,(\beta_1+\beta_2+\beta_3-\gamma)n\}} \sum_{l=0}^{\gamma n} A_l z^l$$

and

$$V_n^{(1)}(z) = (1-z)^{\max\{0,(\beta_1+\beta_2+\beta_3-\gamma)n\}} \sum_{l=0}^{\gamma n} A_l \sum_{t=1}^{\delta n+l} \frac{z^{\delta n+l-t}}{t}.$$

Similarly $S_n^{(2)}(z)$ splits into two parts:

$$\begin{aligned} -S_n^{(2)}(z) &= z^{\delta n}(1-z)^{\max\{0,(\beta_1+\beta_2+\beta_3-\gamma)n\}} \sum_{t\geqslant 1+\delta n} \left(\sum_{l=0}^{\gamma n} \frac{A_l}{(t+l)^2} - B'(t) \right) z^{-t} \\ &= (1-z)^{\max\{0,(\beta_1+\beta_2+\beta_3-\gamma)n\}} \sum_{l=0}^{\gamma n} A_l \sum_{t\geqslant 1+\delta n} \frac{z^{\delta n-l}}{(t+l)^2} \\ &\quad - z^{\delta n}(1-z)^{\max\{0,(\beta_1+\beta_2+\beta_3-\gamma)n\}} \sum_{t\geqslant 1+\delta n} B'(t) z^{-t}. \end{aligned}$$

If $(\beta_1+\beta_2+\beta_3-\gamma)n > 1$ we also have

$$\sum_{t\geqslant 1+\delta n} B'(t) z^{-t} = \frac{\widetilde{U}_n^{(2)}(z)}{z^{\delta n}(1-z)^{\max\{0,(\beta_1+\beta_2+\beta_3-\gamma)n\}-1}}$$

for a suitable polynomial $\widetilde{U}_n^{(2)}(z)$ of degree $< \max\{0,(\beta_1+\beta_2+\beta_3-\gamma)n\} - 1$.

Hence

$$-S_n^{(2)}(z) = W_n(z)\mathrm{Li}_2(1/z) - \big(U_n^{(2)}(z) + V_n^{(2)}(z)\big),$$

where $U_n^{(2)}(z) = (1-z)\widetilde{U}_n^{(2)}(z)$ and

$$V_n^{(2)}(z) = (1-z)^{\max\{0,(\beta_1+\beta_2+\beta_3-\gamma)n\}} \sum_{l=0}^{\gamma n} A_l \sum_{t=1}^{\delta n+l} \frac{z^{\delta n+l-t}}{t^2}.$$

As to $S_n^{(3)}(z)$, we have

$$S_n^{(3)}(z) = (1-z)^{\max\{0,(\beta_1+\beta_2+\beta_3-\gamma)n\}} \sum_{l=0}^{\gamma n} A_l \sum_{t\geqslant 1+\delta n} \frac{z^{\delta n - l}}{(t+l)^3}$$

$$+ z^{\delta n}(1-z)^{\max\{0,(\beta_1+\beta_2+\beta_3-\gamma)n\}} \sum_{t\geqslant 1+\delta n} \frac{1}{2} B''(t) z^{-t},$$

and, if $(\beta_1+\beta_2+\beta_3-\gamma)n > 2$,

$$-\sum_{t\geqslant 1+\delta n} B''(t) z^{-t} = \frac{\widetilde{U}_n^{(3)}(z)}{z^{\delta n}(1-z)^{\max\{0,(\beta_1+\beta_2+\beta_3-\gamma)n\}-2}}$$

for a suitable polynomial $\widetilde{U}_n^{(3)}(z)$ of degree $< (\beta_1+\beta_2+\beta_3-\gamma)n - 2$. It follows that

$$2S_n^{(3)}(z) = 2W_n(z)\mathrm{Li}_3(1/z) - \big(U_n^{(3)}(z) + 2V_n^{(3)}(z)\big),$$

with $U_n^{(3)}(z) = (1-z)^2 \widetilde{U}_n^{(3)}(z)$ and

$$V_n^{(3)}(z) = (1-z)^{\max\{0,(\beta_1+\beta_2+\beta_3-\gamma)n\}} \sum_{l=0}^{\gamma n} A_l \sum_{t=1}^{\delta n + l} \frac{z^{\delta n + l - t}}{t^3}.$$

Moreover,

$$W_n(z) \in \mathbb{Z}[z] \tag{1}$$

and

$$\deg W_n(z), \deg U_n^{(j)}(z), \deg V_n^{(j)}(z) \leqslant (\max\{\beta_1+\beta_2+\beta_3, \gamma\} + \delta)n - 1 \quad (j = 1,2,3).$$

3. Properties of binomial coefficients

In this section we derive the arithmetic properties of the polynomials $V_n^{(j)}(z)$ $(j = 1,2,3)$, and review and extend the methods in [13, Lemata 3,4 and 5] in order to apply them to the polynomials $U_n^{(j)}(z)$ $(j = 1,2,3)$.

In the sequel we denote by $[x]$ the integer part of the real number x, i. e. $[x] \in \mathbb{Z}$ and $[x] \leqslant x < [x]+1$.

LEMMA 3.1. *Let* $0 \leqslant l \leqslant m$ *be integers. Then*

$$D_m \binom{l+m}{m} \in D_{l+m}\mathbb{Z}.$$

PROOF. For a prime number p and a positive integer k, let $\nu_p(k)$ be the p–adic valuation of k. Then

$$\begin{aligned}\nu_p\left(\frac{D_{l+m}}{D_m}\right) &= \left[\frac{\log(l+m)}{\log p}\right] - \left[\frac{\log m}{\log p}\right] \\ &\leqslant \sum_{b=1}^{\infty}\left(\left[\frac{l+m}{p^b}\right] - \left[\frac{l}{p^b}\right] - \left[\frac{m}{p^b}\right]\right) = \nu_p\left(\frac{(l+m)!}{l!m!}\right),\end{aligned}$$

because $[u+v] \geqslant [u]+[v]$ for any u, v integers, and

$$\left[\frac{l+m}{p^b}\right] = 1 \text{ and } \left[\frac{l}{p^b}\right] = \left[\frac{m}{p^b}\right] = 0 \text{ for } b = \left[\frac{\log m}{\log p}\right] + 1, \dots, \left[\frac{\log(l+m)}{\log p}\right]. \square$$

By applying the above lemma we get

$$D_{\max\{\beta_j n, (\alpha_j - \beta_j)n + l\}} \binom{\alpha_j n + l}{\beta_j n} \in D_{\alpha_j n + l}\mathbb{Z} \qquad (l = 0, \dots, \gamma n; j = 1, 2, 3).$$

Since

$$\alpha_j n = a_4 - b_j \geqslant \max\{0, a_4 - a_1, a_4 - a_2, a_4 - a_3\} = \delta n,$$

then

$$\frac{1}{t} D_{\max\{\beta_j n, (\alpha_j - \beta_j)n + l\}} \binom{\alpha_j n + l}{\beta_j n} \in \mathbb{Z} \qquad (t = 1, \dots, \delta n + l; l = 1, \dots, \gamma n; j = 1, 2, 3).$$

In particular

$$D_{n\max\{\beta_1, \alpha_1+\gamma-\beta_1\}} D_{n\max\{\beta_2, \alpha_2+\gamma-\beta_2\}} D_{n\max\{\beta_3, \alpha_3+\gamma-\beta_3\}} V_n^{(j)}(z) \in \mathbb{Z}[z] \quad (j = 1, 2, 3). \tag{2}$$

We now review and extend the method of [13, Lemata 3,4 and 5].

