Reprint from

ISSN 2220-5438

Moscow Journal of Combinatorics and Number Theory

Moscow Journal of Combinatorics and Number Theory Volume 6 • Issue 1 2016

URSS

МФТИ

URSS

Volume 6 • Issue 1 2016

Moscow Journal
of Combinatorics
and Number Theory
2016, vol. 6, iss. 1,
pp. 25–63 [25–63]

An analog of Brooks′ Theorem for dynamic colorings

Dmitriy V. Karpov (St. Petersburg)

Abstract: Let $d \geq 6$ and G be a connected graph with $\Delta(G) \leq d$ different from K_{d+1} and from any graph which can be obtained from K_{d+1} upon substituting some edges of K_{d+1} by chains of two edges (every such chain adds to a graph a new vertex of degree 2). We prove that there is a proper dynamic vertex coloring of G with d colors.

Keywords: graph, proper coloring, dynamic coloring, Brooks' theorem.

AMS Subject Classification: 05C15

Received: 22.12.2015; **revised:** 05.02.2016

1. Introduction

In this paper, we consider undirected graphs without loops and multiple edges. We study proper vertex colorings of such graphs.

The set of vertices of a graph G is denoted by $V(G)$. We denote by $\mathrm{N}_G(v)$ the neighborhood of a vertex v in the graph G, i.e., the set of all vertices that are adjacent to v.

The degree of a vertex $v \in V(G)$ in the graph G is denoted by $d_G(v)$. We denote by $\Delta(G)$ the maximal vertex degree of a graph G.

We study vertex colorings of a graph and always denote the color of a vertex v in a coloring ρ by $\rho(v)$.

DEFINITION 1. *A vertex coloring is* ***proper****, if any two adjacent vertices have different colors.*

A vertex coloring of a graph G is called ***dynamic****, if for any vertex $v \in V(G)$ of degree at least 2 the neighborhood $\mathrm{N}_G(v)$ contains two vertices of distinct colors.*

Recall that a vertex coloring of a hypergraph is called *proper*, if any its hyperedge contains two vertices of different colors.

Consider a hypergraph $\mathcal{G}$ on the vertex set $V(G)$, which hyperedges are neighborhoods of all vertices of degree at least two in the graph G. Thus, a proper dynamic coloring of the graph G is a proper vertex coloring of G and, at the same time, a proper vertex coloring of the hypergraph $\mathcal{G}$.

We define the dynamic chromatic number of a graph similarly to the classic chromatic number.

DEFINITION 2. *The* ***dynamic chromatic number*** *$\chi_2(G)$ of a graph G is the least natural number such that there exists a dynamic proper coloring of vertices of G with $\chi_2(G)$ colors.*

The classic Brooks' Theorem [4] states that for $d \geq 3$ and any connected graph G, such that $\Delta(G) \leq d$ and G is not isomorphic to a complete graph K_{d+1} on $d+1$ vertices, we have $\chi(G) \leq d$.

In [7] it is proved that $\chi_2(G) \leq \Delta(G) + 1$ for any connected graph G with $\Delta(G) \geq 3$. Moreover, for the case $\Delta(G) \leq 3$ the inequality $\chi_2(G) \leq 4$ holds with the only exclusion: $\chi_2(C_5) = 5$ (here C_5 denotes a cycle on 5 vertices). In [2] similar bounds on the list dynamic chromatic number are proved.

In [5] the author has proved an analog of Brooks' Theorem for dynamic colorings: for any connected graph G with $\Delta(G) \leq d$ and $d \geq 8$, except for some exclusions, described in the paper, the inequality $\chi_2(G) \leq d$ holds. In this paper, we decrease the bound on the number of colors we need from 8 to 6.

It is interesting that the constant 6 appears in one more paper on dynamic colorings. In [1] it is proved that if $\chi(G) \geq 6$, than there is a proper vertex coloring of G with $\chi(G)$ colors, where the set of bad vertices is independent (a vertex v is bad if $d_G(v) \geq 2$ and $\mathrm{N}_G(v)$ is colored with one color).

In several papers dynamic chromatic numbers of special classes of graphs are studied. In [3] it is proved that any regular bipartite graph has a dynamic proper coloring with 4 colors and some other bounds.

In [6] it is proved that for any connected planar graph, different from C_5, its dynamic chromatic number is at most 4. It is also proved here that the list dynamic chromatic number of any planar graph is at most 5.

2. Main results and definitions

We formulate two main theorems of this paper similarly to the paper [5], but with new bound on d.

THEOREM 1. *Let $d \geq 6$ be an integer and G be a connected graph without vertices of degree 2, such that $\Delta(G) \leq d$. Assume that G is different from a complete graph on $d+1$ vertices. Then $\chi_2(G) \leq d$.*

In the second theorem we add vertices of degree 2. We need to define our class of exclusions.

DEFINITION 3. *Let $n \geq 3$ and let K_n be a complete graph on n vertices.*

We denote by $\mathcal{K}_n$ the set that consists of K_n and all graphs obtained from K_n upon replacing several edges of the graph K_n by chains of two edges (any such chain adds a new vertex of degree 2).

THEOREM 2. *Let $d \geq 6$.*

1) *If a graph $H \in \mathcal{K}_{d+1}$, then $\chi_2(H) = d+1$.*
2) *If G is a connected graph with $\Delta(G) \leq d$ that is not isomorphic to a graph of the class $\mathcal{K}_{d+1}$, then $\chi_2(G) \leq d$.*

The main result of the paper is Theorem 2. Its derivation from Theorem 1, written in [5] is rather easy and requires the number of colors to be at least 5. In this paper, the number of colors is at least 6, hence, this derivation is valid in our case, too. We will not repeat the text from [5].

In what follows, we prove the new variant of Theorem 1. We start with some necessary definitions.

DEFINITION 4. *Let G be a graph and ρ be its vertex coloring.*

1) *A vertex $v \in V(G)$ is **bad** in the coloring ρ, if $d_G(v) \geq 2$ and $\mathrm{N}_G(v)$ is colored in ρ with one color.*

2) *Let a and b be adjacent vertices. The vertex b is a* ***dangerous neighbor*** *of the vertex a in the coloring ρ of the graph G, if $d_G(b) > 1$ and vertices of the set $\mathrm{N}_G(b) \setminus \{a\}$ are colored in ρ with one color different from $\rho(a)$.*

DEFINITION 5. *Let ρ and ρ' be two colorings of a graph G. We write $\rho' \le_G \rho$ if the following two conditions are satisfied.*

(1) *If u and v are adjacent vertices and $\rho'(u) = \rho'(v)$, then $\rho(u) = \rho(v)$.*

(2) *Any vertex that is bad in ρ' is bad in ρ as well.*

We write $\rho' <_G \rho$ if there exists a bad vertex in the coloring ρ that is not bad in ρ'.

Remark 2.1.

1) It is easily seen that if $d_G(b) \neq 2$, then the vertex b can be a dangerous neighbor for at most one other vertex.
2) Assume that $d_G(a) > 2$ and we can make the vertex a bad by changing color of one of its neighbors. Obviously, such a neighbor of a is unique and is a dangerous neighbor of a.
3) If $\rho \le_G \rho_1$ and $\rho_1 \le_G \rho_2$ for some colorings ρ, ρ_1, and ρ_2, then $\rho \le_G \rho_2$. If at least one of the “inequalities” above is strict, then $\rho <_G \rho_2$.

Let us write down a *plan of the proof* of Theorem 1. This plan coincides with the plan of the proof in [5]. However, almost all details of these proofs are different.

Let G be a connected graph different from the complete graph on $d + 1$ vertices, without vertices of degree 2 and such that $\Delta(G) \le d$. By Brooks' Theorem, there exist proper colorings of vertices of the graph G with d colors. Unfortunately, such colorings may contain bad vertices. We select a proper coloring ρ with d colors having the minimal number of bad vertices.

Let a be a bad vertex in the coloring ρ. We want to change the coloring so that the vertex a is not bad in the new coloring and we get no new bad vertices. For this purpose, we use a method that reminds the construction of a classical alternating chain in the proof of the Brooks' Theorem. At the same time, we meet really more difficulties constructing our *chain of prohibitions*. In Brooks' Theorem, the only prohibition on coloring a vertex a with color i is the existence of a neighbor of color i; in our case, the existence of a dangerous neighbor b such that all the vertices in $\mathrm{N}_G(b) \setminus \{a\}$ have color i is also a prohibition. This fact doubles the number of potential prohibitions. However, we do not increase the number of colors.

For this reason, the main algorithm of constructing the chain of prohibitions calls several times an auxiliary algorithm DN that changes the coloring and reduces the number of prohibitions on the required color for the next considered vertex. We describe the algorithm DN in Sec. 3. In Sec. 4, we describe the main algorithm of constructing the chain of prohibitions.

In the new proof, we managed to save one necessary color in the new algorithm DN and one necessary color in the new main algorithm. Thus, the lower bound on the number of necessary colors is decreased from 8 to 6.

In what follows we need some notation for digraphs.

DEFINITION 6. *Let F be a digraph. The set of its vertices is denoted by $V(F)$, the set of its arcs (i.e., directed edges) is denoted by $A(F)$.*

For a vertex $v \in V(F)$ we use the following notation:

- $\mathrm{N}_F^+(v) = \{x \in V(F) \ : \ vx \in A(F)\}$;
- $\mathrm{N}_F^-(v) = \{x \in V(F) \ : \ xv \in A(F)\}$.

3. Algorithm DN

3.1. Setting of the problem

In this section, we write down an auxiliary algorithm $DN(H, \rho, a, J)$, which will be often called by the main algorithm.

Given:

- A graph H and a coloring ρ of vertices of H with at least 4 colors (this coloring can be not proper).
- A bad vertex a in the coloring ρ of the graph H. For convenience, set the following notation: $\rho(a) = j_0$ and j_1 is the color of vertices of the set $\mathrm{N}_H(a)$ in ρ.
- Ordered set of different colors $J = (j_2, j_3, j_4, j_5)$, such that the color j_0 can coincide only with j_4 and the color j_1 can coincide only with j_5.

The aim of the algorithm DN is changing the coloring ρ to the coloring ρ', such that $\rho' \leq_H \rho$ and one of the conditions $(DN1)$ and $(DN2)$ holds.

$(DN1)$. $\rho'(a) = j_2$. If $x \neq a$ and $\rho'(x) \neq \rho(x)$, then $\{\rho(x), \rho'(x)\} = \{j_2, j_4\}$.

$(DN2)$. There exists unique vertex $v \in \mathrm{N}_H(a)$, such that $\rho'(v) = j_3$. Any vertex of the set $\mathrm{N}_H(a)$ different from v has color j_1 in ρ'. If $x \neq v$ and $\rho'(x) \neq \rho(x)$, then $\{\rho(x), \rho'(x)\} = \{j_3, j_5\}$.

3.2. Construction of the set D

First, we construct a sequence of colors (j_n). For $n \in \{0, 1, 2, 3, 4, 5\}$ this sequence is defined above. For each positive integer k we set:

$$j_{4k+2} = j_2, \quad j_{4k+3} = j_3, \quad j_{4k+4} = j_4, \quad j_{4k+5} = j_5.$$

We pass to constructing auxiliary sets D and $D^* \subset D$, and a digraph $\overline{D}$ on the vertex set D.

We will put into D and into D^* some vertices of H and give orientation to some edges between vertices of the set D. Arcs of the digraph $\overline{D}$ will start only at vertices of the set D^*.

The beginning. First, we set $D = \{a\} \cup \mathrm{N}_H(a)$, $D^* = \{a\}$, $A(\overline{D}) = \{ax : x \in \mathrm{N}_H(a)\}$.

Let us describe a **step of construction**. Consider a vertex $u \in D \setminus D^*$, let $\rho(u) = j_k$. Denote by M the set of all vertices $x \in \mathrm{N}_H(u)$, such that $xu \notin A(\overline{D})$. If $M = \varnothing$ or $M \neq \varnothing$ and $\rho(v) = j_{k+1}$ for every vertex $v \in M$, then we put the vertex u into D^* and all vertices of the set $M \setminus D$ — into D. In this case, we add into $A(\overline{D})$ arcs from u to all vertices of the set M.

If $D \setminus D^* = \varnothing$ or the step described above cannot be performed with any vertex of the set $D \setminus D^*$, the construction is finished.

Clearly, since the graph H is finite, the construction will be finished. Let D, D^* and $\overline{D}$ be the sets and the digraph obtained after the construction is finished.

Remark 3.1.

1) By construction, all vertices of the set D are reachable in the digraph $\overline{D}$ from the root a.
2) If $x \in D^*$, then $\mathrm{N}_H(x) = \mathrm{N}_{\overline{D}}(x) = \mathrm{N}^-_{\overline{D}}(x) \cup \mathrm{N}^+_{\overline{D}}(x)$. If $x \neq a$, then the set $\mathrm{N}^-_{\overline{D}}(x)$ is nonempty.
3) If $x \in D \setminus D^*$, then $\mathrm{N}_{\overline{D}}(x) = \mathrm{N}^-_{\overline{D}}(x) \neq \varnothing$.

Definition 7. *Let us divide vertices of the set D into several levels: **level** D_k consists of all vertices x, for which the length of the shortest ax-path in $\overline{D}$ is equal to k.*

Note that $D_0 = \{a\}$.

Remark 3.2. Let $u \in D_k$.

1) By construction, for each arc $xy \in A(\overline{D})$ if $\rho(x) = j_m$, then $\rho(y) = j_{m+1}$. Since $D_0 = \{a\}$ and $\rho(a) = j_0$, we have $\rho(u) = j_k$.

