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On a conjecture of Schmidt for the parametric geometry of numbers

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Abstract: With the help of the recently introduced parametric geometry of numbers by W. M. Schmidt and L. Summerer, we prove a strong version of a conjecture of Schmidt concerning the successive minima of a lattice.

Keywords: Schmidt's conjecture, Successive minima, Generalized $(n+1)$-system

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1. Introduction

Among the conjectures proposed by W. M. Schmidt in 1983, one is concerned with the parametric geometry of numbers [5, Conjecture 2]. This conjecture was proven in 2012 by N. G. Moshchevitin [2, Theorem 1]. The goal of this paper is to prove a stronger statetement along the same lines and we will show that this generalization is the best possible. We start by recalling Moshchevitin's result, using the notations of D. Roy in [3].

Fix an integer $n \geqslant 2$. For each non-zero $\xi \in \mathbb{R}^{n+1}$, we associate the family of convex bodies

$$\mathcal{C}_\xi(Q) := \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \,;\, \|\mathbf{x}\| \leqslant 1 \,,\, |\mathbf{x} \cdot \xi| \leqslant Q^{-1} \right\} \qquad (Q \geqslant 1),$$

where $\mathbf{x} \cdot \mathbf{y}$ denotes the standard scalar product in $\mathbb{R}^n$ and $||\mathbf{x}|| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$ denotes the euclidean norm of $\mathbf{x}$. Define

$$L_{\xi,j}(q) = \log \lambda_j \left(\mathcal{C}_\xi(e^q); \mathbb{Z}^{n+1}\right) \qquad (q \geqslant 0;\ \ 1 \leqslant j \leqslant n+1),$$

where $\lambda_j(\mathcal{C}; \Lambda)$ is defined for a convex body $\mathcal{C}$ and lattice Λ in $\mathbb{R}^{n+1}$ to be the j-th minimum of $\mathcal{C}$ with respect to Λ, i. e. the smallest $\lambda \geqslant 0$ such that $\lambda\mathcal{C}$ contains at least j linearly independent elements of Λ. Clearly, we have

$$L_{\xi,1}(q) \leqslant \cdots \leqslant L_{\xi,n+1}(q) \qquad (q \geqslant 0).$$

The functions $L_{\xi,j} : [0, \infty) \longrightarrow \mathbb{R}$ $(1 \leqslant j \leqslant n+1)$ are continuous and piecewise linear, with slopes alternating between 0 and 1 (see [3, §2], [7, §3]). Moreover, since the volume of $\mathcal{C}_\xi(e^q)$ is bounded below and above by multiples of e^{-q}, Minkowski's theorem implies that

$$q - \sum_{j=1}^{n+1} L_{\xi,j}(q)$$

is a bounded function in q, and so the average of the $L_{\xi,j}$'s is $q/(n+1)$. If the coordinates of ξ are linearly independent over $\mathbb{Q}$, then for each $j = 1, \ldots, n+1$, there exists arbitrarily large values of q such that

$$L_{\xi,j}(q) = L_{\xi,j+1}(q)$$

(see [6, Theorem 1]). On the other hand, we have the following result.

THEOREM 1 (N. G. Moshchevitin, 2012). *For each integer k with $2 \leqslant k \leqslant n$, there exists $\xi \in \mathbb{R}^{n+1}$ whose coordinates are linearly independent over $\mathbb{Q}$ such that*

$$\lim_{q\to\infty} \left(L_{\xi,k-1}(q) - \frac{q}{n+1}\right) = -\infty \quad \textit{and} \quad \lim_{q\to\infty} \left(L_{\xi,k+1}(q) - \frac{q}{n+1}\right) = \infty.$$

Thus, the functions $L_{\xi,k-1}(q)$ and $L_{\xi,k+1}(q)$ can diverge from each other on each side by $q/(n+1)$. Our main result improves upon these estimates, and is stated as follows.