LEMMA 3.2. *For any nonnegative integer j the following equality between two polynomials in x and y holds*:

$$\binom{x+y}{j} = \sum_{i=0}^{j} \binom{x}{i} \binom{y}{j-i}.$$

PROOF. This is the well-known Chu-Vandermonde identity. □

LEMMA 3.3. *Let $a > b$ be integers. Then the following equality between two polynomials in u and v holds*:

$$\binom{u+a-1}{a-b} - \binom{v+a-1}{a-b} = (u-v) \sum_{i=1}^{a-b} \frac{1}{i} \binom{u-v-1}{i-1} \binom{v+a-1}{a-b-i}.$$

PROOF. This follows from the Chu-Vandermonde identity, with $x = u - v$, $y = v + a - 1$ and $j = a - b$. □

The following identity is a reformulation of the above lemma:

$$\frac{R(a, b; u) - R(a, b; v)}{u - v} = \sum_{i=1}^{a-b} \frac{1}{i} R(i, 1; u - v - i) R(a, b + i; v).$$

LEMMA 3.4. *Let $a > b$ be integers. Then the following equality between two polynomials in u holds*:

$$\frac{\mathrm{d}}{\mathrm{d}u} \binom{u+a-1}{a-b} = \sum_{i=1}^{a-b} \frac{(-1)^{i-1}}{i} \binom{u+a-1}{a-b-i}.$$

PROOF. This follows from letting $v \to u$ in the above lemma. □

The interested reader can see [6, Lemma 3.1] for a similar expression for $\big(R(a, b; u)\big)'$ without alternation of the sign.

The next Lemma is [4, Lemma 2.2]. For completeness we repeat the proof.

LEMMA 3.5. *Let $1 \leqslant i < j \leqslant m$ be integers. Then*

$$D_m D_{[\frac{m}{2}]} \frac{1}{ij} \in \mathbb{Z}.$$

PROOF. If $i \leqslant m/2$ there is nothing to prove. If $i > m/2$, then $1 \leqslant j - i \leqslant m - i < m - m/2 = m/2$ and

$$\frac{1}{ij} = \frac{1}{j-i}\left(\frac{1}{i} - \frac{1}{j}\right). \qquad \square$$

LEMMA 3.6. *Let $1 \leqslant i, j \leqslant m$ be integers such that $i + j \leqslant m$. Then*

$$D_m D_{[\frac{m}{2}]} \frac{1}{ij} \in \mathbb{Z}.$$

PROOF. Either $i \leqslant m/2$, or $j \leqslant m/2$, for otherwise we would have $i + j > m/2 + m/2 = m$. □

LEMMA 3.7. *Let $1 \leqslant i, j < l \leqslant m$ be integers such that $i + j < l$. Then*

$$D_m D_{[\frac{m}{2}]} D_{[\frac{m}{3}]} \frac{1}{ijl} \in \mathbb{Z}.$$

PROOF. If $\min\{i, j\} \leqslant m/3$ we apply our lemma 3.5. If $\min\{i, j\} > m/3$, then $1 \leqslant l - (i + j) \leqslant m - (i + j) < m - (m/3 + m/3) = m/3$. After the decomposition

$$\frac{1}{ijl} = \frac{1}{(i+j)l}\left(\frac{1}{i} + \frac{1}{j}\right) = \frac{1}{l-i-j}\left(\frac{1}{i+j} - \frac{1}{l}\right)\left(\frac{1}{i} + \frac{1}{j}\right)$$

we again apply lemma 3.5. □

We now apply Lemata 3.3–3.7 to the polynomial $B(t)$ and its first and second derivatives. On multiplying

$$\frac{(\gamma n)!}{(t)_{\gamma n+1}} = \sum_{l=0}^{\gamma n} (-1)^l \binom{\gamma n}{l} \frac{1}{t+l},$$

by $C(t)$, we get

$$B(t) = \sum_{l=0}^{\gamma n} (-1)^l \binom{\gamma n}{l} \frac{C(t) - C(-l)}{t+l}, \tag{3}$$

where

$$C(t) = C_1(t)C_2(t)C_3(t), \qquad C_j(t) = \binom{t + (\beta_j - \alpha_j)n - 1}{\beta_j n} \quad (j = 1, 2, 3).$$

By writing

$$
\begin{aligned}
C_1(t)C_2(t)C_3(t) &- C_1(-l)C_2(-l)C_3(-l) \\
&= \big(C_1(t)-C_1(-l)\big)C_2(t)C_3(t) + C_1(-l)\big(C_2(t)-C_2(-l)\big)C_3(t) \\
&\qquad + C_1(-l)C_2(-l)\big(C_3(t)-C_3(-l)\big) \qquad (4)
\end{aligned}
$$

and by applying Lemma 3.3 we obtain

$$D_{n\max\{\beta_1,\beta_2,\beta_3\}}B(k)\in\mathbb{Z} \qquad \text{for any } k\in\mathbb{Z},$$

which confirms [13, Lemma 5].

Let $\beta_1^*,\beta_2^*,\beta_3^*$ be a reordering of β_1,β_2,β_3 such that $\beta_1^*\geqslant\beta_2^*\geqslant\beta_3^*$. Using the decompositions (3) and (4), Leibniz's formula for the derivative and lemata 3.3–3.5 we obtain

$$D_{\beta_1^* n}D_{\max\left\{\left[\frac{\beta_1^* n}{2}\right],\beta_2^* n\right\}}B'(k)\in\mathbb{Z} \qquad \text{for any } k\in\mathbb{Z},$$

whence

$$D_{\beta_1^* n}D_{\max\left\{\left[\frac{\beta_1^* n}{2}\right],\beta_2^* n\right\}}U_n^{(2)}(z)\in\mathbb{Z}[z].$$

Similarly, using the decompositions (3) and (4), Leibniz's formula for the second derivative and lemata 3.3–3.7 we obtain

$$D_{\beta_1^* n}D_{\max\left\{\left[\frac{\beta_1^* n}{2}\right],\beta_2^* n\right\}}D_{\max\left\{\left[\frac{\beta_1^* n}{3}\right],\beta_3^* n\right\}}B''(k)\in\mathbb{Z} \qquad \text{for any } k\in\mathbb{Z},$$

hence

$$D_{\beta_1^* n}D_{\max\left\{\left[\frac{\beta_1^* n}{2}\right],\beta_2^* n\right\}}D_{\max\left\{\left[\frac{\beta_1^* n}{3}\right],\beta_3^* n\right\}}U_n^{(3)}(z)\in\mathbb{Z}[z].$$