2) All vertices of the set $N^{-}_{\overline{D}}(u)$ have color j_{k-1} and all vertices of the set $N^{+}_{\overline{D}}(u)$ have color j_{k+1}.
3) If $u \in D^*$, then by Remark 3.1 the set $N_H(x)$ is colored in ρ with two colors: j_{k-1} and j_{k+1} (maybe, one of these colors is absent).
4) If $x \in D \setminus D^*$, then the set $N_{\overline{D}}(x)$ is colored in ρ with the color j_{k-1}. Moreover, $N_H(x) \setminus N_{\overline{D}}(x) \neq \varnothing$ and there is a vertex in this set which have color different from j_{k+1} in the coloring ρ (otherwise, we can continue the construction of D).

DEFINITION 8.

1) *For each vertex $u \in D^*$ different from a we choose one arc which comes to u from a vertex of previous level. Let A' be the set of all chosen arcs.*
2) *If $uv \in A'$, then u is the **ancestor** of v and v is a **descendant** of u.*
3) *Denote by $\overline{D}'$ the digraph with set of vertices D^* and set of arcs A' (see Figure 1).*

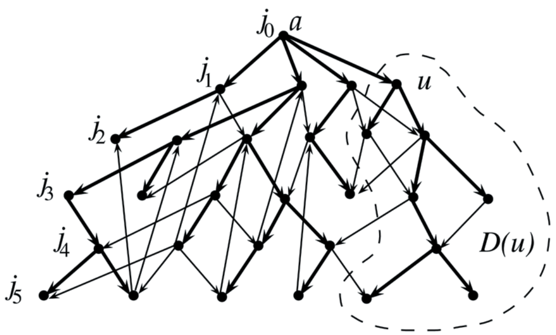

Fig. 1. The digraph $\overline{D}$ and the directed tree $\overline{D}'$. Arcs of the set A' are bold

Remark 3.3.

1) If $uv \in A'$, then $u, v \in D^*$.
2) Note that $\overline{D}'$ is a directed tree rooted at a.
3) Any vertex of D^*, different from a, has unique ancestor.

DEFINITION 9. *For a vertex $u \in D^*$ denote by $D(u)$ the set of all vertices of D^*, which are reachable from u in the digraph $\overline{D}'$ (see Figure 1).*

Clearly, $D(a) = D^*$.

3.3. Recoloring vertices of D^*

In this section, ρ is the given coloring. Studying a certain new coloring ρ^*, we say that a vertex v is *recolored* if $\rho^*(v) \neq \rho(v)$.

LEMMA 1. *Let $v \in D_k$. Then there exists a coloring ρ'_v of vertices of the graph H, satisfying the conditions $(P1)$, $(P2)$ and one of the conditions $(DN1')$ and $(DN2')$.*

$(P1)$. *If $\rho(x) \neq \rho'(x)$, then $x \in D(v)$.*

$(P2)$. *For any vertex $x \in D(v)$, $x \neq v$ either no vertex of the set $\mathrm{N}_{\overline{D'}}(x)$ is recolored or exactly two vertices are recolored: the ancestor of v and one of descendants of v.*

$(DN1')$. *$\rho'_v(v) = j_{k+2}$. If $x \neq v$ and $\rho'_v(x) \neq \rho(x)$, then $\{\rho(x), \rho'_v(x)\} = \{j_{k+2}, j_{k+4}\}$.*

$(DN2')$. *There exists unique vertex $u \in \mathrm{N}^+_{\overline{D'}}(v)$, such that $\rho'_v(u) = j_{k+3}$. Any vertex of the set $\mathrm{N}_H(v)$ different from u has color j_{k+1} in ρ'_v. If $x \neq u$ and $\rho'_v(x) \neq \rho(x)$, then $\{\rho(x), \rho'_v(x)\} = \{j_{k+3}, j_{k+5}\}$.*

PROOF. We call a changing of the coloring ρ, described above, a *recoloring of type* 1, if condition $(DN1')$ holds and a *recoloring of type* 2, if condition $(DN2')$ holds.

We prove Lemma by induction.

Base of induction. *The vertex $v \in D^*$ has no descendants.*
In this case, we change color of the vertex v: let $\rho'_v(v) = j_{k+2}$. All other vertices have the same colors in ρ and ρ'_v. Clearly, conditions $(P1)$, $(P2)$ and $(DN1')$ hold.

Induction step. *The vertex $v \in D^*$ has descendants.*
Recall that $\rho(v) = j_k$. Let $u_1, \dots, u_m$ be all descendants of v.

Note that the vertex sets $D(u_1)$, ..., $D(u_m)$ are pairwise disjoint and their union is $D(v) \setminus \{v\}$ (see Figure 2).

Consider two cases.

1. *For a vertex u_i there exists a recoloring of type* 1.

Perform this recoloring of type 1. Let ρ'_v be obtained coloring. Clearly, for $D(v)$ conditions $(P1)$ and $(DN2')$ hold. It remains to verify that condition $(P2)$ also holds. Consider a vertex $x \in D(v)$, $x \neq v$. Assume that there are recolored vertices in $\mathrm{N}_{\overline{D'}}(x)$. It is clear that then $x \in D(u_i)$. Since all recolored vertices belong to levels of the same parity and u_i is recolored, we have $x \neq u_i$. Then by induction assumption condition $(P2)$ holds for the vertex x.

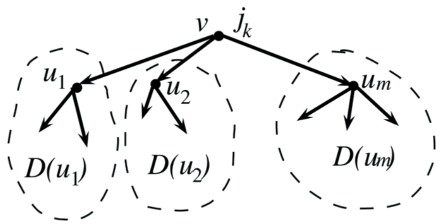

Fig. 2. The vertex v and the sets $D(u_1)$, ..., $D(u_m)$

2. *For each of the vertices* $u_1, \ldots, u_m$ *there exists a recoloring of type* 2.
Perform all these recolorings of type 2 simultaneously, and, after that, change color of the vertex v to j_{k+2}. Prove that the coloring ρ'_v obtained as a result is what we want.

It is clear that for $D(v)$ and ρ'_v conditions $(P1)$ and $(DN1')$ hold. It remains to verify condition $(P2)$. Consider a vertex $x \in D(v)$, $x \neq v$. Assume that there are recolored vertices in $\mathrm{N}_{\overline{D'}}(x)$ and $x \in D(u_i)$. Then only vertices of the set $D(u_i)$ and the vertex v can be the ancestor and descendants of x. Thus, we can consider only vertices of the set D_i and their recoloring of type 2. If $x \neq u_i$, then condition $(P2)$ for x holds by induction assumption for the recoloring of vertices of $D(u_i)$. Consider the vertex u_i. Exactly one of its descendants changed its color in the recoloring of type 2 of $D(u_i)$. The ancestor of u_i is v and this vertex is also recolored. Hence, condition $(P2)$ holds for u_i. □

3.4. Recoloring for the algorithm DN

Consider a coloring ρ'_a from Lemma 1 and denote it by ρ'.

Let us verify that this coloring satisfies requirements of the algorithm DN. Conditions $(P1)$ and $(DN1')$ for the coloring ρ'_a imply condition $(DN1)$ for the coloring ρ'. Conditions $(P1)$ and $(DN2')$ for the coloring ρ'_a imply condition $(DN2)$ for the coloring ρ'.

It remains to check that $\rho' \leq_H \rho$. Assume that a vertex $x \in V(H)$ is bad in ρ', but is not bad in ρ. Then $\mathrm{N}_H(x)$ contains at least one recolored vertex y. By condition $(P1)$ only vertices of the set D^* can be recolored. All vertices adjacent to D^* belong to D. Hence, $x \in D$. Let $x \in D_k$. Consider two cases.

1. $x \in D \setminus D^*$.
By Remark 3.2, all vertices of the set $\mathrm{N}_{\overline{D}}(x)$ are colored in ρ with color j_{k-1}. As we know, among these vertices there is a recolored one, which has color j_{k+1}

in ρ'. Thus, there is a vertex of color j_{k+1} in the coloring ρ' in $\mathrm{N}_H(x)$. By Remark 3.2, the set $M = \mathrm{N}_H(x) \setminus \mathrm{N}_{\overline{D}}(x) \neq \varnothing$ and this set contains a vertex colored in ρ with a color $s \neq j_{k+1}$. The colorings ρ and ρ' coincide on the set M, hence, there is a vertex of color s in the coloring ρ' in $\mathrm{N}_H(x)$. Thus, the vertex x is not bad in ρ'. We obtain a contradiction.

2. $x \in D^*$.
By Remark 3.1, then $\mathrm{N}_H(x) = \mathrm{N}_{\overline{D}}(x) = \mathrm{N}^-_{\overline{D}}(x) \cup \mathrm{N}^+_{\overline{D}}(x)$. By Remark 3.2, all vertices of $\mathrm{N}^-_{\overline{D}}(x)$ have color j_{k-1} in ρ and all vertices of $\mathrm{N}^+_{\overline{D}}(x)$ have color j_{k+1} in ρ.

Since the vertex x is not bad in ρ, we have $x \neq a$. Then by condition $(P2)$ for the coloring ρ_a exactly two vertices of the set $\mathrm{N}_H(x)$ are recolored in ρ': the ancestor y and one of descendants z. Since $\rho(y) = j_{k-1}$ and $\rho(z) = j_{k+1}$, we have $\rho'(y) = j_{k+1}$ and $\rho'(z) = j_{k+3}$. Thus, the vertex x is not bad in the coloring ρ'. We obtain a contradiction.

Let x and y be adjacent vertices such that $\rho'(x) = \rho'(y)$, but $\rho(x) \neq \rho(y)$. Then at least one of this vertices is recolored, let it be x. Hence, $x \in D^*$ and $y \in D$. Let $x \in D_k$. By conditions $(DN1)$ and $(DN2)$ then $\rho'(x) \in \{j_k, j_{k+2}\}$. However, by Remark 3.1 we have $\rho'(y) \in \{j_{k-1}, j_{k+1}\}$. We obtain a contradiction.

Thus, we have proved that $\rho' \leq_H \rho$. The coloring ρ' is the result of performing the algorithm $DN(H, \rho, a, J)$.

3.5. Algorithm DN: properties

We start with useful property of recoloring made by algorithm DN.

LEMMA 2. *Assume that the algorithm $DN(H, \rho, u, J)$ has changed the coloring ρ to the coloring ρ'. Then there is no bad vertex all neighbors of which are recolored.*

PROOF. Assume that x is such vertex. Clearly, $x \neq a$. Note that $\mathrm{N}_H(x) \subset D^*$, since all recolored vertices belong to D^*. Hence, $x \in D$. Since $\mathrm{N}_{\overline{D}}(x) = \mathrm{N}_H(x)$, we have that $x \in D^*$ (see Remark 3.1).

Without loss of generality assume that $x \in D_k$. Since $x \in D^*$, we know that $\mathrm{N}_H(x) = \mathrm{N}^-_{\overline{D}}(x) \cup \mathrm{N}^+_{\overline{D}}(x)$. Moreover, $\mathrm{N}^-_{\overline{D}}(x) \neq \varnothing$ and all vertices of this set have color j_{k-1} in the coloring ρ and vertices of the set $\mathrm{N}^+_{\overline{D}}(x)$ have color $j_{k+1} \neq j_{k-1}$ in ρ. Since the vertex x is bad in the coloring ρ, we obtain $\mathrm{N}^+_{\overline{D}}(x) = \varnothing$.

Thus, the vertex x has the ancestor, but has no descendants. Recall the construction of the coloring ρ': the coloring ρ' coincides with ρ'_a, and ρ'_a satisfies condition $(P2)$ (see Lemma 1). For the vertex $x \in D^*$, having no descendants, condition $(P2)$ means that the ancestor of x is not recolored. We obtain a contradiction. □

Let us return to the initial graph G.

LEMMA 3. *Let $b \in V(G)$ be such that $d_G(b) \geq 3$ and let $B \subset \mathrm{N}_G(b)$. Consider a new graph G', obtained from G upon deleting edges, joining b with B. Let ρ be a coloring of vertices of the graph G (ρ may be not proper).*

Let $u \in B$, $\rho(u) = j_0$, and suppose that all vertices of the set $\mathrm{N}_{G'}(u)$ are colored in ρ with color j_1, $j_0 \neq j_1$. Consider an ordered set of different colors $J = (j_2, j_3, j_4, j_5)$, which is suited to algorithm DN (i.e., j_0 can coincide only with j_4, and j_1 can coincide only with j_5).

Assume that $\rho(b) \notin J$ and the following condition holds:

(J) either b is a bad vertex in the coloring ρ of the graph G, or there exists a vertex $x \in \mathrm{N}_G(b)$, such that $\rho(x) \notin \{j_0, j_2, j_4\}$.

Assume that the algorithm $DN(G', \rho, u, J)$ changes the coloring ρ to ρ'. Then $\rho' \leq_G \rho$ and $\rho'(b) = \rho(b)$.

PROOF. By properties of algorithm DN, we have $\rho' \leq_{G'} \rho$. Since $\rho(b) \notin J$, the vertex b can be recolored only if $b = u$ or $b \in \mathrm{N}_{G'}(u)$. Since none of these conditions holds, $\rho'(b) = \rho(b)$.

Let x and y be adjacent in G, $\rho'(x) = \rho'(y)$, but $\rho(x) \neq \rho(y)$. Since $\rho' \leq_{G'} \rho$, the vertices x and y are not adjacent in G'. Hence without loss of generality we can set $x = b$, $y \in B$. Since $s = \rho'(b) = \rho(b) \notin J$, and all recolored vertices have in ρ' colors of the set J, the vertex y must have color s in ρ. Thus, $\rho(b) = \rho(y)$, we obtain a contradiction. Hence, if the vertices x and y are adjacent in G and $\rho'(x) = \rho'(y)$, then $\rho(x) = \rho(y)$.