THEOREM 2. *For each integer k with $2 \leqslant k \leqslant n$, there exist uncountably many vectors $\xi \in \mathbb{R}^{n+1}$ whose coordinates are linearly independent over $\mathbb{Q}$ such that*

$$\lim_{q\to\infty} \frac{L_{\xi,k-1}(q)}{q} = 0 \quad \textit{and} \quad \liminf_{q\to\infty} \frac{L_{\xi,k+1}(q)}{q} = \frac{1}{n-k+2}.$$

Further, this result is the best possible in the following sense.

THEOREM 3. *Let k be an integer with $2 \leqslant k \leqslant n$, and suppose that ξ is a point in $\mathbb{R}^{n+1}$ whose coordinates are linearly independent over $\mathbb{Q}$ and which satisfies*

$$\lim_{q\to\infty} \frac{L_{\xi,k-1}(q)}{q} = 0.$$

Then, we have

$$\liminf_{q\to\infty} \frac{L_{\xi,k+1}(q)}{q} \leqslant \frac{1}{n-k+2}.$$

In the following section, we state Schmidt's original conjecture, and we justify the above reformulation of Moshchevitin's result. In section 3, we use the results of [4, §4] to prove Theorem 2. Finally, section 4 provides a proof of theorem 3 by using Schmidt and Summerer's parametric geometry of numbers.

2. Link with Schmidt's original conjecture

For each $N \in \mathbb{R}$ with $N \geqslant 1$ and for each $\xi = (1, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$, Schmidt [5] introduced the lattice $\Lambda(\xi, N) \subset \mathbb{R}^{n+1}$ generated by the vectors

$$\mathbf{v}_0 = (N^{-1}, N^{1/n}\xi_1, \dots, N^{1/n}\xi_n),\ \mathbf{v}_1 = (0, -N^{1/n}, \dots, 0),\ \dots,\ \mathbf{v}_n = (0, 0, \dots, -N^{1/n}),$$

and defined

$$\mu_j(\xi, N) = \lambda_j(\mathcal{B}; \Lambda(\xi, N)) \qquad (1 \leqslant j \leqslant n+1)$$

where $\mathcal{B} = \big\{(y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1};\ |y_i| \leqslant 1,\ i = 0, \dots, n\big\}$ is the unit hypercube in $\mathbb{R}^{n+1}$.

With these notations, he conjectured the following result, later proven by Moshchevitin.

THEOREM 4 (N. G. Moshchevitin, 2012). *Let k be an integer with $2 \leqslant k \leqslant n$. There exists real numbers $\xi_1, \dots, \xi_n \in [0, 1)$ such that*

- $1, \xi_1, \dots, \xi_n$ *are linearly independent over* $\mathbb{Q}$;
- $\lim_{N\to\infty} \mu_{k-1}(\xi, N) = 0$ *and* $\lim_{N\to\infty} \mu_{k+1}(\xi, N) = \infty$, *where* $\xi = (1, \xi_1, \dots, \xi_n)$.

In fact, Schmidt's original conjecture omits the linear independence condition, but as Moshchevitin mentions in his article, (see [2, §3]), the conjecture is trivial without this hypothesis.

To show the equivalence between Theorems 1 and 4, fix a point $\xi = (1, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$ whose coordinates are linearly independent over $\mathbb{Q}$, and fix an integer k with $2 \leqslant k \leqslant n$. In [2, §1], Moshchevitin begins by observing that

$$\mu_j(\xi, N) = \lambda_j(\mathcal{K}_\xi(N); \mathbb{Z}^{n+1}) \qquad (N \geqslant 1,\ 1 \leqslant j \leqslant n+1),$$

where

$$\mathcal{K}_\xi(N) = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1}\ ;\ |x_0| \leqslant N\,,\ |x_0\xi_j - x_j| \leqslant N^{-1/n},\ j = 1, \dots, n \right\}.$$

Consequently, the second statement of Theorem 4 can be rewritten as

$$\lim_{N\to\infty} \lambda_{k-1}(\mathcal{K}_\xi(N); \mathbb{Z}^{n+1}) = 0 \quad \text{and} \quad \lim_{N\to\infty} \lambda_{k+1}(\mathcal{K}_\xi(N); \mathbb{Z}^{n+1}) = \infty. \qquad (1)$$