Let $(\alpha_1,\beta_1)^*,(\alpha_2,\beta_2)^*,(\alpha_3,\beta_3)^*$ be a reordering of $(\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_3,\beta_3)$ such that β_j^* is the second component of $(\alpha_j,\beta_j)^*$ $(j=1,2,3)$, and let $\alpha_1^*,\alpha_2^*,\alpha_3^*$ be a reordering of $\alpha_1,\alpha_2,\alpha_3$ such that α_j^* is the first component of $(\alpha_j,\beta_j)^*$ $(j=1,2,3)$. In other words, $\alpha_i=\alpha_j^*$ if and only if $\beta_i=\beta_j^*$. Then

$$
\begin{aligned}
D_{\max\{\beta_1^* n,(\alpha_1^*+\gamma-\beta_1^*)n\}}D_{\max\left\{\left[\frac{\beta_1^* n}{2}\right],\beta_2^* n,(\alpha_2^*+\gamma-\beta_2^*)n\right\}}D_{\max\left\{\left[\frac{\beta_1^* n}{3}\right],\beta_3^* n,(\alpha_3^*+\gamma-\beta_3^*)n\right\}}X_n^{(j)}(z)& \\
\in\mathbb{Z}[z], \qquad (X{=}U,V;j{=}1,2,3). \quad (5)&
\end{aligned}
$$

4. Refined arithmetic

From now on we suppose that

$$b_1, b_2, b_3 < a_1, a_2, a_3, a_4 < b_4 - 1.$$

Hence $n = 1, 2, 3, \ldots$, and $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma$ are positive integers such that

$$\alpha_i + \gamma - \beta_i, \alpha_i + \beta_j - \alpha_j > 0 \qquad (i, j = 1, 2, 3).$$

Let $\sigma \in \mathfrak{S}_4$ be an element of the symmetric group $\mathfrak{S}_4$, i. e. a permutation of $1, 2, 3, 4$. Let

$$\widehat{R}_\sigma(t) = \prod_{i=1}^{4} R(a_{\sigma(i)}, b_i; t).$$

Then

$$\widehat{R}(t) = \frac{(a_{\sigma(1)} - b_1)!(a_{\sigma(2)} - b_2)!(a_{\sigma(3)} - b_3)!(b_4 - a_4 - 1)!}{(a_1 - b_1)!(a_2 - b_2)!(a_3 - b_3)!(b_4 - a_{\sigma(4)} - 1)!} \widehat{R}_\sigma(t). \tag{6}$$

We remark that $A_l = 0$ for $l < 0$ or $l > \gamma n$, whence

$$\widehat{R}(t - a_4) = \sum_{l=-\infty}^{+\infty} \frac{A_l}{t + l} + B(t).$$

For any $\sigma \in \mathfrak{S}_4$ we also introduce the coefficients $A_{l,\sigma}$ and the polynomial $B_\sigma(t)$ by the relation

$$\widehat{R}_\sigma(t - a_{\sigma(4)}) = \sum_{l=-\infty}^{+\infty} \frac{A_{l,\sigma}}{t + l} + B_\sigma(t).$$

For any σ, there are only finitely many l such that $A_{l,\sigma} \neq 0$. Moreover, $A_{l,\sigma} \in \mathbb{Z}$ for any l and any σ. By the uniqueness of the decomposition in partial fractions and the relation (6) we get

$$A_{l-a_4} = \frac{(a_{\sigma(1)} - b_1)!(a_{\sigma(2)} - b_2)!(a_{\sigma(3)} - b_3)!(b_4 - a_4 - 1)!}{(a_1 - b_1)!(a_2 - b_2)!(a_3 - b_3)!(b_4 - a_{\sigma(4)} - 1)!} A_{l-a_{\sigma(4)},\sigma} \tag{7}$$

and

$$B(t+a_4)=\frac{(a_{\sigma(1)}-b_1)!(a_{\sigma(2)}-b_2)!(a_{\sigma(3)}-b_3)!(b_4-a_4-1)!}{(a_1-b_1)!(a_2-b_2)!(a_3-b_3)!(b_4-a_{\sigma(4)}-1)!}B_\sigma(t+a_{\sigma(4)}). \quad (8)$$

We rewrite the series $S_n^{(j)}(z)$ $(j=1,2,3)$ as follows:

$$S_n^{(j)}(z)=z^{\delta n}(1-z)^{\max\{0,(\beta_1+\beta_2+\beta_3-\gamma)n\}}\sum_{t\geqslant 1+\delta n}\frac{1}{(j-1)!}\frac{\mathrm{d}^{j-1}}{\mathrm{d}t^{j-1}}\widehat{R}(t-a_4)z^{-t}.$$

By changing the index t into $\widetilde{t}=t-a_4$ we also have

$$S_n^{(j)}(z)=z^{\max\{-a_1,-a_2,-a_3,-a_4\}}(1-z)^{\max\{0,d+1\}}\sum_{\widetilde{t}\geqslant 1+\max\{-a_1,-a_2,-a_3,-a_4\}}\frac{1}{(j-1)!}\frac{\mathrm{d}^{j-1}}{\mathrm{d}\widetilde{t}^{j-1}}\widehat{R}(\widetilde{t})z^{-\widetilde{t}}.$$

Now the exponents of z and $1-z$, as well as the range of the summations, are symmetric in a_1,a_2,a_3,a_4. The same remark also applies to the polynomials $W_n(z)$ and $X_n^{(j)}(z)$ $(X=U,V;j=1,2,3)$. For example:

$$W_n(z)=z^{\max\{-a_1,-a_2,-a_3,-a_4\}}(1-z)^{\max\{0,d+1\}}\sum_{l=-\infty}^{+\infty}A_{l-a_4}z^l;$$

$$U_n^{(1)}(z)=-z^{\max\{-a_1,-a_2,-a_3,-a_4\}}(1-z)^{\max\{0,d+1\}}\sum_{t\geqslant 1+\max\{-a_1,-a_2,-a_3,-a_4\}}B(t+a_4)z^{-t};$$

and so on. For any $\sigma\in\mathfrak{S}_4$ we introduce the parameters $\alpha_{1,\sigma},\alpha_{2,\sigma},\alpha_{3,\sigma},\beta_{1,\sigma},\beta_{2,\sigma},\beta_{3,\sigma},\gamma_\sigma$ analogue to $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma$:

$$\gamma_\sigma n+1=b_4-a_{\sigma(4)},\quad \alpha_{j,\sigma}n=a_{\sigma(4)}-b_j,\quad \beta_{j,\sigma}n=a_{\sigma(j)}-b_j \qquad (j=1,2,3)$$

and the polynomials $W_{n,\sigma}(z)$ and $X_{n,\sigma}^{(j)}(z)$ $(X=U,V;j=1,2,3)$ associated to the rational function $\widehat{R}_\sigma(t)$ as we did in section 3 for $\widehat{R}(t)$. By (7) and (8),

$$W_n(z)=\frac{(\beta_{1,\sigma}n)!(\beta_{2,\sigma}n)!(\beta_{3,\sigma}n)!(\gamma n)!}{(\beta_1 n)!(\beta_2 n)!(\beta_3 n)!(\gamma_\sigma n)!}W_{n,\sigma}(z)$$

and

$$X_n^{(j)}(z)=\frac{(\beta_{1,\sigma}n)!(\beta_{2,\sigma}n)!(\beta_{3,\sigma}n)!(\gamma n)!}{(\beta_1 n)!(\beta_2 n)!(\beta_3 n)!(\gamma_\sigma n)!}X_{n,\sigma}^{(j)}(z) \qquad (X=U,V;j=1,2,3).$$