Let us prove than no new bad vertex appears in ρ'. If we consider ρ' and ρ as colorings of the graph G', then all vertices bad in ρ' are bad in ρ (recall that $\rho' \leq_{G'} \rho$).

Now consider ρ' and ρ as colorings of the graph G. Let a vertex x be bad in ρ', but not bad in ρ (as in colorings of the graph G). Clearly, x has in G and G' different neighborhoods. Thus, it is enough to verify vertices of the set B and the vertex b.

Check of the vertex b.

It is enough to consider the case where b is not a bad vertex in the coloring ρ of the graph G. By condition (J) there exists a vertex $x \in \mathrm{N}_G(b)$, such that $\rho(x) \notin \{j_0, j_2, j_4\}$. Then $\rho'(x) \notin \{j_0, j_2, j_4\}$. On the other side, the vertex $u \in \mathrm{N}_G(b)$ has color $\rho(u) = j_0$, consequently, $\rho'(u) \in \{j_0, j_2\}$. Thus, the set $\mathrm{N}_G(b)$ contains vertices having different colors in ρ'. Hence, the vertex b is not bad in the coloring ρ' of the graph G.

Check of the set B.

Consider a vertex $x \in B$. It is adjacent to b and $\rho'(b) = \rho(b)$. In the coloring ρ' no new vertices of color $\rho(b)$ appear. Hence, if x is bad in the coloring ρ' of the graph G, then x is bad in the coloring ρ of the graph G.

Thus, $\rho' \leq_G \rho$. □

4. Construction of a dynamic coloring for a graph without vertices of degree 2

In this section, $d \geq 6$ is an integer and G is a connected graph without vertices of degree 2, such that $\Delta(G) \leq d$ and G differs from the complete graph K_{d+1}. This section is devoted to a proof of Theorem 1; i.e., we construct a dynamic coloring of the graph G with d colors.

4.1. Algorithm for construction of a chain. Basic principles of choice of vertices and the start of the construction

By Brooks' Theorem, there exists a proper coloring of the graph G with d colors. Assume that the coloring ρ contains a bad vertex a. Let $\rho(a) = 0$, let all the vertices in $\mathrm{N}_G(a)$ have color 1, and let color 2 be different from 0 and 1. We want to replace the coloring ρ by a coloring $\rho' \leq_G \rho$ such that some (but not all) of the vertices in $\mathrm{N}_G(a)$ have color 2 in ρ', while the remaining vertices in $\mathrm{N}_G(a)$ preserve their color. Then the vertex a is not bad in the coloring ρ'; hence, $\rho' <_G \rho$.

Set $c(k) = 1$ for odd k and $c(k) = 2$ for even k. We construct a sequence of different vertices $b_1, b_2, \ldots, b_k, \ldots$. For any vertex b_k we will define a unique *ancestor* $\mathrm{asc}(b_k)$, which will be always a vertex adjacent to b_k.

DEFINITION 10.

1) *Set* $\mathrm{N}'(b_k) = \mathrm{N}_G(b_k) \setminus \{\mathrm{asc}(b_k)\}$.

2) *For* $u \in \mathrm{N}'(b_k)$ *set* $\mathrm{N}'(u) = \mathrm{N}_G(u) \setminus \{b_k\}$.

The idea of our construction reminds the classical method of alternating chains: every next vertex b_{k+1} is a "prohibition" that forbids to recolor b_k with color $c(k+1)$. At the same time, the construction is significantly more complicated than in the classical proof of the Brooks' Theorem.

DEFINITION 11. *Let* $i \neq c(k)$.

1) *If there exists a vertex* $v \in \mathrm{N}'(b_k)$ *of color* $\rho(v) = i$, *then such a situation is called a* ***prohibition of type*** 1 ***on color*** i *for the vertex* b_k. *We say that the vertex* a ***imposes*** *this prohibition.*
2) *If a vertex* $u \in \mathrm{N}'(b_k)$ *is such that all vertices of the set* $\mathrm{N}'(u)$ *are colored with color* i *in the coloring* ρ, *then such a situation is called a* ***prohibition of type*** 2 ***on color*** i *for the vertex* b_k, *and the vertex* u *is called the* ***basic vertex*** *of this prohibition.*

In this situation, we will use the notation $z2(\rho, u) = i$. *In the case where it is clear what coloring we deal with, we will write simply* $z2(u)$.

Remark 4.1. A vertex b_k can have several prohibitions of type 2 on color i with different basic vertices.

Let us pass to construction of the chain. We take as b_1 an arbitrary vertex in $\mathrm{N}_G(a)$; the vertex $a_0 = a$ is the ancestor of b_1. For $k \geq 1$, the vertex b_{k+1} satisfies the following conditions:

- $\rho(b_{k+1}) = c(k+1)$;
- one of the following two situations is possible:
  - — the vertices b_k and b_{k+1} are adjacent, $\mathrm{asc}(b_{k+1}) = b_k$ and the vertex $b_{k+1} \in \mathrm{N}'(b_k)$ imposes a prohibition of type 1 on color $c(k+1)$ for the vertex b_k;
  - — $\mathrm{asc}(b_{k+1}) = a_k \neq b_k$, the vertex $a_k \in \mathrm{N}'(b_k)$ is the basic vertex of a prohibition of type 2 on color $c(k+1)$ for the vertex b_k and $b_{k+1} \in \mathrm{N}'(a_k)$.

DEFINITION 12. *A sequence of vertices* $b_1, \ldots, b_p$ *constructed for a coloring* ρ *according to the above rules is called a* ***chain of prohibitions*** *in the coloring* ρ.

In Figure 3, $c(k) = 1$ and $c(k+1) = 2$. On the left, b_{k+1} imposes a prohibition of type 1 on color $c(k+1) = 2$ for the vertex b_k; on the right, a prohibition of type 2 with basic vertex a_k is shown.

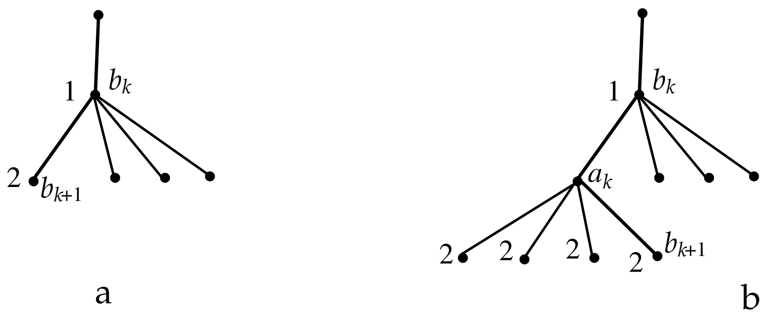

Fig. 3. Prohibitions of type 1 and of type 2

Remark 4.2. If $\mathrm{asc}(b_k) = a_{k-1} \neq b_{k-1}$, then the vertex a_{k-1} is adjacent to both vertices b_{k-1} and b_k; hence, $\rho(a_{k-1}) \notin \{c(k-1), c(k)\} = \{1, 2\}$.

4.2. Gluing and ungluing of colors 1 and 2

Let ρ be a proper vertex coloring of the graph G.

Definition 13. *Construct a new coloring ρ_I from a coloring ρ by joining colors* 1 *and* 2 *into one new color I.*

We will say that the coloring ρ_I is obtained from ρ by ***gluing*** *colors* 1 *and* 2.

Note some properties of gluing colors.

- The coloring ρ_I may be not proper. If x, y are two adjacent vertices of the same color in ρ_I, then $\rho_I(x) = \rho_I(y) = I$, one of vertices x and y has color 1 in ρ and the other has color 2.
- If v is a bad vertex in ρ_I then either v is a bad vertex in ρ, or vertices of the set $\mathrm{N}_G(v)$ are colored in ρ with colors 1 and 2.

Definition 14. *Let a coloring ρ'_I is obtained from a coloring ρ_I upon recoloring some vertices and no new vertices of color I are added.*

Since each vertex x of color $\rho'_I(x) = I$ has color $\rho_I(x) = I$, this vertex also has color $\rho(x) \in \{1, 2\}$. For each such vertex we set $\rho'(x) = \rho(x)$. For each vertex y of color $\rho'_I(y) \neq I$ we set $\rho'(y) = \rho'_I(y)$.

We say that the coloring ρ' defined above is obtained from ρ'_I by ***ungluing*** *the color I.*

LEMMA 4.

1) *Let a coloring ρ' be obtained from ρ upon recoloring some vertices. Assume that no vertex was recolored with colors 1 and 2. Let colorings ρ_I and ρ'_I be obtained upon gluing colors 1 and 2 from ρ and ρ', respectively. Assume that $\rho' \le_G \rho$. Then $\rho'_I \le_G \rho_I$.*
2) *Let ρ be a vertex coloring of the graph G. Let a coloring ρ'_I be obtained from ρ_I upon recoloring some vertices. Assume that no vertex was recolored with color I and at most one vertex of color I was recolored with another color. Let a coloring ρ' be obtained from ρ'_I by ungluing the color I. Assume that $\rho'_I \le_G \rho_I$. Then $\rho' \le_G \rho$.*

PROOF. 1) It follows from the condition that $\rho'_I(x) = I$ is possible only if $\rho_I(x) = I$.

Let $xy \in E(G)$ and $\rho'_I(x) = \rho'_I(y)$, but $\rho_I(x) \ne \rho_I(y)$. Then at least one of the vertices x and y is recolored, therefore $\rho'_I(x) = \rho'_I(y) \ne I$. Hence, $\rho'(x) = \rho'_I(x) = \rho'_I(y) = \rho'(y)$. However, it is clear that $\rho(x) \ne \rho(y)$. We obtain a contradiction with $\rho' \le_G \rho$.

Let w be a bad vertex in the coloring ρ'_I of the graph G, but not a bad vertex in ρ_I. Let all vertices of the set $\mathrm{N}_G(w)$ have color i in ρ'_I. At least one vertex of the set $\mathrm{N}_G(w)$ was recolored, therefore, $i \ne I$. Hence, all vertices of the set $\mathrm{N}_G(w)$ in the coloring ρ' also have color i. However, it is clear that vertices of $\mathrm{N}_G(w)$ are not colored with one color in the coloring ρ. Therefore, w is a bad vertex in the coloring ρ' of the graph G, but not a bad vertex in the coloring ρ. We obtain a contradiction with $\rho' \le_G \rho$.

2) Let $xy \in E(G)$ and $\rho'(x) = \rho'(y) = i$. If $i \in \{1, 2\}$, then $\rho'(x) = \rho(x)$ and $\rho'(y) = \rho(y)$, hence, $\rho(x) = \rho(y)$.

Let $i \notin \{1, 2\}$. Then $\rho'_I(x) = \rho'(x) = \rho'(y) = \rho'_I(y)$. It follows from $\rho'_I \le_G \rho_I$ that $\rho_I(x) = \rho_I(y)$. If $\rho_I(x) = \rho_I(y) = I$, then at least two vertices of color I were recolored with color $i \ne I$. We have a contradiction. Hence, $\rho_I(x) = \rho_I(y) = j \ne I$ and, therefore, $\rho(x) = \rho_I(x) = \rho_I(y) = \rho(y)$.

Assume that w is a bad vertex in the coloring ρ' of the graph G and all vertices of the set $\mathrm{N}_G(w)$ have in ρ' color i. If $i \in \{1, 2\}$, then all vertices of the set $\mathrm{N}_G(w)$ have color i in the coloring ρ, too.

Let $i \notin \{1, 2\}$. Then all vertices of the set $\mathrm{N}_G(w)$ in the coloring ρ'_I also have color i, i.e., w is a bad vertex in the coloring ρ'_I of the graph G. It follows from

$\rho'_I \le_G \rho_I$ that w is a bad vertex in the coloring ρ_I of the graph G. Assume that vertices of $\mathrm{N}_G(w)$ have color I in ρ_I. Since $d_G(w) \ge 3$, at least three vertices of color I were recolored. This contradicts the condition of Lemma. Hence, vertices of $\mathrm{N}_G(w)$ have color $j \ne I$ in ρ_I. Then in the coloring ρ vertices of the set $\mathrm{N}_G(w)$ also have color i, i.e., w is a bad vertex in the coloring ρ of the graph G. □

4.3. Conditions ($C1$) and ($C2$)

Let $b_1, \dots, b_p$ be a chain of prohibitions for a coloring ρ. Before describing a step of the algorithm (choice of the next vertex of the chain), we formulate two important conditions and indicate several properties of a chain of prohibitions that satisfies these conditions.

($C1(p)$) For any $i \in \{1, \dots, p\}$ there exists a unique prohibition on color $c(i+1)$ for the vertex b_i in the coloring ρ.

($C2(p)$) Assume that $i \in \{1, \dots, p\}$, $v \in \mathrm{N}'_G(b_i)$, and all neighbors of v have colors 1 and 2. Then v is a basic vertex for a prohibition of type 2 on color $c(i+1)$ for the vertex b_i.

Before the step of algorithm, choosing a vertex b_k, we assume that conditions ($C1(k-1)$) and ($C2(k-1)$) hold for the current coloring and the chain $b_1, \dots, b_{k-1}$.

LEMMA 5. *Let $b_1, \dots, b_p$ be a chain of prohibitions for a coloring ρ, which satisfies conditions ($C1(p)$) and ($C2(p)$). Then none of vertices $b_2, \dots, b_p$ belongs to $\mathrm{N}_G(a)$.*

PROOF. Assume that $\ell \ge 2$ and $b_\ell \in \mathrm{N}_G(a)$. Let $\mathrm{asc}(b_\ell) = a$. Since $\rho(a) \notin \{1, 2\}$ we have $a \ne b_{\ell-1}$. Then by Remark 4.2 we have $b_\ell, b_{\ell-1} \in \mathrm{N}_G(a)$, i.e. $\mathrm{N}_G(a)$ contains vertices of two different colors in ρ. We obtain a contradiction.