Meanwhile, Mahler's duality theorem in [1, Chapter 8, p. 219] yields

$$\lambda_j(\mathcal{K}_\xi(N); \mathbb{Z}^{n+1})\lambda_{n-j+2}(\mathcal{K}^\star_\xi(N); \mathbb{Z}^{n+1}) \asymp 1 \qquad (1 \leqslant j \leqslant n+1),$$

where

$$\mathcal{K}^\star_\xi(N) = \left\{ \mathbf{x} \in \mathbb{R}^{n+1}\ ;\ |\mathbf{x}\cdot\xi| \leqslant N^{-1}\,,\ ||\mathbf{x}|| \leqslant N^{1/n} \right\}$$

is essentially the convex body dual to $\mathcal{K}_\xi(N)$. Thus, the conditions in (1) become

$$\lim_{N\to\infty} \lambda_{n+3-k}(\mathcal{K}^\star_\xi(N); \mathbb{Z}^{n+1}) = \infty \quad \text{and} \quad \lim_{N\to\infty} \lambda_{n+1-k}(\mathcal{K}^\star_\xi(N); \mathbb{Z}^{n+1}) = 0. \qquad (2)$$

On the other hand, since $\mathcal{C}_\xi(e^q) = e^{-q/(n+1)}\mathcal{K}^\star_\xi(e^{nq/(n+1)})$, it follows that

$$L_{\xi,j}(q) = \frac{q}{n+1} + \log \lambda_j(\mathcal{K}^\star_\xi(e^{nq/(n+1)}); \mathbb{Z}^{n+1}) \qquad (1 \leqslant j \leqslant n+1).$$

Thus, the conditions in (2) can be rewritten as

$$\lim_{q\to\infty}\left(L_{\xi,n+3-k}(q)-\frac{q}{n+1}\right)=\infty \quad \text{and} \quad \lim_{q\to\infty}\left(L_{\xi,n+1-k}(q)-\frac{q}{n+1}\right)=-\infty.$$

The equivalence between Theorems 1 and 4 follows.

3. Proof of the main result

In order to prove Theorem 2, we need to etablish some prelimanary results which relie on the following basic construction.

PROPOSITION 1. *Let $a, b, c, \alpha, \beta, \gamma \in (0,\infty)$ with $a < b < c$. There exists a unique choice of real numbers $r, s, t, u \in (0,\infty)$ with $r < s < t < u$ and a unique triplet of continuous and piecewise linear functions (A, B, C) on $[r, u]$ such that the union of their graphs is as in Figure 1, i. e.*

i) for all $q \in [r, u]$, we have

$$A(q) \leqslant B(q) \leqslant C(q) \quad \text{and} \quad \frac{1}{\alpha}A(q)+\frac{1}{\beta}B(q)+\frac{1}{\gamma}C(q)=q; \tag{3}$$

ii) the function A is constant equal to a on $[r, t]$, has slope α on $[t, u]$, and satisfies $A(u) = b$;

iii) the function B has slope β on $[r, s]$, is constant equal to b on $[s, u]$, and satisfies $B(r) = a$;

iv) the function C is constant equal to b on $[r, s]$, has slope γ on $[s, t]$, and is constant equal to c on $[t, u]$.

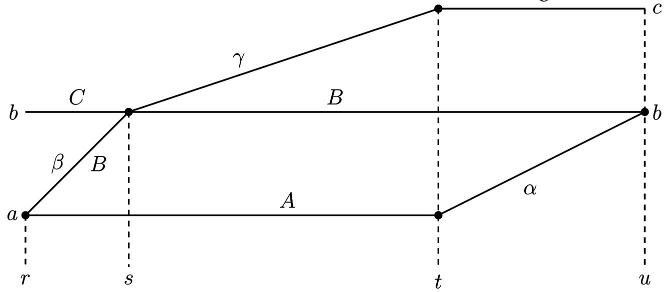

Fig. 1

PROOF. If there exist real numbers r, s, t, u and functions A, B, C as in the claim, then substituting q by r, s, t, u in the second condition of (3) yields, respectively,

$$r = \frac{a}{\alpha} + \frac{a}{\beta} + \frac{b}{\gamma}; \quad s = \frac{a}{\alpha} + \frac{b}{\beta} + \frac{b}{\gamma}; \quad t = \frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma}; \quad u = \frac{b}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma}, \tag{4}$$

which uniquely determines them all.