Let $\beta^*_{1,\sigma}, \beta^*_{2,\sigma}, \beta^*_{3,\sigma}$ be a reordering of $\beta_{1,\sigma}, \beta_{2,\sigma}, \beta_{3,\sigma}$ such that $\beta^*_{1,\sigma} \geqslant \beta^*_{2,\sigma} \geqslant \beta^*_{3,\sigma}$; let $\alpha^*_{1,\sigma}, \alpha^*_{2,\sigma}, \alpha^*_{3,\sigma}$ be the corresponding reordering of $\alpha_{1,\sigma}, \alpha_{2,\sigma}, \alpha_{3,\sigma}$. By applying the argument of section 3 we get $W_{n,\sigma}(z) \in \mathbb{Z}[z]$ and

$$\begin{aligned} &D_{\max\{n\beta^*_{1,\sigma}, n(\alpha^*_{1,\sigma}+\gamma-\beta^*_{1,\sigma})\}} D_{\max\left\{\left[\frac{n\beta^*_{1,\sigma}}{2}\right], n\beta^*_{1,\sigma}, n(\alpha^*_{2,\sigma}+\gamma-\beta^*_{2,\sigma})\right\}} \\ &\quad \times D_{\max\left\{\left[\frac{n\beta^*_{1,\sigma}}{3}\right], n\beta^*_{3,\sigma}, n(\alpha^*_{3,\sigma}+\gamma-\beta^*_{3,\sigma})\right\}} X^{(j)}_{n,\sigma}(z) \in \mathbb{Z}[z] \qquad (X = U, V;\, j = 1, 2, 3). \end{aligned}$$

Let

$$\theta = \max_{\sigma \in \mathfrak{S}_4} \max\{\gamma_\sigma, \beta_{1,\sigma}, \beta_{2,\sigma}, \beta_{3,\sigma}\}.$$

For any $\omega \in [0, 1)$ let

$$\mu(\omega) = \max_{\sigma \in \mathfrak{S}_4} \big([\beta_{1,\sigma}\omega]+[\beta_{2,\sigma}\omega]+[\beta_{3,\sigma}\omega]-[\gamma_\sigma\omega]\big)-[\beta_1\omega]-[\beta_2\omega]-[\beta_3\omega]+[\gamma\omega] \geqslant 0.$$

Clearly, $\mu(\omega) = 0$ for $\omega < 1/\theta$ and for $\omega \geqslant 1 - 1/\theta$. Let us assume $\theta \geqslant 3$.

Let $\{x\} = x - [x]$ be the fractional part of a real number x. For any integer $n \geqslant 0$ let

$$\Delta_n = \prod_{\substack{\omega=\frac{n}{p},\ p \text{ prime} \\ \sqrt{\theta n}<p\leqslant \varrho n}} p^{\mu(\omega)} \in \mathbb{Z},$$

where

$$\varrho=\min\left\{\max\{\beta^*_1,\alpha^*_1+\gamma-\beta^*_1\},\max\left\{\frac{\beta^*_1}{2},\beta^*_2,\alpha^*_2+\gamma-\beta^*_2\right\},\max\left\{\frac{\beta^*_1}{3},\beta^*_3,\alpha^*_3+\gamma-\beta^*_3\right\}\right\}$$

We get the following refinements of (1), (2) and (5):

$$\Delta_n^{-1} W_n(z) \in \mathbb{Z}[z],$$

$$\begin{aligned} &\Delta_n^{-1} D_{\max\{\beta_1 n, (\alpha_1+\gamma-\beta_1)n\}} D_{\max\{\beta_2 n, (\alpha_2+\gamma-\beta_2)n\}} \\ &\qquad\qquad \times D_{\max\{\beta_3 n, (\alpha_3+\gamma-\beta_3)n\}} V^{(j)}_n(z) \in \mathbb{Z}[z] (j = 1, 2, 3), \end{aligned}$$

$$\begin{aligned} &\Delta_n^{-1} D_{\max\{\beta^*_1 n, (\alpha^*_1+\gamma-\beta^*_1)n\}} D_{\max\left\{\left[\frac{\beta^*_1 n}{2}\right], \beta^*_2 n, (\alpha^*_2+\gamma-\beta^*_2)n\right\}} \\ &\qquad \times D_{\max\left\{\left[\frac{\beta^*_1 n}{3}\right], \beta^*_3 n, (\alpha^*_3+\gamma-\beta^*_3)n\right\}} X^{(j)}_n(z) \in \mathbb{Z}[z] \qquad (X=U,V; j=1,2,3). \end{aligned}$$

Moreover, $[0,1)=\cup_{i=1}^{m}[u_i,u_{i+1})$ for suitable rational numbers

$$0=u_1<u_2=\frac{1}{\theta}<\cdots<u_m=1-\frac{1}{\theta}<u_{m+1}=1$$

such that $\mu(\omega)$ is constant on any interval $[u_i,u_{i+1})$, and we may assume that the sequence $u_2,\dots,u_{m+1}$ includes $u_{l+1}=1/\varrho$ for some $l=1,\dots,m-2$. By a standard lemma (see e. g. [9, Lemma 6] and [13, Lemma 7])

$$\Delta=\lim_{n\to\infty}\frac{1}{n}\log\Delta_n=\sum_{i=2}^{m-1}\mu(u_i)\big(\psi(u_{i+1})-\psi(u_i)\big)-\sum_{i=2}^{l}\mu(u_i)\left(\frac{1}{u_i}-\frac{1}{u_{i+1}}\right),$$

where $\psi(x)=\Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of Euler's gamma function.

Using the Prime Number Theorem we also have

$$\lim_{n\to\infty}\frac{1}{n}\log\left(D_{\max\{\beta_1^*n,(\alpha_1^*+\gamma-\beta_1^*)n\}}D_{\max\left\{\left[\frac{\beta_1^*n}{2}\right],\beta_2^*n,(\alpha_2^*+\gamma-\beta_2^*)n\right\}}D_{\max\left\{\left[\frac{\beta_1^*n}{3}\right],\beta_3^*n,(\alpha_3^*+\gamma-\beta_3^*)n\right\}}\right)$$
$$=\max\{\beta_1^*,\alpha_1^*+\gamma-\beta_1^*\}+\max\left\{\left[\frac{\beta_1^*}{2}\right],\beta_2^*,\alpha_2^*+\gamma-\beta_2^*\right\}$$
$$+\max\left\{\left[\frac{\beta_1^*}{3}\right],\beta_3^*,\alpha_3^*+\gamma-\beta_3^*\right\}.$$

5. Saddle point method

We recall from section 2 that the rational function $\widehat{R}(t-a_4)$ vanishes with multiplicity 3 for $t=\delta n+1,\dots,\min\{\alpha_1,\alpha_2,\alpha_3\}n$. Let τ_0 be a real number such that $\delta<\tau_0<\min\{\alpha_1,\alpha_2,\alpha_3\}$. We introduce three integrals:

$$I_n^{(j)}(z)=\frac{z^{\delta n}(1-z)^{\max\{0,(\beta_1+\beta_2+\beta_3-\gamma)n\}}}{2\pi i}\int_{t_0-i\infty}^{t_0+i\infty}\widehat{R}(t-a_4)\chi_j(t)z^{-t}\,dt\quad(j=1,2,3),$$

where

$$\chi_j(t)=\frac{\pi^j\cos\pi t}{\sin^j\pi t}\ \text{ for } j=1,3,\qquad \chi_2(t)=\frac{\pi^2}{\sin^2\pi t}.$$