Let $a \in \mathrm{N}'(b_\ell)$. All vertices of $\mathrm{N}_G(a)$ have in ρ color 1. However it is clear that a is a not a basic vertex of a prohibition of type 2 on color 1 for b_ℓ. This contradicts condition ($C2(p)$). □

The following two lemmas, proved in [5], show possibilities of recoloring vertices in a chain of prohibitions.

LEMMA 6. *Let $b_1, \dots, b_s$ be a chain of prohibitions for a proper coloring ρ that satisfies conditions ($C1(s-1)$) and ($C2(s-1)$). Let ρ' be a new coloring such that $\rho'(b_i) = c(i+1)$ for $i \in \{1, \dots, s\}$ and the colors of the remaining vertices are the same as in the coloring ρ. Then the following statements hold.*

1) *If vertices* u *and* v *are adjacent and* $\rho'(v) = \rho'(u)$, *then either* $u = b_s$, $v \in \mathrm{N}'(b_s)$ *and* $\rho'(v) = \rho(v) = c(s+1)$, *or* $v = b_s$, $u \in \mathrm{N}'(b_s)$ *and* $\rho'(u) = \rho(u) = c(s+1)$.
2) *The vertex* a *is not bad in the coloring* ρ'.
3) *If a vertex* v *is bad in* ρ' *and is not bad in* ρ, *then* $v \in \mathrm{N}'(b_s)$, *and* v *is the basic vertex of a prohibition of type* 2 *on color* $c(s+1)$ *for the vertex* b_s *in* ρ.

Lemma 7. *Let* $b_1, \dots, b_s$ *be a chain of prohibitions for a proper coloring* ρ *that satisfies conditions* $(C1(s-1))$ *and* $(C2(s-1))$. *Assume that there is no prohibition on color* $c(s+1)$ *for the vertex* b_s. *Then there exists a coloring* ρ' *of vertices of the graph* G *such that* $\rho' <_G \rho$.

We need one more lemma.

Lemma 8. *Let* $b_1, \dots, b_s$ *be a chain of prohibitions for a proper coloring* ρ *that satisfies conditions* $(C1(s-1))$ *and* $(C2(s-1))$. *Let a coloring* $\rho' \leq_G \rho$ *be obtained from* ρ *upon recoloring some vertices, such that at least one vertex of the set* $\mathrm{N}_G(a)$ *is not recolored. Assume that* $\rho'(v) \in \{1, 2\}$ *implies* $\rho(v) \in \{1, 2\}$. *Then the following statements hold.*

1) *Either* $b_1, \dots, b_s$ *is a chain of prohibitions for the coloring* ρ', *satisfying conditions* $(C1(s-1))$ *and* $(C2(s-1))$, *or there exists a coloring* $\rho'' <_G \rho$.
2) *If one of the vertices* $b_1, \dots, b_s$ *is recolored, then there exists a coloring* $\rho'' <_G \rho$.

Proof. Recall that $\rho' \leq_G \rho$. If at least one vertex of the set $\mathrm{N}_G(a)$ is recolored, then a is not a bad vertex in the coloring ρ' of the graph G. Then $\rho' <_G \rho$ and both statements 1 and 2 are proved. Thus, in what follows we assume that for any vertex $x \in \mathrm{N}_G(a)$ we have $\rho'(x) = \rho(x) = 1$.

Let t be the maximal index such that $t \leq s$ and all vertices b_1, ..., b_t were not recolored. Prove by induction for $\ell < t$ that either conditions $(C1(\ell))$ and $(C2(\ell))$ hold, or there exists a coloring ρ'', such that $\rho'' <_G \rho$. Assume that conditions $(C1(\ell-1))$ and $(C2(\ell-1))$ hold for ρ' and prove the statement for ℓ.

Consider the vertex b_ℓ, where $\ell < t$. This vertex has unique prohibition on color $c(\ell+1)$ in the coloring ρ. Assume that this is a prohibition of type 1 and a vertex $w \in \mathrm{N}'(b_\ell)$ is such that $\rho(w) = c(\ell+1)$. By conditions $(C1(\ell))$ and $(C2(\ell))$ for the coloring ρ, in the case we consider there is no vertex $v \in \mathrm{N}'(b_i)$, all neighbors of which have colors 1 and 2 in ρ. Hence, there is no such vertex in the coloring ρ'. Thus, condition $(C2(\ell))$ holds for the coloring ρ'. If $w' \in \mathrm{N}'(b_\ell)$ and $\rho'(w') \in \{1,2\}$, then $\rho(w') \in \{1,2\}$, and, therefore, $w' = w$. If $\rho'(w) \notin \{1,2\}$ then there is no prohibition on color $c(\ell+1)$ for the vertex b_ℓ in the coloring ρ' and by Lemma 7 there is a coloring $\rho'' <_G \rho$. Let $\rho'(w) \in \{1,2\}$. If $\rho'(w) \neq \rho(w)$ then $\rho'(w) = \rho(b_\ell) = \rho'(b_\ell)$ and the coloring ρ' is not proper. We obtain a contradiction. Hence, $\rho'(w) = \rho(w) = c(\ell+1)$ and condition $(C1(\ell))$ holds for the coloring ρ'.

Assume that the prohibition on color $c(\ell+1)$ for the vertex b_ℓ in ρ has type 2 and basic vertex v. Then all neighbors of v have colors 1 and 2 in ρ and such vertex in $\mathrm{N}'(b_\ell)$ is unique up to condition $(C2(\ell))$ for the coloring ρ. Hence, if a vertex of $\mathrm{N}'_G(b_\ell)$ have in the coloring ρ' all neighbors of colors 1 and 2, then this vertex is v. Thus, condition $(C2(\ell))$ holds for ρ' and the vertex b_ℓ has at most one prohibition on color $c(\ell+1)$ in ρ'. If there is such prohibition, then condition $(C1(\ell))$ holds for ρ'. If there is no such prohibition, then by Lemma 7 there exists a coloring $\rho'' <_G \rho$.

In the case where $t = s$ Lemma is proved. If $t < s$, then we consider the vertex b_{t+1}, which was recolored. Hence, the only prohibition on the color $c(t+1)$ for the vertex b_t disappears in ρ'. By condition $(C2(t))$ for the coloring ρ, a new prohibition on color $c(t+1)$ for the vertex b_t cannot appear. Hence, by Lemma 7 there exists a coloring $\rho'' <_G \rho$. □

LEMMA 9. *Let G' be a subgraph of G, obtained upon deleting several edges, such that $d_{G'}(a) \geq 2$ (maybe, $G' = G$). Let ρ and ρ' be vertex colorings of the graph G. Assume that the coloring ρ' is obtained from the coloring ρ in one of the following ways:*

- *as a result of applying the algorithm $DN(G', \rho, x, J)$, where either $1, 2 \notin J$ or $\{j_0 = j_4, j_2\} = \{1,2\}$, or $\{j_1 = j_5, j_3\} = \{1,2\}$;*
- *as a result of applying the algorithm $DN(G', \rho_I, x, J)$, where $I \notin J$, and unglying of the color I.*

Then the pair of colorings ρ and ρ' satisfies the condition of Lemma 8.

PROOF. Since $d_{G'}(a) \geq 2$, the vertex a is a bad vertex in the coloring ρ of the graph G'. By Lemma 2, there is a vertex in $\mathrm{N}_G(a)$ which is not recolored by the algorithm DN.

Let $\rho'(v) = j \in \{1,2\}$ and $\rho'(v) \neq \rho(v)$. Then the vertex v was recolored, we have the first case of condition of Lemma and $j \in J = \{j_2, j_3, j_4, j_5\}$. By the condition of Lemma and by properties of the algorithm DN it is clear that $\rho(v) \in \{1,2\}$. □

4.4. A step of the main algorithm

The current coloring of vertices which the algorithm deals with we denote by ρ^1. Before the beginning of the first step (choice of the vertex b_2), set $\rho^1 = \rho$. On some steps, the algorithm will change the current coloring such that the condition $\rho^1 \leq_G \rho$ always holds.

Assume that vertices $b_1, \dots, b_{k-1}$ (where $k \geq 2$) are chosen, pairwise different and $\rho^1(b_i) = c(i)$ for all $i \in \{1, \dots, k-1\}$. The chain of prohibitions is constructed such that before the beginning of the step on which b_k will be chosen conditions $(C1(k-1))$ and $(C2(k-1))$ hold for the current coloring ρ^1.

There are two possible results of the step of the main algorithm:

- a vertex b_k will be chosen and the coloring ρ^1 will be changed such that the chain of prohibitions $b_1, \dots, b_k$ will satisfy conditions $(C1(k))$ and $(C2(k))$;
- a coloring ρ'' will be constructed, such that $\rho'' <_G \rho$.

In the second case the algorithm stops, in the first case the algorithm will pass to the next step — choice of b_{k+1}.

A. Choice of the vertex b_k

By condition $(C1(k-1))$, the vertex b_{k-1} has unique prohibition on the color $c(k)$. Consider this prohibition.

If it is a prohibition of type 1, then there exists a unique vertex $u \in \mathrm{N}'_G(b_{k-1})$, colored with $c(k)$. In this case, set $b_k = u$, $\mathrm{asc}(b_k) = b_{k-1}$.

Suppose b_{k-1} has a prohibition of type 2 on color $c(k)$ with basic vertex v. Then all vertices of the set $\mathrm{N}'(v)$ have color $c(k)$. We choose as b_k any of these vertices and set $\mathrm{asc}(b_k) = a_{k-1} = v$.

If vertices $b_1, \dots, b_k$ are pairwise different, we pass to step B. Otherwise, we pass to step A.1, where the algorithm will stop.

A.1. *Repetition of a vertex*

Assume that $b_k = b_t$, where $t < k$. Recall that the vertices $b_1, \dots, b_{k-1}$ are pairwise different. Consider the following coloring ρ': we set $\rho'(b_i) = c(i+1)$ for i from 1 to $k-1$ and $\rho'(v) = \rho^1(v)$ for the remaining vertices v.

Since the coloring ρ^1 satisfies conditions $(C1(k-1))$ and $(C2(k-1))$, the coloring ρ' satisfies the conditions of Lemma 6 for $s = k-1$. We claim that ρ' is a proper coloring and $\rho' <_G \rho$.

Assume that the coloring ρ' contains two adjacent vertices of the same color. By Lemma 6, one of these vertices is b_{k-1} whose color became $c(k)$ and the second vertex is a vertex $v \in \mathrm{N}'(b_{k-1})$ such that $\rho'(v) = \rho^1(v) = c(k)$, i.e., the vertex v imposes a prohibition of type 1 on color $c(k)$ for the vertex b_{k-1} in the coloring ρ^1. By our construction, this means that $v = b_k = b_t$, and this vertex was recolored with color $c(t+1) = c(k+1)$. Then Lemma 6 implies that ρ' is a proper coloring.

Assume that a new bad vertex v appeared in the coloring ρ'. By Lemma 6, $v = a_{k-1}$, and all vertices in $\mathrm{N}'(a_{k-1})$ have color $c(k)$ in the coloring ρ'. By construction, the vertex a_{k-1} is adjacent to $b_k = b_t$, and $\rho'(b_t) = c(t+1) = c(k+1) \neq c(k)$. Then the vertices $b_t, b_{k-1} \in \mathrm{N}_G(v)$ have different colors in the coloring ρ', and we get a contradiction.

By Lemma 6, we have $\rho' <_G \rho^1 \leq_G \rho$. The algorithm stops.

In what follows we consider the case where the vertices $b_1, \dots, b_{k-1}, b_k$ are pairwise different.

C. Changing the coloring ρ^1

The step C will provide one of the three following results:

- the coloring ρ^1 will be modified such that b_k has exactly one prohibition on color I in ρ^1_I;
- the coloring ρ^1 will be modified such that b_k has no prohibition on a certain color i in ρ^1_I;
- a coloring $\rho'' <_G \rho$ will be constructed, the algorithm will stop.

Step C has complicated structure shown on Figure 4. In what follows we write down details of this step.

Let us start. Consider a coloring ρ^1_I, obtained from ρ^1 upon gluing colors 1 and 2.

Definition 15. *Denote by U the set of all vertices of* $\mathrm{N}'(b_k)$, *which are basic vertices of prohibitions of type* 2 *for b_k in the coloring ρ^1_I. Set $T = \mathrm{N}'(b_k) \setminus U$ and $p = |U|$.*

Remark 4.3. In particular, if $y \in \mathrm{N}'(b_k)$ is such that all vertices of the set $\mathrm{N}'(y)$ have colors 1 and 2 in ρ^1, then $y \in U$ and $z2(\rho^1_I, y) = I$.

Definition 16. *Define a collection of colors Z as follows: for each vertex $u \in U$ we put in Z the color $z2(\rho^1_I, u)$, and for each vertex $t \in T$ we put in Z its color $\rho^1_I(t)$.*

Remark 4.4. Clearly, $|Z| = |\mathrm{N}'(b_k)| \leq d - 1$. Note that a color can occur in the collection Z more than once.