Now, let r, s, t, u be given by (4). Since $r < s < t < u$, there exists a unique triplet of continuous functions (A, B, C) on $[r, u]$ with constant slopes on $[r, s]$, $[s, t]$ and $[t, u]$, and with

$$\begin{aligned} A(r) = A(s) = A(t) = a \quad &\text{and} \quad A(u) = b, \\ B(r) = a \quad &\text{and} \quad B(s) = B(t) = B(u) = b, \\ C(r) = C(s) = b \quad &\text{and} \quad C(t) = C(u) = c. \end{aligned}$$

Thus, the function $F = \frac{1}{\alpha}A + \frac{1}{\beta}B + \frac{1}{\gamma}C$ is continuous and of constant slope on each of the interval $[r, s]$, $[s, t]$, and $[t, u]$. by construction, we have that $F(q) = q$ for $q = r, s, t, u$. Thus,

$$F(q) = q \quad \text{for all } q \in [r, u].$$

Since A and C are constant on $[r, s]$, this implies that B has slope β on $[r, s]$. Similarly, we deduce that C has slope γ on $[s, t]$, and that A has slope α on $[t, u]$. □

PROPOSITION 2. *With the same notation as above, suppose that $b/a < c/b$. Then, we have*

$$\max_{q\in[r,u]} \frac{A(q)}{q} = \frac{a}{r} \qquad \textit{and} \qquad \min_{q\in[r,u]} \frac{C(q)}{q} = \frac{b}{s}. \tag{5}$$

PROOF. First, using (4) note that

$$\frac{a}{t} < \frac{b}{u} < \frac{a}{r} \quad \text{and} \quad \frac{b}{s} < \frac{b}{r} < \frac{c}{u} < \frac{c}{t}.$$

Since $a/r < \alpha$ and $b/s < \gamma$, it follows that the ratio $A(q)/q$ is decreasing on $[r, t]$ and increasing on $[t, u]$, and that the ratio $C(q)/q$ is decreasing on $[r, s]$, increasing on $[s, t]$ and decreasing on $[t, u]$. The conclusion follows straightforwardly. □

Let Δ denote the set of sequences $(a_m)_{m\in\mathbb{Z}}$ of positive reals which satisfy

$$1 < \frac{a_{m+1}}{a_m} < \frac{a_{m+2}}{a_{m+1}} \qquad (m \in \mathbb{Z}),$$

$$\lim_{m\to-\infty} a_m = 0 \quad \text{and} \quad \lim_{m\to\infty} \frac{a_{m+1}}{a_m} = +\infty.$$

The following result further extends the preceding propositions.

Proposition 3. *Let $(a_m)_{m\in\mathbb{Z}} \in \Delta$ and let $\alpha, \beta, \gamma \in (0, \infty)$. Define*

$$r_m = \frac{a_m}{\alpha} + \frac{a_m}{\beta} + \frac{a_{m+1}}{\gamma} \qquad (m \in \mathbb{Z}). \tag{6}$$

Then, there exists a unique triplet of continuous and piecewise linear functions (A, B, C) on $(0, \infty)$ whose restriction to the interval $[r_m, r_{m+1}]$ fulfills the conditions of Proposition 1 with $a = a_m$, $b = a_{m+1}$ and $c = a_{m+2}$ for each $m \in \mathbb{Z}$. Moreover, we have

$$\lim_{q\to\infty} A(q) = \infty, \quad \limsup_{q\to\infty} \frac{A(q)}{q} = 0 \quad \textit{and} \quad \liminf_{q\to\infty} \frac{C(q)}{q} = \frac{\beta\gamma}{\beta+\gamma}. \tag{7}$$

Proof. Let $(a_m)_{m\in\mathbb{Z}} \in \Delta$, and define

$$s_m = \frac{a_m}{\alpha} + \frac{a_{m+1}}{\beta} + \frac{a_{m+1}}{\gamma} \quad \text{and} \quad t_m = \frac{a_m}{\alpha} + \frac{a_{m+1}}{\beta} + \frac{a_{m+2}}{\gamma} \qquad (m \in \mathbb{Z}). \tag{8}$$

by setting $a = a_m$, $b = a_{m+1}$ and $c = a_{m+2}$, Proposition 1 and (4) yield for each $m \in \mathbb{Z}$ a triplet of continuous and piecewise linear functions $(A^{(m)}, B^{(m)}, C^{(m)})$ on $[r, u] = [r_m, r_{m+1}]$.