The integration domain, i. e. the vertical line $\operatorname{Re} t = t_0 = \tau_0 n$, separates the decreasing and the increasing sequences of poles of the integrand

$$\widehat{R}(t-a_4)\chi_j(t)z^{-t} \quad (j=1,2,3). \tag{9}$$

By a standard argument (see e. g. [9, Proposition 1]), after remarking that for $|z|>1$ the absolute value of (9) is small on a large half-circle $\{t_0+\rho e^{i\pi\theta} : -\pi/2 \leqslant \theta \leqslant \pi/2\}$, by the residue theorem we get

$$I_n^{(j)}(z) = z^{\delta n}(1-z)^{\max\{0,(\beta_1+\beta_2+\beta_3-\gamma)n\}} \sum_{k=\delta n+1}^{\infty} \operatorname*{Res}_{t=k} \left(\widehat{R}(t-a_4)\chi_j(t)z^{-t}\right). \tag{10}$$

It's easy to see that for any $m \in \mathbb{Z}$ we have, for $t \to m$,

$$\chi_j(t) = \frac{1}{(t-m)^j} + O(1) \quad (j=1,2,3),$$

$$\widehat{R}(t) = \widehat{R}(m) + \widehat{R}'(m)(t-m) + \frac{1}{2}\widehat{R}''(t-m)^2 + O\big((t-m)^3\big),$$

$$z^{-t} = z^{-m}\Big(1-(t-m)\log z + \frac{1}{2}(t-m)^2\log^2 z + O\big((t-m)^3\big)\Big).$$

Hence (for $t \to m$)

$$\widehat{R}(t-a_4)\chi_1(t)z^{-t} = \widehat{R}(m-a_4)\,\frac{z^{-m}}{t-m} + O(1),$$

$$\widehat{R}(t-a_4)\chi_2(t)z^{-t} = \widehat{R}(m-a_4)\frac{z^{-m}}{(t-m)^2} + \big(\widehat{R}'(m-a_4) - \widehat{R}(m-a_4)\log z\big)\frac{z^{-m}}{t-m} + O(1),$$

$$\begin{aligned}\widehat{R}(t-a_4)\chi_3(t)z^{-t} &= \widehat{R}(m-a_4)\,\frac{z^{-m}}{(t-m)^3} + \big(\widehat{R}'(m-a_4) - \widehat{R}(m-a_4)\log z\big)\frac{z^{-m}}{(t-m)^2}\\ &\quad + \Big(\frac{1}{2}\widehat{R}''(m-a_4) - \widehat{R}'(m-a_4)\log z + \frac{1}{2}\log^2 z\Big)\frac{z^{-m}}{t-m} + O(1).\end{aligned}$$

From (10) we get

$$\begin{aligned} I_n^{(1)}(z) &= S_n^{(1)}(z),\\ I_n^{(2)}(z) &= S_n^{(2)}(z) - S_n^{(1)}(z)\log z,\\ I_n^{(3)}(z) &= \frac{1}{2}S_n^{(3)}(z) - S_n^{(2)}(z)\log z + \frac{1}{2}S_n^{(1)}(z)\log^2 z. \end{aligned}$$

Moreover, by the residue theorem

$$W_n(z) = \frac{z^{\delta n}(1-z)^{\max\{0,(\beta_1+\beta_2+\beta_3-\gamma)n\}}}{2\pi i}\oint_{\Omega} \widehat{R}(t-a_4)z^{-t}dt,$$

where Ω is a contour enclosing the poles $t = 0, 1, \dots, \gamma n$ of $\widehat{R}(t-a_4)$.

Similarly to [9, Propositions 2 and 3] and [13, Lemma 6]), the upper bounds for

$$\limsup_{n\to\infty} \frac{\log|I_n^{(j)}(z)|}{n} \quad (j = 1, 2, 3)$$

and the value of

$$\lim_{n\to\infty} \frac{\log W_n(z)}{n}$$

can be determined through the roots of the algebraic equation of degree four

$$(\tau+\beta_1-\alpha_1)(\tau+\beta_2-\alpha_2)(\tau+\beta_3-\alpha_3)\tau-(\tau-\alpha_1)(\tau-\alpha_2)(\tau-\alpha_3)(\tau+\gamma)z = 0. \quad (11)$$

In all the numerical examples given in the last section of the present paper we make

$$\alpha_1 < \beta_1 = \alpha_2 < \beta_2 < \alpha_3 = \beta_3 \text{ and } \gamma = \beta_1 + \beta_2 + \beta_3. \quad (12)$$

5.1. The case $z > 0$

If $z > 0$, then the equation (11) has four real roots $\tau_1, \tau_2, \tau_3, \tau_4$ satisfying $\tau_4 < -\gamma < \alpha_1 < \tau_1 < \tau_2 < \alpha_2 < \alpha_3 < \tau_3$. The condition $\tau_0 < \alpha_1$ indicates that here the vertical line $\operatorname{Re} t = t_0 = \tau_0 n$ is not admissible for the application of the saddle point method. We need to modify the half-line $\operatorname{Re}\tau = \tau_0$ in the τ-half-plane $\operatorname{Im}\tau > 0$ near the real line, by taking a curve from a suitable τ_0 close to α_1, tangent to the real line at the point τ_1 and remaining on the half-plane $\operatorname{Re}\tau > 0$ except for the point τ_1. The same modification is due in the τ-half-plane $\operatorname{Im}\tau < 0$.

Since τ_1 is irrational, by the residue theorem the value of the integrals $I_n^{(j)}(z)$ remains unchanged. By straightforward computations one can verify a curve exists such that the function Re $f(\tau)$ has a unique maximum at the point τ_1 on the curve from τ_0 to $i\infty$ (resp. from τ_0 to $-i\infty$), where

$$\begin{aligned} f(\tau) &= (\tau+\beta_1-\alpha_1)\log(\tau+\beta_1-\alpha_1) + (\tau+\beta_2-\alpha_2)\log(\tau+\beta_2-\alpha_2) \\ &\quad + (\tau+\beta_3-\alpha_3)\log(\tau+\beta_3-\alpha_3) + \tau\log\tau - (\tau-\alpha_1)\log(\tau-\alpha_1) \\ &\quad - (\tau-\alpha_2)\log(-\tau+\alpha_2) - (\tau-\alpha_3)\log(-\tau+\alpha_3) - (\tau+\gamma)\log(\tau+\gamma) \\ &\quad - \beta_1\log\beta_1 - \beta_2\log\beta_2 - \beta_3\log\beta_3 + \gamma\log\gamma - \tau\log z. \end{aligned} \tag{13}$$

In our numerical examples, $f(\tau_2) < f(\tau_3) < f(\tau_1)$ and $f(\tau_3)$ is very close to $f(\tau_1)$, and the curve above must cross the vertical line $\operatorname{Re}\tau = \tau_3$ very closely to the point τ_3.

In the integrals $I_n^{(1)}(z)$ and $I_n^{(2)}(z)$ we may relax the condition $\tau_0 < \alpha_1$ to $\tau_0 < \alpha_2$. In those cases we can modify the line $\operatorname{Re}\tau = \tau_0$, with τ_0 close to α_2, by taking in the half plane $\operatorname{Im}\tau > 0$ a curve from τ_0, tangent to the real line in the point τ_3, and symmetrically in the half-plane $\operatorname{Im}\tau < 0$.