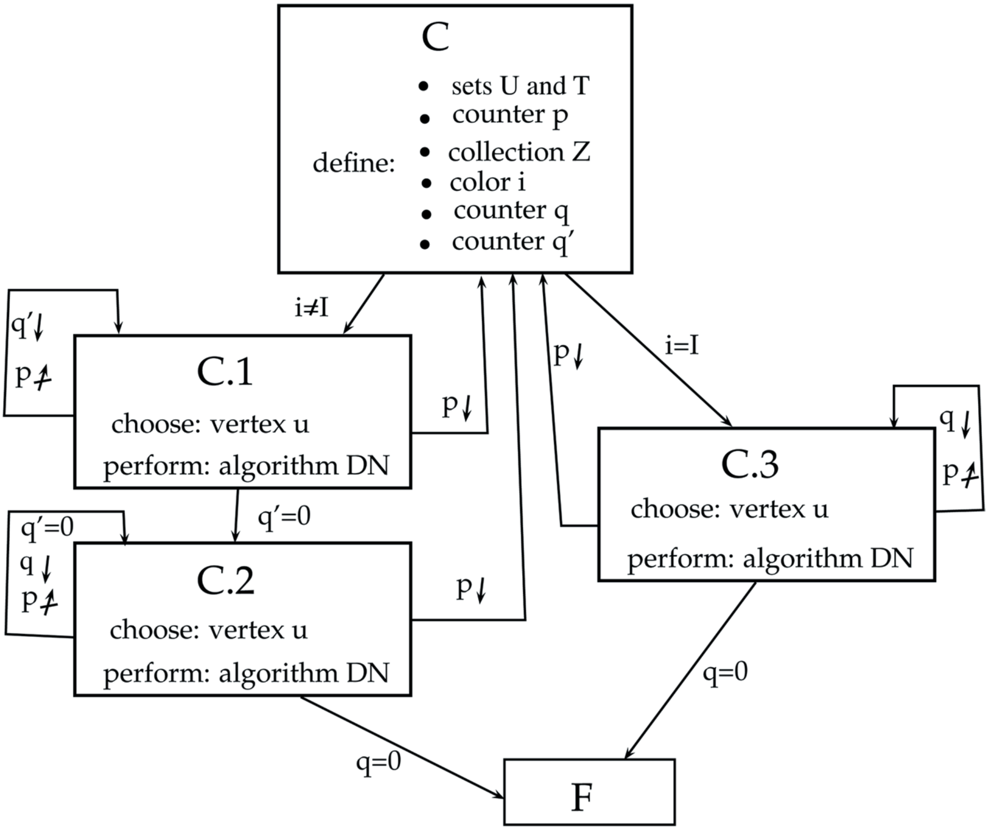

Fig. 4. Scheme of Step C

DEFINITION 17.

1) *If the color I occurs in Z at most once, we set $i = I$.*
   Let I occurs in Z at least twice. Since $|Z| \leq d - 1$, and the number of colors in ρ_I^1 is $d - 1$ (recall that colors 1 and 2 are glued together into the color I), one of the colors does not occur in Z, let it be i.
2) *Denote by q the number of vertices $u \in U$ such that $\rho_I^1(u) = i$.*
3) *A vertex $u \in \mathrm{N}'(b_k)$ is called i-*regular *for the coloring ρ_I^1, if $\rho_I^1(u) = i$ and $z2(\rho_I^1, u) \neq I$. Denote by q' the number of i-regular vertices for ρ_I^1.*

Remark 4.5. If x is an i-regular vertex for ρ_I^1, then $x \in U$. Hence, $q' \leq q$.

The counters p, q and q' play an important role in Step C of the algorithm.

Remark 4.6. During Step C we will many times change the current coloring, applying algorithm DN to a graph G', obtained from G upon deleting several edges incident to the vertex b_k and to the coloring ρ^1 or to the coloring ρ^1_I. Clearly, at most one deleted edge can be incident to the vertex a, therefore, $d_{G'}(a) \geq 2$. In all cases, the set of colors J contains none of colors 1, 2 and I. Such graph G' and set J satisfy the conditions of Lemma 9. Hence, Lemma 8 can be applied to current and new colorings.

We will always consider the case, where b_1, ..., b_k is a chain of prohibitions in the new coloring, satisfying conditions $(C1(k-1))$ and $(C2(k-1))$. In the other case, by Lemma 8 there exists a coloring ρ', such that $\rho' <_G \rho$ and the algorithm stops.

C.1. $i \neq I$

If $q' = 0$, then we pass to Step C.2. If $q' \neq 0$, then we choose a vertex $u \in U$ such that $\rho^1_I(u) = i$ and $z2(\rho^1_I, u) \neq I$.

We want to change the coloring ρ^1 such that in a new coloring $b_1, \dots, b_k$ will be a chain of prohibitions satisfying conditions $(C1(k-1))$ and $(C2(k-1))$, and either p will decrease, or p will be preserved and q' will decrease. (Note that the color i can be changed in one of the cases. The counter q may increase.)

C.1.1. *Choice of colors for the algorithm DN*

Set $j_4 = j_0 = i$, $j_5 = j_1 = z2(\rho^1_I, u)$. Let C^* be the set of all colors of ρ^1_I, except for I, j_0 and j_1. Then $|C^*| = d - 4 \geq 2$. It remains to choose colors $j_2, j_3 \in C^*$. We want to do it such that condition (J) holds (see Lemma 3) and condition (P) doesn't hold.

(P) There exists a vertex $x \in T$, such that $\rho^1(x) = j_2$ and vertices of the set $\mathrm{N}'(x)$ are colored in ρ^1_I with colors j_1 and j_3.

Let T^* be the set of all vertices of T which color in ρ^1_I is not I. Since color I occurs in the collection Z at least twice and b_k has at least one prohibition of type 2 on color $j_1 \neq I$ (with basic vertex u), then $|T^*| \leq d - 4$.

Assume that $T^* = \varnothing$. Then either $T = \varnothing$ or all vertices of T have color I in ρ^1_I. If there is a vertex $y \in \mathrm{N}_G(b_k)$ of color $s \neq i$ (maybe $s = I$), then we choose as j_2 any color of C^* different from s and after that we choose as j_3 any color of C^* different from j_2. Since $|C^*| \geq 2$ this choice is possible. Clearly, condition (J) holds.

Assume that $T^* \neq \varnothing$. If there is a color of the set C^* which is not presented in the coloring ρ^1 among vertices of the set T^*, let j_2 be this color. Since $|C^*| \geq 2$ we can choose color $j_3 \in C^*$ different from j_2. Clearly, condition (P) does not hold. Recall that color $j_4 = i \notin Z$. Hence, there is no prohibition of type 1 on this color, and, consequently, j_4 is not presented in the coloring ρ^1 in $T^* \subset T$. Since $T^* \neq \varnothing$, then any its vertex have in ρ^1_I color different from j_2 and j_4 and condition (J) holds.

In the remaining case, all colors of the set C^* are presented in ρ^1_I among vertices of the set T^*. Then $|T^*| = |C^*| = d - 4$ and for any color of C^* there is exactly one vertex of this color in T^*. Since $|C^*| = d - 4 \geq 2$ and $j_4 = j_0 \notin C^*$, in this case, condition (J) holds for any choice of j_2 and j_3.

Lemma 10. *Assume that for any choice of colors $j_2, j_3 \in C^*$ condition (P) holds. Then the number of colors $d = 6$ and one can enumerate vertices in $\mathrm{N}'(b_k)$ and colors such that*

$$\begin{aligned} \mathrm{N}'(b_k) = \{x_1, x_2, x_3, x_4, x_5\}, \quad i = j_0 = 3, \quad j_1 = 6, \quad \rho^1_I(x_3) = i = 3,\\ T^* = \{x_4, x_5\}, \quad \rho^1_I(x_4) = 4, \quad \rho^1_I(x_5) = 5. \end{aligned} \tag{4.1}$$

Moreover, in the coloring ρ^1_I:

- *all vertices of the set $\mathrm{N}'(x_3)$ have color $j_1 = 6$;*
- *all vertices of the set $\mathrm{N}'(x_4)$ have colors 5 and 6;*
- *all vertices of the set $\mathrm{N}'(x_5)$ have colors 4 and 6;*
- *each of vertices x_1 and x_2 either imposes a prohibition of type 1 on color I for b_k or is a basic vertex of a prohibition of type 2 on color I for b_k.*

Proof. Since for any choice of colors j_2 and j_3 condition (P) holds, for any vertex $x \in T^*$ the set $\mathrm{N}'(x)$ is colored in ρ^1_I with two colors and j_1 is among them. We choose any color of the set C^* as j_2, let $x \in T^*$, $\rho^1_I(x) = j_2$. If condition (P) holds for a certain choice of the color j_3, then this color must be the color presented in the coloring ρ^1_I among vertices of $\mathrm{N}'(x)$ and different from j_1. There is only one such color. Hence, if we cannot choose j_3 such that (P) does not hold, the number of colors in ρ^1_I is equal to 5 (we have colors I, j_0, j_1, j_2 and the only variant for j_3). Therefore, $d = 6$.

Let's enumerate colors and vertices. Set $u = x_3$, $i = j_0 = \rho^1(u) = 3$, $z2(\rho^1, u) = 6$ (see Figure 5a). Recall that $x_3 \notin T^*$, $|T^*| = d - 4$ and $|\mathrm{N}'(b_k)| = d - 1$.

Therefore, $\mathrm{N}'(b_k)$ contains at most two vertices outside $T^* \cup \{x_3\}$. However, b_k has at least two prohibitions on color I (since we have chosen $i \neq I$). Vertices of the set $T^* \cup \{x_3\}$ can neither impose a prohibition of type 1 on color I for b_k, nor be a basic vertex of a prohibition of type 2 on color I for b_k. Hence, there are exactly two prohibitions on color I for b_k, let each of vertices x_1 and x_2 either imposes a prohibition of type 1 on color I for b_k or is a basic vertex of a prohibition of type 2 on color I for b_k.

Two remaining vertices of the set $\mathrm{N}'(b_k)$ form the set $T^* = \{x_4, x_5\}$. Let $\rho_I^1(x_4) = 4$, $\rho_I^1(x_5) = 5$.

As we know, $\mathrm{N}'(x_4)$ is colored in ρ_I^1 with two colors, one of them is $j_1 = 6$. The other belong to C^*, i.e., it is different from I and $i = 3$. Moreover, it is also different from $\rho_I^1(x_4) = 4$, therefore, this color is 5. Similarly, $\mathrm{N}'(x_5)$ is colored in ρ_I^1 with colors 4 and 6.

The resulting configuration is shown on Figure 5a. □

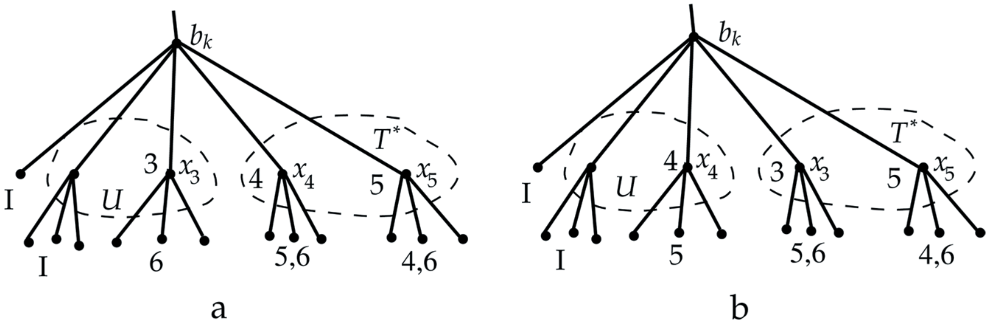

Fig. 5. Problems with condition (P). On the left side the coloring ρ_I^1 is shown, on the right — the coloring ρ_I^2

In the case described in Lemma 10, we set $j_2 = 4$, $j_3 = 5$.

C.1.2. *Construction of the graph G' and applying the algorithm DN*

Let T' be the set of all vertices $x \in T$, such that vertices of the set $\mathrm{N}'(x)$ are colored in ρ_I^1 with colors j_0 and j_2. Let U' be the set of all vertices $x \in U$, such that $z2(\rho_I^1, x) = j_2$. Set

$$F = \{b_k u\} \cup \{b_k x \ : \ x \in T' \cup U'\}, \quad G' = G - F.$$

Let the algorithm $DN(G', \rho_I^1, J, u)$ change the coloring ρ_I^1 to ρ_I^2. Note that $\rho_I^1(b_k) = I \notin J$ and condition (J) holds. Hence by Lemma 3 we have

$\rho_I^2(b_k) = \rho_I^1(b_k)$ and $\rho_I^2 \leq_G \rho_I^1$. Since $I \notin J$, we can unglue the color I and by Lemma 4 obtain a proper coloring ρ^2 of the graph G, such that $\rho^2 \leq_G \rho^1$.

Consider two cases.

C.1.2.1. *Condition* $(DN1)$ *holds*
Then $\rho_I^2(u) = j_2$. By condition $(DN1)$, the only possible recolorings are from the color $i = j_4$ to the color j_2 and vice versa.

Only vertices of the color j_2 can be recolored with color i. Moreover, the neighborhood of any such vertex must be colored with colors $j_1 = j_5$ and j_3, which are different from I. Thus, in $\mathrm{N}'(b_k)$ only vertices which are not adjacent in G' to the vertex b_k of color I can be recolored with color i — namely, the vertex u and vertices of the set $T' \cup U'$. But each vertex of the set $T' \cup U'$ is adjacent to at least one vertex having in ρ_I^1 color j_2. Hence, vertices of $T' \cup U'$ cannot have in ρ_I^1 color j_2. Therefore, no new prohibition of type 1 on color i for b_k appears in ρ_I^2.

Prove that no new prohibition of type 2 for b_k appears in ρ_I^2. Assume the converse, let such prohibition with basic vertex x appear. Clearly, $x \in T$. Then $\mathrm{N}_{G'}(x)$ must be colored in ρ_I^1 with colors i and j_2, i.e., $x \in T'$. Hence, in the coloring ρ_I^2 of the graph G' a new bad vertex x appears after applying the algorithm DN. This is impossible.

Prove that no new prohibition of type 2 on color i for b_k appears in ρ_I^2. Assume the converse, let such prohibition with basic vertex x appear. By the proved above, $x \in U$ (otherwise, a new prohibition of type 2 for b_k appears). Then, clearly, $z2(\rho_I^1, x) = j_2$, i.e. $x \in U'$. Thus, x is a bad vertex in the coloring ρ_I^1 of the graph G' and the algorithm DN has recolored all vertices in $\mathrm{N}_{G'}(x)$. This is impossible by Lemma 2.

Prove that if a vertex x is i-regular for ρ_I^2 it is also i-regular for ρ_I^1. Let $x \in \mathrm{N}'(b_k)$, $\rho_I^2(x) = i$ and $z2(\rho_I^2, x) = j \neq I$. By the proved above, then $x \in U$ and $\rho_I^1(x) = i$. Let $z2(\rho_I^1, x) = j'$. Since both colorings ρ^1 and ρ^2 are proper, we have $j \neq i$ and $j' \neq i$. If $j' \neq j$, then by condition $(DN1)$ we have $\{j', j\} = \{j_2, i\}$. This contradicts proved above. Hence, $j' = j \neq I$ and the vertex x is i-regular for ρ_I^1.

Thus, the new coloring $\rho^2 \leq_G \rho^1$ has the same value of counter p as ρ^1. By the proved above, for the same color i we have the counter q' decreased (recall that u is recolored). Set $\rho^1 = \rho^2$ and return to Step C.1.

C.1.2.2. *Condition* $(DN2)$ *holds*

In this case, the only possible recolorings are from the color $j_1 = j_5$ to the color j_3 and vice versa. Moreover, the neighborhood of any recolored vertex must be colored in ρ^1_I with colors j_2 and $j_4 = j_0 = i$, different from I.

By condition $(DN2)$, in the coloring ρ^2_I exactly one vertex of the set $\mathrm{N}'(u)$ has color j_3, all others are colored with j_1. Therefore, u is not a basic vertex of a prohibition of type 2 for b_k in the coloring ρ^2_I. If no new prohibition of type 2 for b_k appears in ρ^2_I, then the counter $p = |U|$ is decreased. In this case, set $\rho^1 = \rho^2$ and return to Step C.

Assume that a new prohibition of type 2 for b_k appears, let $x \in T$ be its basic vertex. Then in the coloring ρ^1 the set $\mathrm{N}'(x)$ must be colored with colors j_1 and j_3 and at least one vertex of the set $\mathrm{N}'(x)$ must be recolored. Since neighbors of a recolored vertex have colors j_2 and $i = j_4$ in the coloring ρ^1_I, and color i is not presented in the coloring ρ^1_I among the vertices of T, we have $\rho^1_I(x) = j_2$. Therefore, condition (P) holds. By the choice of colors for the algorithm DN and Lemma 10, this is possible only in the case described in this Lemma. Hence, $d = 6$. Moreover, vertices of $\mathrm{N}'(b_k)$ and colors can be enumerated such that we obtain the configuration shown on Figure 5a. Then for the coloring ρ^2_I the vertex $u = x_3$ occurs in the set T and the vertex $x = x_4$ occurs in the set U (see Figure 5b). Thus, the counter $p = |U|$ for colorings ρ^1 and ρ^2 is the same.

Vertices of the set $\mathrm{N}'(x_4)$ in ρ^1_I have colors j_1 and j_3. Let's return to details of the algorithm DN (Lemma 1) and remember that only vertices of the set D^* were recolored, and all neighbors of such vertices belong to the set D. In particular, $x_4 \in D$. However, the vertex $x_4 \notin T'$ is adjacent in the graph G' to b_k and $\rho^1_I(b_k) = I \notin \{j_1, j_3\}$. Therefore, $x_4 \notin D^*$. Hence, only neighbors of x_4 having in ρ^1_I color $j_1 = j_5$ can belong to D^* (recall that $\rho^1_I(x_4) = j_2$). Therefore, in the coloring ρ^2_I all vertices of the set $\mathrm{N}'(x_4)$ have color $j_3 = 5$. The configuration obtained is shown on Figure 5b.

Now set $\rho^1 = \rho^2$. As it was written above, the counter p is not changed. Construct the collection of colors of prohibitions Z for the new coloring. Clearly, color 4 is absent in this collection. Then we set $i = 4 \neq I$, $u = x_4$. Hence, $j_1 = z2(\rho^1_I, x_4) = 5$. Note that we have $q' = 1$ for both previous and new coloring. Thus, the parameters p and q' are preserved.

Return to Step C.1.1 with the new coloring. Now $T^* = \{x_3, x_5\}$ (see Figure 5b) and color $6 \notin \{j_0, j_1, I\}$ is not presented among vertices of T^*. Then we can set $j_2 = 6$, $j_3 = 3$ and condition (P) does not hold. Let us perform Step C.1.2 (i.e., the next algorithm DN). Since condition (P) does not hold, by the proved above in this case we can either decrease p or preserve p and, at the same time, decrease q'.

Remark 4.7. Let us show that each Step C.1 finishes its work. The main operation of the step is applying of the algorithm DN. Except for the case described in Lemma 10, each applying of DN in Step C.1 either leads to decrease of the counter p (in this case, we return to the beginning of the step C), or leads to decrease of the counter q' together with preserving p (in this case, we return to the beginning of Step C.1). The only case when no counter decreases is the case described in Lemma 10 and shown on fig. 5a. However, in this case, we perform one more algorithm DN which leads to desired decrease of counters.

C.2. $i \neq I$, $q' = 0$

If $q = 0$, then we pass to step F. If $q \neq 0$, we choose a vertex $u \in U$ of color i. Let $j_1 = z2(\rho_I^1, u)$. Since $q' = 0$, the only possible variant is $j_1 = I$.

We want to modify the coloring ρ^1, such that in the new coloring $b_1, \dots, b_k$ will be a chain of prohibitions satisfying conditions $(C1(k-1))$ and $(C2(k-1))$, and either p will decrease, or the counter p and the color i will be preserved, $q' = 0$ and q will decrease.

C.2.1. *Choice of colors for the algorithm* DN

Set $j_4 = j_0 = i$. In our case, $j_1 = I$. We want to choose colors j_2, j_3 and j_5, such that condition (J) holds.

Let T^* be the set of all vertices of T, which color in ρ_I^1 is not I. Since color I occurs in the collection Z at least twice, we have $|T^*| \leq d - 3$.

Let C^* be the set of all colors of ρ_I^1, different from I and i. Then $|C^*| = d - 3 \geq 3$.

Assume that $T^* = \varnothing$. Then either $T = \varnothing$ or all vertices of T have color I in ρ_I^1. In this case for any choice of colors $j_2, j_3, j_5 \in C^*$ condition (P) doesn't hold. If there is a vertex $y \in \mathrm{N}_G(b_k)$ of color $s \neq i$ (maybe $s = I$), then we choose as j_2 any color of C^* different from s and after that we choose as j_3 and j_5 any two distinct colors of C^* different from j_2. Since $|C^*| \geq 3$ this choice is possible. Clearly, condition (J) holds.

Assume that $T^* \neq \varnothing$. If there is a color of the set C^* which is not presented in the coloring ρ_I^1 among vertices of the set T^*, then let j_2 be this color. Since we have at least 5 colors, we can choose colors $j_3, j_5 \notin \{I, j_0, j_2\}$. Recall that color $j_4 = i \notin Z$. Hence, there is no prohibition of type 1 on this color for the vertex b_k in ρ_I^1, and, consequently, j_4 is not presented in the coloring ρ_I^1 in $T^* \subset T$. Since $T^* \neq \varnothing$, then any its vertex have in ρ_I^1 color different from j_2 and j_4 and condition (J) holds.

In the remaining case, all colors of the set C^* are presented in ρ_I^1 among vertices of the set T^*. Then $|T^*| = |C^*| = d - 3$ and for any color of C^* there is exactly one vertex of this color in T^*. Since $|C^*| \geq 3$, in this case, condition (J) will be satisfied for any choice of j_2, j_3 and j_5. Choose any three different colors $j_2, j_3, j_5 \in C^*$.

C.2.2. *Construction of the graph G' and applying the algorithm DN*
Let T' be the set of all vertices $x \in T$, such that vertices of the set $\mathrm{N}'(x)$ are colored in ρ_I^1 with two colors: $j_0 = i$ and j_2. Let U' be the set of all vertices $x \in U$, such that $z2(\rho_I^1, x) = j_2$. Set

$$F = \{b_k u\} \cup \{b_k x \ : \ x \in T' \cup U'\}, \quad G' = G - F.$$

Let the algorithm $DN(G', \rho_I^1, J, u)$ change the coloring ρ_I^1 to ρ_I^2. Note that $\rho_I^1(b_k) = I \notin J$ and condition (J) holds. Hence by Lemma 3 we have $\rho_I^2(b_k) = \rho_I^1(b_k)$ and $\rho_I^2 \leq_G \rho_I^1$. Since $I \notin J$, we can unglue the color I and by Lemma 4 obtain a proper coloring ρ^2 of the graph G, such that $\rho^2 \leq_G \rho^1$.

Consider two cases.

C.2.2.1. *Condition $(DN1)$ holds*
In this case, $\rho_I^2(u) = j_2$. By condition $(DN1)$, the only possible recolorings are from the color $i = j_0 = j_4$ to the color j_2 and vice versa.

Only vertices of the color j_2 can be recolored with color i. Recall that $\rho_I^1(b_k) = I$ and b_k is not adjacent in the graph G' to u. Therefore, the neighborhood of any vertex recolored with color i cannot contain the vertex b_k. Thus, in $\mathrm{N}'(b_k)$ only vertices which are not adjacent in G' to the vertex b_k of color I can be recolored with color i — namely, the vertex u and vertices of the set $T' \cup U'$. But each vertex of the set $T' \cup U'$ is adjacent to at least one vertex having in ρ_I^1 color j_2. Hence, vertices of $T' \cup U'$ cannot have in ρ_I^1 color j_2. Therefore, no new prohibition of type 1 on color i for b_k appears in ρ_I^2.

Prove that no new prohibition of type 2 for b_k appears in ρ_I^2. Assume the converse, let such prohibition with basic vertex x appears. Clearly, $x \in T$. Then $\mathrm{N}_{G'}(x)$ must be colored in ρ_I^1 with colors i and j_2, i.e., $x \in T'$. Hence, in the coloring ρ_I^2 of the graph G' a new bad vertex x appears after applying the algorithm DN. This is impossible.

Prove that no new prohibition of type 2 on color i for b_k appears in ρ_I^2. Assume the converse, let such prohibition with basic vertex x appears. By the proved above, $x \in U$. Then, clearly, $z2(\rho_I^1, x) = j_2$, i.e. $x \in U'$. Thus, x is a bad vertex in the coloring ρ_I^1 of the graph G' and the algorithm DN has recolored all vertices in $\mathrm{N}_{G'}(x)$. This is impossible by Lemma 2.

Prove that there is no i-regular for ρ_I^2 vertex. Assume the converse, let $x \in \mathrm{N}'(b_k)$, $\rho_I^2(x) = i$ and $z2(\rho_I^2, x) = j \neq I$. By the proved above, then $x \in U$ and $\rho_I^1(x) = i$. Let $z2(\rho_I^1, x) = j'$. Since both colorings ρ^1 and ρ^2 are proper, we have $j \neq i$ and $j' \neq i$. If $j' \neq j$, then by condition $(DN1)$ we have $\{j', j\} = \{j_2, i\}$. This contradicts proved above. Thus $j' = j$, whence x is an i-regular vertex for the coloring ρ_I^1. This contradicts $q' = 0$.

Thus, the new coloring $\rho^2 \leq_G \rho^1$ has the same value of the counter p as ρ^1. For the same color i we have the counter q decreased (recall that u is recolored). Set $\rho^1 = \rho^2$ and return to Step C.2. It follows fom the proved above that $q' = 0$ for the new coloring. Therefore, the return to Step C.2 is correct.

C.2.2.2. *Condition* $(DN2)$ *holds*

In this case, the only possible recolorings are from colors j_1 and j_5 to the color j_3 and from color j_3 to color j_5. Moreover, the neighborhood of any recolored vertex must be colored in ρ_I^1 with colors j_2 and $j_4 = j_0 = i$, different from I.

By condition $(DN2)$, in the coloring ρ_I^2 exactly one vertex of the set $\mathrm{N}'(u)$ has color j_3, all others are colored with $j_1 = I$. Therefore, u is not a basic vertex of a prohibition of type 2 on color I for b_k in the coloring ρ_I^2. If no new prohibition of type 2 appears in ρ_I^2, then the counter $p = |U|$ is decreased. In this case, set $\rho^1 = \rho^2$ and return to Step C.

Assume that a new prohibition of type 2 for b_k appears, let $x \in T$ be its basic vertex. Since neighbors of a recolored vertex have colors j_2 and $i = j_4$ in the coloring ρ_I^1, and color i is not presented in the coloring ρ_I^1 among vertices of T, we have $\rho_I^1(x) = j_2$. By the construction, it is possible only in the case, where $|T^*| = d - 3$ and for each color $j \in C^*$ there is exactly one vertex in T^* colored with j in the coloring ρ_I^1. Then $|\mathrm{N}'(b_k) \setminus T^*| \leq 2$.