One can also choose a contour Ω enclosing the poles that $t = 0, 1, \dots, \gamma n$ of $\widehat{R}(t-a_4)$, containing the point τ_4 and such that the function Re $f(\tau)$ has a unique maximum in $\tau = \tau_4$.

By the saddle point method we get

$$\limsup_{n\to\infty} \frac{\log|I_n^{(j)}(z)|}{n} \leqslant f(\tau_1) = \max\{f(\tau_1), f(\tau_2), f(\tau_3)\} \quad (j = 1, 2, 3) \tag{14}$$

and

$$\lim_{n\to\infty} \frac{\log W_n^{(j)}(z)}{n} = f(\tau_4). \tag{15}$$

5.2. The case $z < 0$

In all the numerical examples given in the last section of the present paper, if $z < 0$ the equation (11) has two real roots τ_1, τ_4 with $-\gamma < \tau_4 < 0 < \tau_1 < \alpha_1$, and two complex conjugate roots τ_2 and $\overline{\tau_2}$ with $\beta_2 < \operatorname{Re}\tau_2 < \alpha_3$. In this case the line $\operatorname{Re}\tau = \tau_1$ is admissible in the application of the saddle point method for $I_n^{(j)}(z)$ $(j = 1, 2, 3)$, and the function $f(\tau)$ is now defined by

$$f(\tau) = (\tau + \beta_1 - \alpha_1)\log(\tau + \beta_1 - \alpha_1) + (\tau + \beta_2 - \alpha_2)\log(\tau + \beta_2 - \alpha_2)$$
$$+ (\tau + \beta_3 - \alpha_3)\log(\tau + \beta_3 - \alpha_3) + \tau\log\tau - (\tau - \alpha_1)\log(-\tau + \alpha_1)$$
$$- (\tau - \alpha_2)\log(-\tau + \alpha_2) - (\tau - \alpha_3)\log(-\tau + \alpha_3) - (\tau + \gamma)\log(\tau + \gamma)$$
$$- \beta_1\log\beta_1 - \beta_2\log\beta_2 - \beta_3\log\beta_3 + \gamma\log\gamma - \tau\log(-z).$$

The formulas (14) and (15) still hold.

6. A non-vanishing determinant

From now on, we suppose that $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma$ satisfy the conditions (12). In addition, we assume that

$$\beta_1 - \alpha_1 > \beta_2 - \alpha_2.$$

It is easy to check that $\deg W_n(z) = \gamma n$, $\operatorname{ord}_{z=0} W_n(z) \geqslant (\beta_1 - \alpha_1)n$, $\operatorname{ord}_{z=0} V_n^{(1)}(z) \geqslant (\beta_2 - \alpha_2)n$, $S_n^{(j)}(z) = O(|z|^{-\alpha_j n - 1})$ $(|z| \to \infty; j = 1, 2, 3)$.

Besides the polynomial $W_{n,0}(z) = W_n(z)$ and the three series $S_{n,0}^{(j)}(z) = S_n^{(j)}(z)$ $(j = 1, 2, 3)$ associated to the rational function

$$\widehat{R}_0(t - a_4) = \widehat{R}(t - a_4) = \frac{(t - \alpha_1 n)_{\beta_1 n}}{(\beta_1 n)!}\frac{(t - \alpha_2 n)_{\beta_2 n}}{(\beta_2 n)!}\frac{(t - \alpha_3 n)_{\beta_3 n}}{(\beta_3 n)!}\frac{(\gamma n)!}{(t)_{\gamma n + 1}},$$

we introduce three more polynomials $W_{n,h}(z)$ and nine more series $S_{n,h}^{(j)}(z)$ $(h, j = 1, 2, 3)$, associated to the rational functions $\widehat{R}_h(t - a_4)$ defined by $\widehat{R}_1(t - a_4) = \frac{t - \alpha_3 n - 1}{\beta_3 n + 1}\widehat{R}_0(t - a_4)$, $\widehat{R}_2(t - a_4) = \frac{t - \alpha_2 n - 1}{\beta_2 n + 1}\widehat{R}_1(t - a_4)$ and $\widehat{R}_3(t - a_4) = \frac{t - \alpha_1 n - 1}{\beta_1 n + 1}\widehat{R}_2(t - a_4)$. By a standard argument (see e. g. [5, Lemma 4]), we have

$$z^{(\alpha_1 + \alpha_2 - \beta_1 - \beta_2)n} \det\begin{pmatrix} W_{n,0}(z) & S_{n,0}^{(1)}(z) & S_{n,0}^{(2)}(z) & S_{n,0}^{(3)}(z) \\ W_{n,1}(z) & S_{n,1}^{(1)}(z) & S_{n,1}^{(2)}(z) & S_{n,1}^{(3)}(z) \\ W_{n,2}(z) & S_{n,2}^{(1)}(z) & S_{n,2}^{(2)}(z) & S_{n,2}^{(3)}(z) \\ W_{n,3}(z) & S_{n,3}^{(1)}(z) & S_{n,3}^{(2)}(z) & S_{n,3}^{(3)}(z) \end{pmatrix} \equiv \text{const.} \neq 0.$$

Similarly to $S^{(j)}_{n,h}(z)$ we may introduce the corresponding integrals $I^{(j)}_{n,h}(z)$ $(h = 0,\dots,3; j = 1,2,3)$, and the relations (14) and (15) hold for $I^{(j)}_{n,h}(z)$ and $W_{n,h}(z)$ as well.

7. Linear independence of -, di- and trilogarithms

Along with the series $S^{(j)}_{n,h}(z)$ $(h = 0,\dots,3; j = 1,2,3)$, we introduce the polynomials $V^{(j)}_{n,h}(z)$ $(h, j = 1,2,3)$, and the integers $\Delta_{n,h}$ $(h = 0,\dots,3)$ as in section 4. Note that by (12) we have $\delta = 0$ and $U^{(j)}_n(z) \equiv 0$ $(j = 1,2,3)$. By section 6 we have

$$z^{(\alpha_1+\alpha_2-\beta_1-\beta_2)n} \det \begin{pmatrix} W_{n,0}(z) & V^{(1)}_{n,0}(z) & V^{(2)}_{n,0}(z) & V^{(3)}_{n,0}(z) \\ W_{n,1}(z) & V^{(1)}_{n,1}(z) & V^{(2)}_{n,1}(z) & V^{(3)}_{n,1}(z) \\ W_{n,2}(z) & V^{(1)}_{n,2}(z) & V^{(2)}_{n,2}(z) & V^{(3)}_{n,2}(z) \\ W_{n,3}(z) & V^{(1)}_{n,3}(z) & V^{(2)}_{n,3}(z) & V^{(3)}_{n,3}(z) \end{pmatrix} \equiv \text{const.} \neq 0.$$

In particular, for any $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}$ not all zero, for any $z \neq 0$ and for any $n \in \mathbb{N}!$, at least one among the four quantities $\lambda_0 W_{n,h}(z) + \lambda_1 V^{(1)}_{n,h}(z) + \lambda_2 V^{(2)}_{n,h}(z) + \lambda_3 V^{(3)}_{n,h}(z)$ $(h = 0,\dots,3)$ is non-zero. So we have defined a sequence $\{h_n\}$ depending on $\lambda_0, \lambda_1, \lambda_2, \lambda_3$, and therefore a sequence of integers $\{4n + h_n\}$ $(n \in \mathbb{N})$. One of the seven integers $m, m+1, \dots, m+6$ belongs to the sequence $\{4n + h_n\}$.