Let's study the coloring ρ_I^1. As we know, the vertex b_k has at least two prohibitions on color I in ρ_I^1. Vertices of the set T^* do not impose such prohibitions. Hence, $|\mathrm{N}'(b_k) \setminus T^*| = 2$. Moreover, the vertex b_k has exactly two prohibitions on color I in the coloring ρ_I^1, and each of two vertices of the set $\mathrm{N}'(b_k) \setminus T^*$ either imposes a prohibition of type 1 on color I for b_k in the coloring ρ_I^1, or is a basic vertex of a prohibition of type 2 on color I for b_k in the coloring ρ_I^1 (the vertex u is just such basic vertex). In particular, in the coloring ρ_I^1 there is no vertex of color I in the set U (see Figure 6a).

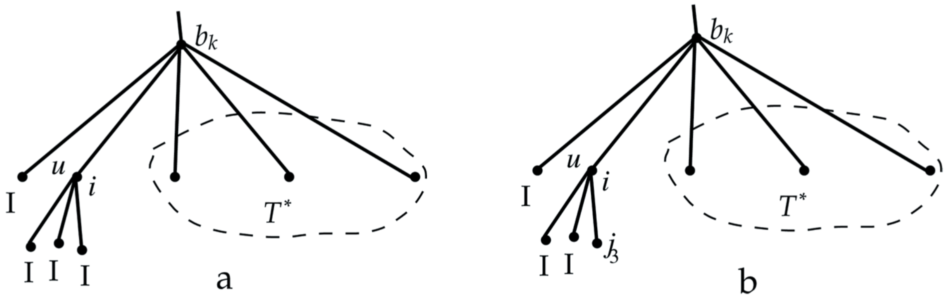

Fig. 6. Prohibitions on color I in the colorings ρ_I^1 (on the left) and ρ_I^2 (on the right)

Thus, the vertex b_k in the coloring ρ_I^1 has exactly two prohibitions on color I. One of these prohibitions (of type 2 with basic vertex u) disappears in the coloring ρ_I^2 (see Figure 6b). Since no vertex was recolored with color I, no new prohibition on color I appears in ρ_I^2. Moreover, in the coloring ρ_I^2 no vertex of the set U has color I. Thus, b_k has exactly one prohibition on color I in the coloring ρ_I^2. Then set $\rho^1 = \rho^2$, $i = I$ and pass to Step F.

Remark 4.8. Let us show that each Step C.2 finishes its work. The main operation of the step is applying of the algorithm DN. Except for one exclusion, each applying of DN in Step C.2 either leads to decrease of the counter p (in this case, we return to the beginning of Step C), or leads to decrease of the counter q together with preserving of color i and counter p. In the last case, we have $q' = 0$ for the new coloring and return to the beginning of Step C.2. The only case where no counter decreases is the case shown on fig. 6a. However, in this case, we obtain a coloring in which the vertex b_k has exactly one prohibition on color I and we pass to Step F with this coloring.

C.3. $i = I$

Denote by q the number of vertices $u \in U$, such that $\rho_I^1(u) = I$. If $q = 0$, then we pass to step F. If $q \neq 0$, we choose a vertex $u \in U$ of color I. Let $j_1 = z2(\rho_I^1, u)$.

We want to modify the coloring ρ^1, such that in a new coloring $b_1, \dots, b_k$ will be a chain of prohibitions satisfying conditions $(C1(k-1))$ and $(C2(k-1))$, and either p will decrease, or p will be preserved and q will decrease.

Consider two cases.

C.3.1. $j_1 \neq I$

Set $j_0 = I$ and $j_5 = j_1$. It remains to choose colors j_2, j_3 and j_4, different from I and j_1, such that condition (J) holds.

Let C^* be the set of all colors of the coloring ρ_I^1, different from I and j_1. Then $|C^*| = d - 3 \geq 3$. If there is a vertex $y \in \mathrm{N}_G(b_k)$ of color $\rho_I^1(y) = s \neq I$, then choose as j_2 and j_4 two colors of C^*, different from s and any other color $j_3 \in C^*$. Clearly, condition (J) holds in this case. Otherwise, all vertices of $\mathrm{N}_G(b_k)$ have in ρ_I^1 color I, i.e. b_k is a bad vertex in the coloring ρ_I^1 of the graph G and condition (J) also holds.

Let G' be the graph obtained from G upon deleting all edges joining b_k to vertices of T.

Let the algorithm $DN(G', \rho_I^1, J, u)$ change the coloring ρ_I^1 to ρ_I^2. Since condition (J) holds, by Lemma 3 we have $\rho_I^2(b_k) = \rho_I^1(b_k)$ and $\rho_I^2 \leq_G \rho_I^1$. Since $I \notin J$, we can unglue the color I and by Lemma 4 obtain a proper coloring ρ^2 of the graph G, such that $\rho^2 \leq_G \rho^1$.

Prove that no new prohibition of type 2 for the vertex b_k appears in the coloring ρ_I^2. Assume the converse, let such a prohibition with basic vertex x appear. Then $x \in T$. Hence a new bad vertex x appears in the graph G' after applying the algorithm DN. This is impossible.

Consider two cases.

C.3.1.1. *Condition* $(DN1)$ *holds*

Since $J \not\ni I$, no new vertex of color I appears in $\mathrm{N}'(b_k)$. Hence, no new prohibition of type 1 or type 2 on color I for the vertex b_k appears in the coloring ρ_I^2. Since $\rho_I^2(u) = j_2$, the counter q is decreased by 1.

It is proved above that no new prohibitions of type 2 for b_k appear in ρ_I^2. Thus, we obtained a new coloring $\rho_I^2 \leq_G \rho_I^1$ with the same value of the counter p as ρ_I^1. For the same color $i = I$ we have the counter q decreased by 1 (recall that u is recolored). Set $\rho^1 = \rho^2$ and return to Step C.3.

C.3.1.2. *Condition* $(DN2)$ *holds*

In the coloring ρ_I^2 exactly one vertex of the set $\mathrm{N}'(u)$ has color $j_3 \notin \{1,2\}$, all other vertices are colored with $j_1 = I$. Hence u is not a basic vertex of a prohibition of type 2 for b_k in the coloring ρ_I^2.

It is proved above that no new prohibitions of type 2 for b_k appear in ρ_I^2. Hence, $p = |U|$ is decreased. Then set $\rho^1 = \rho^2$ and return to Step C.

C.3.2. $j_1 = I$

Since ρ^1 is a proper coloring of the graph G, in this case $\rho^1(u) = c(k+1)$ and all vertices of the set $\mathrm{N}_G(b_k)$ have the same color as b_k — color $c(k)$.

In this case we will apply the algorithm DN to the coloring ρ^1. Set $j_0 = c(k+1)$, $j_1 = c(k)$. Having 6 colors, we can choose pairwise distinct colors j_2, j_3, j_4 and j_5, different from 1 and 2. We want to do this choice such that condition (J) will hold.

Let C^* be the set of all colors of the coloring ρ^1, different from 1 and 2. Then $|C^*| = d - 2 \geq 4$. If there is a vertex $y \in \mathrm{N}_G(b_k)$ of color $\rho^1(y) = s \in C^*$, then choose as j_2, j_3 and j_4 three colors of C^*, different from s. Clearly, condition (J) holds in this case. Otherwise, every vertex of $\mathrm{N}_G(b_k)$ have in ρ_I^1 color 1 or 2. Since $\rho^1(b_k) = c(k) \in \{1,2\}$ and ρ^1 is a proper coloring, all vertices of the set $\mathrm{N}_G(b_k)$ have in ρ^1 color $c(k+1)$, i.e., b_k is a bad vertex in the coloring ρ^1 of the graph G and condition (J) also holds.

Let G' be the graph obtained from G upon deleting all edges joining b_k to vertices of the set T.

Let the algorithm $DN(G', \rho^1, J, u)$ change the coloring ρ^1 to ρ^2. Then $\rho^1 \leq_{G'} \rho^2$. By Lemma 3 we have $\rho^2(b_k) = \rho^1(b_k)$ and $\rho^2 \leq_G \rho^1$. Since $1, 2 \notin J$ by Lemma 4 we have $\rho_I^2 \leq_G \rho_I^1$ and $\rho_I^2 \leq_{G'} \rho_I^1$.

Prove that no new prohibition of type 2 for the vertex b_k appears in the coloring ρ_I^2. Assume the converse, let such a prohibition with basic vertex x appear. Then $x \in T$. Hence x is a bad vertex in the coloring ρ_I^2 of the graph G' and not a bad vertex in the coloring ρ_I^1. This contradicts to $\rho_I^2 \leq_{G'} \rho_I^1$.

Consider two cases.

C.3.2.1. *Condition* $(DN1)$ *holds*

Since $1, 2 \notin J$, no new vertices of colors 1 and 2 appear. Hence, no new prohibition of type 1 or 2 on color I for the vertex b_k appears in the coloring ρ_I^2. Since $\rho_I^2(u) = j_2$, the counter q is decreased.

It is proved above that no new prohibition of type 2 for b_k appears in ρ_I^2. Thus, we obtain a new coloring $\rho^2 \leq_G \rho^1$ with the same value of the counter p as ρ^1.

For the same color $i = I$ we have the counter q decreased by 1 (recall that u is recolored). Set $\rho^1 = \rho^2$ and return to Step C.3.

C.3.2.2. *Condition* $(DN2)$ *holds*
In the coloring ρ^2, exactly one vertex of the set $\mathrm{N}'(u)$ has color $j_3 \notin \{1, 2\}$, all other vertices are colored with $j_1 \in \{1, 2\}$. Hence u is not a basic vertex of a prohibition of type 2 for b_k in the coloring ρ^2_I.

It is proved above that no new prohibition of type 2 for b_k appears in ρ^2_I. Hence, $p = |U|$ is decreased. Then set $\rho^1 = \rho^2$ and return to Step C.

Remark 4.9. Let us show that each Step C.3 finishes its work. The main operation of the step is applying of the algorithm DN. Each applying of DN in Step C.3 either leads to decrease of the counter p (in this case, we return to the beginning of the step C), or leads to decrease of the counter q together with preserving of color i and counter p (in this case, we return to the beginning of Step C.3), see Figure 4.

Remark 4.10. 1) Every return from one of the steps C.1, C.2 and C.3 to the beginning of Step C decreases the counter p by at least 1. Since $p \le d - 1$, Step C is repeated at most $d - 2$ times.

Look at the last applying of Step C. If $i \ne I$, we begin with Step C.1. Since we don't return to the beginning of Step C, each iteration of Step C.1 decreases the counter q' by at least 1 (see Figure 4). Since $q' \le p$, after at most p steps C.1 we obtain $q' = 0$ and pass on to Step C.2.

Each iteration of Step C.2, except for one exclusion, decreases the counter q with preserving $q' = 0$ (otherwise, we must return to the beginning of Step C, but we have done the last return before). In the exceptional case we obtain a coloring with exactly one prohibition on color I for b_k and pass on to Step F with this coloring and $i = I$. Since $q \le p$, after at most p steps C.2 we obtain $q = 0$ and pass on to Step F with the coloring ρ^1, such that $i \notin Z$ and no vertex $u \in U$ has $\rho^1_I(u) = i$.

If $i = I$, we perform only Step C.3. Since we don't return to Step C, each iteration of Step C.3 decreases the counter q by at least 1. Since $q \le p$, after at most p steps C.3 we obtain $q = 0$ and pass on to Step F with the coloring ρ^1, such that color I occurs in the collection Z at most once and no vertex $u \in U$ has $\rho^1_I(u) = I$.

2) Note that if we pass to Step F with the coloring ρ^1 and the color i then b_k has at most one prohibition on color i in ρ^1_I. Indeed, i occurs in Z at most once and no vertex of the set U has color i in ρ^1_I.

F. End of the Step of the main algorithm

We pass to this step after performing Step C, in the case, where the vertex b_k has in the coloring ρ^1_I no prohibition on a certain color i or exactly one prohibition on color I. Recall that the coloring ρ^1 satisfies conditions $(C1(k-1))$ and $(C2(k-1))$.

Consider two cases.

F.1. *The vertex b_k has at most one prohibition on color I in ρ^1_I*

A prohibition of type 1 on color $c(k+1)$ in the coloring ρ^1 corresponds to a vertex $t \in \mathrm{N}'(b_k)$, such that $\rho^1_I(t) = I$. A prohibition of type 2 on color $c(k+1)$ in the coloring ρ^1 has basic vertex $u \in U$, such that $z2(\rho^1_I, u) = I$. Thus, each prohibition on color $c(k+1)$ for the vertex b_k in ρ^1 corresponds to a prohibition on color I for b_k in ρ^1_I. Recall that there is at most one such prohibition. Hence, there is at most one prohibition on color $c(k+1)$ for the vertex b_k in the coloring ρ^1.

If the vertex b_k has no prohibition on color $c(k+1)$ in the coloring ρ^1, then we apply Lemma 7 to $s = k$ and the coloring ρ^1 and obtain that there exists a coloring ρ', such that $\rho' <_G \rho^1 \leq_G \rho$. In this case, the main algorithm stops.

Assume that there is a prohibition on color $c(k+1)$ for b_k. Then condition $(C1(k))$ holds.

We claim that condition $(C2(k))$ also holds. Let $x \in \mathrm{N}'(b_k)$ be a vertex, such that all vertices of the set $\mathrm{N}_G(x)$ are colored with colors 1 and 2 in the coloring ρ^1. Then v is a basic vertex of a prohibition of type 2 on color I for the vertex b_k in the coloring ρ^1_I. By the condition of Step F.1 there is exactly one such prohibition. By condition $(C1(k))$ and proved above this prohibition must correspond to a prohibition of type 2 on color $c(k+1)$ for b_k in the coloring ρ^1_I, i.e., condition $(C2(k))$ holds.