We introduce the notation $\widetilde{W}_{4n+h}(z) = W_{n,h}(z)$ and $\widetilde{V}^{(j)}_{4n+h}(z) = V^{(j)}_{n,h}(z)$ $(h = 0,\dots,3; j = 1,2,3)$. For any $m \in \mathbb{N}$, at least

We have

$$\Delta^{-1}_{n,h} W_{n,h}(z) \in \mathbb{Z}[z] \qquad (h = 0,\dots,3)$$

and

$$\Delta^{-1}_{n,h} D_{\max\{\beta_1 n+1,(\alpha_1+\gamma-\beta_1)n\}} D_{\max\{\beta_2 n+1,(\alpha_2+\gamma-\beta_2)n\}} D_{\max\{\beta_3 n+1,(\alpha_3+\gamma-\beta_3)n\}} \times V^{(j)}_{n,h}(z) \in \mathbb{Z}[z] (h=0,\dots,3;\ j=1,2,3),$$

Let $z = a/b$ be a rational number, with $a \in \mathbb{Z}$, $b \in \mathbb{N}$. We may apply [4, Lemma 6.1] to the sequence

$$b^{\gamma n+h} \Delta^{-1}_{n,h} D_{\max\{\beta_1 n+1,(\alpha_1+\gamma-\beta_1)n\}} D_{\max\{\beta_2 n+1,(\alpha_2+\gamma-\beta_2)n\}} D_{\max\{\beta_3 n+1,(\alpha_3+\gamma-\beta_3)n\}}$$

$$\times \left(\widetilde{W}_m(z), \widetilde{V}_m^{(1)}(z), \widetilde{V}_m^{(2)}(z), \widetilde{V}_m^{(3)}(z)\right) \quad (m = 0, 1, 2, \ldots), \quad (16)$$

with the convention that $n = [\frac{m}{4}]$ and $h = m - 4n$. It is convenient to repeat Hata's Lemma here.

LEMMA 7.1. *Let M be a positive integer, let $\gamma_1, \ldots, \gamma_M$ be real numbers, and let d be a positive real number. Let $\{q_n\}, \{p_{1,n}\}, \ldots, \{p_{M,n}\}$ be sequences in $\mathbb{Z} + id\mathbb{Z}$ satisfying $q_n \neq 0$ for all n, and*

$$\limsup_{n\to\infty} \frac{1}{n} \log |q_n| \leqslant \sigma, \quad \limsup_{n\to\infty} \frac{1}{n} \log |q_n \gamma_j - p_{j,n}| \leqslant -\tau \quad (j = 1, \ldots, M)$$

for suitable positive real numbers τ and σ. Suppose further that there exists a positive integer N such that

$$\sum_{k=0}^{N} \left|\lambda_0 q_{n+k} + \lambda_1 p_{1,n+k} + \cdots + \lambda_M p_{M,n+k}\right| > 0$$

for any $n \in \mathbb{N}$ and for any $\lambda_0, \lambda_1, \ldots, \lambda_M \in \mathbb{Z}$ not all zero.

Then for any $\varepsilon > 0$ there exists an effectively computable constant $H = H(\varepsilon)$ such that for any $w, v_1, \ldots, v_M \in \mathbb{Z}$ with $\max\{|v_1|, \ldots, |v_M|\} \geqslant H$ we have

$$|w + v_1\gamma_1 + \cdots + v_M\gamma_M| > H^{-\frac{\sigma}{\tau} - \varepsilon}.$$

With $M = 3$ and $N = 6$ we obtain the following

THEOREM 7.1. *With the notation above, if*

$$\gamma \log b + \max\{\beta_1, \alpha_1 + \gamma - \beta_1\} + \max\{\beta_2, \alpha_2 + \gamma - \beta_2\} + \max\{\beta_3, \alpha_3 + \gamma - \beta_3\} - \Delta + f(\tau_1) < 0,$$

then the four numbers $1, \mathrm{Li}_1(b/a), \mathrm{Li}_2(b/a), \mathrm{Li}_3(b/a)$ are linearly independent over $\mathbb{Q}$, and for any $\varepsilon > 0$ there exists an effectively computable constant $H = H(\varepsilon)$ such that for any $w, v_1, v_2, v_3 \in \mathbb{Z}$ with $\max\{|v_1|, |v_2|, |v_3|\} \geqslant H$ we have

$$|w + v_1\mathrm{Li}_1(b/a) + v_2\mathrm{Li}_2(b/a) + v_3\mathrm{Li}_3(b/a)| \geqslant H^{-\mu-\varepsilon},$$

where μ equals

$$\frac{f(\tau_4) - f(\tau_1)}{\Delta - \gamma\log b - \max\{\beta_1, \alpha_1 + \gamma - \beta_1\} - \max\{\beta_2, \alpha_2 + \gamma - \beta_2\} - \max\{\beta_3, \alpha_3 + \gamma - \beta_3\} - f(\tau_1)}.$$

Alternatively, we could adapt the argument used in [5].

In the table below we give a few numerical applications of the above theorem. Remarkably, all the values of the parameters giving the optimal values of μ satisfy

$$\alpha_2 = \beta_1 = \alpha_1 + 4k, \beta_2 = \alpha_1 + 5k, \alpha_3 = \beta_3 = \alpha_1 + 7k, \gamma = 3\alpha_1 + 16k$$

for some $k \in \mathbb{N}$. For comparison, we note that Hata's method (see [2, Corollary 2.2]) implies the linear independence of $1, \mathrm{Li}_1(1/a), \mathrm{Li}_2(1/a), \mathrm{Li}_3(1/a)$ only for any $a \in \mathbb{Z}$ with $|a| \geqslant 1038$. For fixed α_1 and k, the measure μ is a decreasing function of a if $a > 0$, and is an increasing function of a if $a < 0$. In particular we obtain that $1, \mathrm{Li}_1(1/a), \mathrm{Li}_2(1/a), \mathrm{Li}_3(1/a)$ are linearly independent over $\mathbb{Q}$ for any $a \geqslant 486$ and for any $a \leqslant -471$.

α_1	β_1	α_2	β_2	α_3	β_3	γ	a	μ
144	172	172	179	193	193	544	486	33634.75742462...
597	713	713	742	800	800	2255	487	12033.98026474...
474	566	566	589	635	635	1790	488	7317.00150985...
165	197	197	205	221	221	623	489	5233.10245543...
289	345	345	359	387	387	1091	490	4096.44058030...
333	397	397	413	445	445	1255	500	1283.33382990...
167	195	195	202	216	216	613	700	105.55027709...
477	565	565	587	631	631	1783	−471	15691.27259092...
415	491	491	510	548	548	1549	−480	1760.88115153...
307	363	363	377	405	405	1145	−485	1187.90395198...
22	26	26	27	29	29	82	−490	903.96836632...
111	131	131	136	146	146	413	−500	607.15490293...
194	222	222	229	243	243	694	−900	62.18233153...

8. Linear independence of logarithms and dilogarithms

Here we briefly review the results found in [10] by Rhin and Viola. The authors introduce three families of double integrals and construct simultaneous approximations to $\mathrm{Li}_1(1/z)$ and $\mathrm{Li}_2(1/z)$. By applying their permutation group method, Rhin and Viola obtain the new result $\mathrm{Li}_2(1/6) \notin \mathbb{Q}$, and also new irrationality measures of $\mathrm{Li}_2(r/s)$ for suitable positive rational numbers r/s.

By the special case $m = 3$, $r = 2$ of [8, Theorem 2], for $|z| > 1$ the double integral

$$I_z^{(0)}(h, j, k, l, m) = z^{-l-m} \int_0^1 \int_0^1 \frac{x^j(1-x)^h y^k (1-y)^l}{(x(1-y) + yz)^{j+k-m+1}}$$

in [10] equals

$$\sum_{t \geqslant 1} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{(t-j)_{j+k-m}}{(j+k-m)!} \frac{(t-m)_{l+m-j}}{(l+m-j)!} \frac{h!}{(t)_{h+1}} z^{-t} \right),$$

provided that $j + k - m, l + m - j \geqslant 0$.

By analogy with our construction in section 2, one may introduce two series $(j = 1, 2)$

$$S_n^{(j)}(z) = z^{\delta n}(1-z)^{\max\{0,(\beta_1+\beta_2-\gamma)n\}} \sum_{t \geqslant 1+\delta n} \frac{1}{(j-1)!} \frac{\mathrm{d}^{j-1}}{\mathrm{d}t^{j-1}} \left(\frac{(t-\alpha_1 n)_{\beta_1 n}}{(\beta_1 n)!} \frac{(t-\alpha_2 n)_{\beta_2 n}}{(\beta_2 n)!} \frac{(\gamma n)!}{(t)_{\gamma n+1}} \right) z^{-t},$$

where $\delta = \max\{0, \alpha_1 - \beta_1, \alpha_2 - \beta_2\}$, and repeat the same discussion as before.

For brevity, we just give the numerical values of the linear independence measure μ of $1, \mathrm{Li}_1(1/z), \mathrm{Li}_2(1/z)$ for some rational and algebraic numbers z. Some of them can be slightly refined by increasing the size of the parameters. We indicate that by applying the saddle point method to the corresponding integrals $I_n^{(j)}(z)$ as we did in section 5 one can remove the analytic condition in [10, inequality (5.23)] and obtain slightly better irrationality measures of dilogarithms than those in [10, Theorem 5.2]. One can also apply this method to $z < 0$ (the numerical results in the table below should be compared with those obtained by Hata in [3]) and more generally to suitable algebraic numbers. For example, one can prove that $1, \mathrm{Li}_1(i/q), \mathrm{Li}_2(i/q)$ are linearly independent over $\mathbb{Q}(i)$ for any $q \geqslant 6$. Moreover, by using a non-vanishing determinant as we did in section 6 we also obtain linear independence measures of $1, \mathrm{Li}_1(1/z), \mathrm{Li}_2(1/z)$ instead of irrationality measures of $\mathrm{Li}_2(1/z)$ only (see [7, Lemma 2.6] for a linear independence criterion over a number field).

α_1	β_1	α_2	β_2	γ	z	μ
125	75	230	192	250	6	372.86747 ...
54	35	97	82	108	7	65.74235...
37	23	62	51	74	8	42.99008...
56	36	91	76	112	9	29.45327...
20	13	32	27	40	10	22.97703...
107	88	146	126	214	51/2	348.69415...
126	104	171	148	252	53/2	185.72122...
117	96	158	138	234	55/2	128.64066...
96	80	129	112	192	57/2	99.99291...
18	13	21	18	27	−5	81.21374...
18	13	21	18	27	−6	45.67753...
27	24	30	25	45	−7	33.97683...
27	24	30	25	45	−8	27.18769...
41	36	51	46	82	−49/2	459.60280...
36	20	60	51	72	$6i$	128.00276...
154	111	277	222	308	$2i\sqrt{7}$	1395.262930...

We finally remark that we could also apply the linear forms $S_n^{(j)}$ $(j = 1, 2)$ of sections 2–7, and disregard $S_n^{(3)}$, to obtain results on the linear independence of $1, \mathrm{Li}_1(1/z), \mathrm{Li}_2(1/z)$ only (as we did in [6] for the linear independence measure of $1, \log 2, \log^2 2$), but we would not improve the above results in this way.

Acknowledgements

This research was performed during a recent stay at the Department of Mathematics of the University of Genova. I am pleased to thank Alberto Perelli for the kind hospitality. The author was supported by the grant PRIN 2010–11 "Arithmetic Algebraic Geometry and Number theory".

I am thankful to Wadim Zudilin for pertinent remarks. I am indebted with the referee for useful comments and corrections which helped me to improve the presentation of this work.

Bibliography

1. ***L. A. Gutnik,*** *The linear independence over* $\mathbb{Q}$ *of dilogarithms at rational points.* Uspekhi Mat. Nauk. **37** (1982), no.5 (227) [Russian Math. Surveys **37**:5], 179–180 [176–177].
2. ***M. Hata,*** *On the linear independence of the values of polylogarithmic functions.* J. Math. Pures Appl. (9) **69**:2 (1990), 133–173.
3. ***M. Hata,*** *Rational approximations to the dilogarithm.* Trans. Amer. Math. Soc. **336**:1 (1993), 363–387.
4. ***M. Hata,*** *The irrationality of* $\log(1+1/q)\log(1-1/q)$. Trans. Amer. Math. Soc. **350**:6 (1998), 2311–2327.
5. ***T. Hessami Pilehrood, Kh. Hessami Pilehrood,*** *Lower bounds for linear forms in values of polylogarithms.* Mat. Zametki **77** (2005) [Math. Notes **77**], 573–579 [623–629].
6. ***R. Marcovecchio,*** *Multiple Legendre polynomials in Diophantine approximation.* Int. J. Number Theory **10** (2014), 1829–1855.
7. ***R. Marcovecchio, C. Viola,*** *Irrationality and nonquadraticity measures for logarithms of algebraic numbers.* J. Aust. Math. Soc. **92** (2012), 237–267.
8. ***Yu. V. Nesterenko,*** *Integral identities and constructions of approximations to zeta-values.* J. Théor. Nombres Bordeaux **15**:2 (2003), 535–550.
9. ***Yu. V. Nesterenko,*** *On the irrationality exponent of the number* $\ln 2$. Mat. Zametki **88** (2010) [Math. Notes **88**], 549–564 [530–543].
10. ***G. Rhin, C. Viola,*** *The permutation group method for the dilogarithm.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) **4** (2005), 389–437.
11. ***C. Viola,*** *On the coefficients of linear forms in polylogarithms.* Oberwolfach Rep. **9**:2 (2012), 1313--1316.
12. ***W. Zudilin,*** *Approximations to -, di- and tri-logarithms.* J. Comput. Appl. Math. **202**:2 (2007), 450–459.
13. ***W. Zudilin,*** *Two hypergeometric tales and a new irrationality measure of* $\zeta(2)$. Ann. Math. Qué. **38**:1 (2014), 101–117.

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