In this case, the algorithm returns to the beginning of Step A with the chain of prohibitions b_1, ..., b_k (its length is increased by 1 and is equal to k).

F.2. *The vertex b_k has no prohibition on color $i \neq I$*

Hence, b_k has no prohibition on color $i \notin \{1, 2\}$ in the coloring ρ^1. However, this does not mean that b_k can be recolored with color i: the ancestor of b_k can prohibit this recoloring. We know that current coloring $\rho^1 \leq_G \rho$. Our aim is to present a coloring $\rho' <_G \rho$ and stop the main algorithm.

Recall that $\{c(k-1), c(k)\} = \{1, 2\}$ and consider two cases.

F.2.1. $\mathrm{asc}(b_k) = b_{k-1}$

Consider a coloring ρ^2, such that $\rho^2(b_k) = i$, and all other vertices are colored as in ρ^1. Since $\rho^1(b_{k-1}) = c(k-1) \neq i$, the coloring ρ^2 is proper. If no new bad vertex appears, then $\rho^2 \leq_G \rho^1 \leq_G \rho$. By item 2 of Lemma 8, there exists a coloring $\rho' <_G \rho^2$, in this case the main algorithm stops.

If a vertex $x \in \mathrm{N}'(b_k)$ is not bad in the coloring ρ^1, but is bad in the coloring ρ^2, then b_k has a prohibition of type 2 on color i with the basic vertex x. This contradicts the condition of Step F.2. Hence, the only vertex which can be bad in ρ^2 and not bad in ρ^1 is b_{k-1} (in the case, where all vertices of the set $\mathrm{N}_G(b_{k-1}) \setminus \{b_k\}$ have color i in the coloring ρ^1). Consider this case, the colorings ρ^1 and ρ^2 are shown on Figures 7a and 7b, respectively.

We will apply the algorithm DN to the graph G, the coloring ρ^2 and the bad vertex b_{k-1}. Let's choose the set of colors J. We have $j_0 = c(k-1)$, $j_1 = i$. Set also $j_5 = i$. Let j_2, j_3 and j_4 be three pairwise distinct colors, different from 1, 2 and i (recall that there are $d \geq 6$ colors). Note that the set of colors J satisfies the condition of Lemma 9. Hence, Lemma 8 can be applied.

Let ρ^3 be the coloring, obtained upon applying the algorithm $DN(G, \rho^2, b_{k-1}, J)$. Then $\rho^3 \leq_G \rho^2$. Consider two cases.

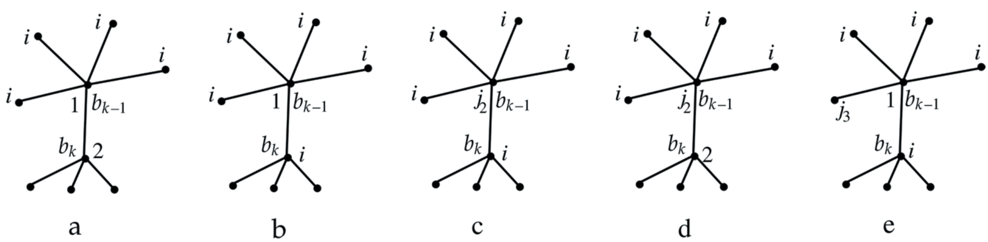

Fig. 7. The case $\mathrm{asc}(b_k) = b_{k-1}$. Here $c(k-1) = 1$, $c(k) = 2$

F.2.1.1. *Condition* $(DN1)$ *holds*

In this case, $\rho^3(b_{k-1}) = j_2 \notin \{1, 2, i\}$ (see Figure 7c). Note that no new vertices of color $c(k) \notin J$ appear in ρ^3. If we change color of the vertex b_k from i to $c(k)$ in the coloring ρ^2, then no new bad vertex appears and we obtain the proper coloring ρ^1. Hence, if we change color of the vertex b_k from i to $c(k)$ in the coloring ρ^3, then no new bad vertex appears and we also obtain a proper coloring, let it be ρ^4 (see Figure 7d). The colorings ρ^2 and ρ^3 have one bad vertex which is not bad in ρ^1 — this is the vertex b_{k-1}. Since in the coloring ρ^4 the vertex b_{k-1} is not bad, $\rho^4 \leq_G \rho^1$.

Since $\rho^4(b_{k-1}) = j_2 \neq c(k-1)$, by item 2 of Lemma 8 there exists a coloring ρ', such that $\rho' <_G \rho^4 \leq_G \rho$. The algorithm stops.

F.2.1.2. *Condition* $(DN2)$ *holds*

In this case, exactly one vertex of the set $\mathrm{N}_G(b_{k-1})$ was recolored with color j_3 (see Figure 7e). Hence, the vertex b_{k-1} is not bad in ρ^3 and $\rho^3 \leq_G \rho^1$. We know that $\rho^3(b_k) \in \{i, j_3\}$, i.e., $\rho^3(b_k) \neq c(k)$. Then by item 2 of Lemma 8 there exists a coloring ρ', such that $\rho' <_G \rho^3 \leq_G \rho$. The algorithm stops.

F.2.2. $\mathrm{asc}(b_k) = a_{k-1} \neq b_{k-1}$

Consider a coloring ρ^2, such that $\rho^2(b_k) = i$ and all other vertices are colored as in ρ^1. If $\rho^2 \leq \rho^1$, then by item 2 of Lemma 8 there exists a coloring ρ', such that $\rho' <_G \rho^2 \leq_G \rho$. In this case the algorithm stops.

Assume that $\rho^2 \not\leq_G \rho^1$. By the construction, b_{k-1} is the only vertex of color different from $c(k)$ in $\mathrm{N}_G(a_{k-1})$ in the coloring ρ^1 and $d_G(a_{k-1}) \geq 3$. Hence, the vertex a_{k-1} cannot be bad in ρ^2. Other neighbors of b_k are not bad in ρ^2, since b_k has no prohibitions on color i in ρ^1_I. However, it is possible that $\rho^2(a_{k-1}) = \rho^1(a_{k-1}) = i$ (see Figure 8a). In this case, the coloring ρ^2 is not proper.

Thus, the remaining case is where $\rho^1(a_{k-1}) = i$. In this case, we consider a graph $G' = G - b_{k-1}a_{k-1}$. In this graph, a_{k-1} is a bad vertex in coloring ρ^1. We will apply the algorithm DN to the graph G', the vertex a_{k-1} and the coloring ρ^1.

Let's choose the set of colors J. Set $j_0 = i$, $j_1 = j_5 = c(k)$, $j_3 = c(k-1)$.

If $\mathrm{N}_G(b_{k-1})$ is colored in ρ^1 with one color, then the vertex b_k is bad. In this case we choose distinct colors j_2 and j_4 different from i, 1 and 2.

If $\mathrm{N}_G(b_{k-1})$ is colored in ρ^1 with two colors, then we choose j_2 different from these colors, i, 1 and 2 (six colors is enough for this). After that, we choose j_4 different from j_2, i, 1 and 2.

Finally, if at least three colors are presented among vertices of the set $\mathrm{N}_G(b_{k-1})$ in the coloring ρ^1, then we choose arbitrary distinct colors j_2 and j_4, different from i, 1 and 2.

It is easy to verify that in all cases condition (J) is satisfied. Moreover, the set of colors J satisfies the condition of Lemma 9. Hence, Lemma 8 can be applied.

Let ρ' be a coloring, obtained upon applying the algorithm $DN(G', \rho^1, b_{k-1}, J)$. Since $J \not\ni i = \rho^1(a_{k-1})$ and condition (J) holds, by Lemma 3 we have $\rho' \le_G \rho^1 \le_G \rho$.

By item 1 of Lemma 8 either there exists a coloring ρ'', such that $\rho'' <_G \rho$ (in this case, the algorithm stops), or b_1, ..., b_k is a chain of prohibitions in the coloring ρ', satisfying conditions $(C1(k-1))$ and $(C2(k-1))$. In the last case, consider two subcases.

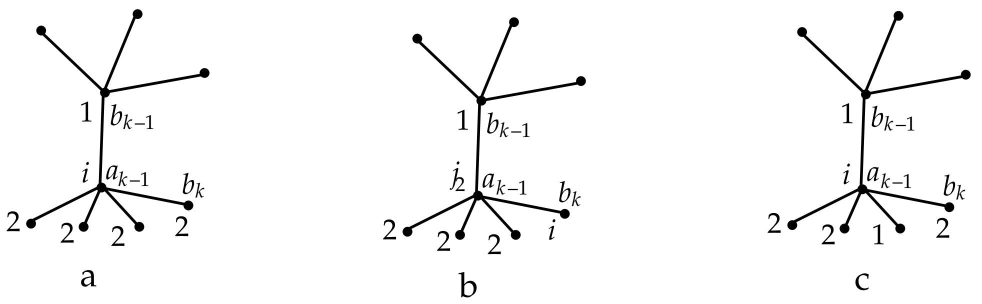

Fig. 8. The case $\mathrm{asc}(b_k) = a_{k-1}$. Here $c(k-1) = 1$, $c(k) = 2$

F.2.2.1. *Condition* $(DN1)$ *holds*
There is no prohibition on color i for the vertex b_k in the coloring ρ' (since $i \notin J$ such prohibition cannot appear after applying the algorithm DN). We recolor b_k with color i and obtain a coloring ρ^*. Recall that in the case we consider $\rho'(a_{k-1}) = j_2 \neq i$, see Figure 8b. Hence, ρ^* is a proper coloring of the graph G and $\rho^* \le_G \rho$. Then by item 2 of Lemma 8 there exists a coloring ρ'', such that $\rho'' <_G \rho' \le_G \rho$. The algorithm stops.

F.2.2.2. *Condition* $(DN2)$ *holds*
By condition $(C1(k-1))$, there is exactly one prohibition on color $c(k)$ for the vertex b_{k-1} in the coloring ρ^1: a prohibition of type 2 with basic vertex a_{k-1}. However, in the coloring ρ' there is exactly one vertex of color $j_3 = c(k-1)$ in $\mathrm{N}_G(a_{k-1}) \setminus \{b_{k-1}\}$, other vertices of this set have color $c(k)$ (see Figure 8c). Hence, this prohibition on the color $c(k)$ for the vertex b_k disappears in the coloring ρ'.

Recall that any vertex of color $c(k)$ in ρ' has color $c(k)$ or $c(k-1)$ in ρ^1. By condition $(C1(k-1))$ for the coloring ρ^1, there is no vertex of color $c(k)$

in $N'_G(b_{k-1})$ in the coloring ρ^1. Since ρ^1 is a proper coloring, there is no vertex of color $c(k-1)$ in $N'_G(b_{k-1})$ in the coloring ρ^1. Finally, by condition $(C2(k-1))$ for the coloring ρ^1 there is exactly one vertex in $N'_G(b_{k-1})$, which neighborhood is colored in the coloring ρ^1 with colors 1 and 2 — this is a_{k-1}. Hence, no new prohibition on color $c(k)$ can appear in ρ'. Thus, the vertex b_{k-1} has no prohibition on color $c(k)$ in the coloring ρ' and by Lemma 7 there exists a coloring ρ'', such that $\rho'' <_G \rho$. The algorithm stops.

4.5. The end of the proof of Theorem 1

Proof of Theorem 1. Let us consider a proper coloring ρ of the graph G with the minimal number of bad vertices; assume that a is a bad vertex in ρ. Start the algorithm that constructs a chain of prohibitions beginning at the bad vertex a. We have shown that vertices of the chain cannot repeat. Since the graph is finite, this means that, after some time, the algorithm stops and we get a coloring $\rho' <_G \rho$, which contradicts the minimality of ρ. Thus the coloring ρ does not contain bad vertices; hence, ρ is a dynamic proper coloring. □

5. Acknowledgements

This research is partially supported by Russian Foundation for Basic Research (grant 14-01-00545-a) and by the Government of the Russian Federation (grant 14.Z50.31.0030).

Bibliography

1. ***A. Ahadi, S. Akbari, A. Dehghan, M. Ghanbari.*** *On the difference between chromatic number and dynamic chromatic number of graphs*, Discrete Mathematics **312**:17 (2012), 2579–2583.
2. ***S. Akbari, M. Ghanbari, S. Jahanbekam*** *On the list dynamic coloring of graphs*, Discrete Applied Mathematics **157**:14 (2009), 3005–3007.
3. ***S. Akbari, M. Ghanbari, S. Jahanbekam*** *On the Dynamic Chromatic Number of Graphs*, Contemp. Math. **531** (2010), 11–18.
4. ***R. L. Brooks.*** *On coloring the nodes of network*, Proc. Cambridge Philos. Soc. **37** (1941), 194–197.

5. ***D. V. Karpov.*** *Dynamic proper colorings of a graph*, J. Math. Sciences **179**:5 (2011), 601–615. Translated from Zap. Nauchn. Semin. POMI **381** (2010).
6. ***S.-J. Kim, S. J. Lee, W.-J Park.*** *Dynamic coloring and list dynamic coloring of planar graphs*, Discrete Applied Mathematics **161** (2013), 2207–2212.
7. ***H.-J. Lui, B. Montgomery, H. Poon.*** *Upper bounds of dynamic chromatic number*, Ars Combinatoria **68** (2003), 193–201.

Karpov D. V.

St. Petersburg Department
of V.A.Steklov Institute of Mathematics
of the Russian Academy of Sciences
191023, 27 Fontanka,
St. Petersburg, Russia

and

St. Petersburg University,
198504, Universitetsky prospekt,
28, Peterhof, St. Petersburg,
Russia

dvk0@yandex.ru