Since the triplets $(A^{(m-1)}, B^{(m-1)}, C^{(m-1)})$ and $(A^{(m)}, B^{(m)}, C^{(m)})$ coincide at the point r_m and are equal to (a_m, a_m, a_{m+1}) for each $m \in \mathbb{Z}$, it follows that the sequence of triplets of functions $(A^{(m)}, B^{(m)}, C^{(m)})$ with $m \in \mathbb{Z}$ determine a unique triplet of continuous and piecewise linear functions (A, B, C) on $\bigcup_{m\in\mathbb{Z}} [r_m, r_{m+1}] = (0, \infty)$. Now, Proposition 2 gives

$$\max_{q\in[r_m, r_{m+1}]} \frac{A(q)}{q} = \frac{a_m}{r_m} \quad \text{and} \quad \min_{q\in[r_m, r_{m+1}]} \frac{C(q)}{q} = \frac{a_{m+1}}{s_m}, \tag{9}$$

and so

$$\limsup_{q\to\infty} \frac{A(q)}{q} = \lim_{m\to\infty} \frac{a_m}{r_m} = 0 \quad \text{and} \quad \liminf_{q\to\infty} \frac{C(q)}{q} = \lim_{m\to\infty} \frac{a_{m+1}}{s_m} = \frac{\beta\gamma}{\beta+\gamma}. \qquad \square$$

Our next result uses the notion of *generalized* $(n+1)$*-system* introduced by D. Roy in [4]. It provides a good approximation of the functions $\mathbf{L}_\xi$ for non-zero point $\xi \in \mathbb{R}^{n+1}$ (see [4] for more details). We recall here the definition.

DEFINITION. *Let I be a subinterval of $[0,\infty)$ with non-empty interior. A* generalized $(n+1)$-system *on I is a continuous piecewise linear map $\mathbf{P}=(P_1,\dots,P_{n+1}) : I \longrightarrow \longrightarrow \mathbb{R}^{n+1}$ with the following properties.*

$(G1)$ *For each $q \in I$, we have $0 \leqslant P_1(q) \leqslant \cdots \leqslant P_{n+1}(q)$ and $P_1(q)+\cdots+P_{n+1}(q)=q$.*

$(G2)$ *If H is a non-empty open subinterval of I on which $\mathbf{P}$ is differentiable, then there are integers $\underline{r},\overline{r}$ with $1 \leqslant \underline{r} \leqslant \overline{r} \leqslant n+1$ such that $P_{\underline{r}}, P_{\underline{r}+1},\dots,P_{\overline{r}}$ coincide on the whole interval H and have slope $1/(\overline{r}-\underline{r}+1)$ while any other component P_j of P is constant on H.*

$(G3)$ *If q is an interior point of I at which $\mathbf{P}$ is not differentiable, if $\underline{r},\overline{r}$, $\underline{s},\overline{s}$ are the integers for which*

$$P_j'(q^-)=\frac{1}{\overline{r}-\underline{r}+1} \quad (\underline{r}\leqslant j\leqslant\overline{r}) \quad \textit{and} \quad P_j'(q^+)=\frac{1}{\overline{s}-\underline{s}+1} \quad (\underline{s}\leqslant j\leqslant\overline{s}) \quad (10)$$

and if $\underline{r} \leqslant \overline{s}$, then we have $P_{\underline{r}}(q)=P_{\underline{r}+1}(q)=\cdots=P_{\overline{s}}(q)$.

We now combine the previous Propositions to etablish the following result.

PROPOSITION 4. *Let k be an integer with $2 \leqslant k \leqslant n$. With the notation of Proposition* 3, *suppose that $\alpha = 1/(k-1)$, $\beta=1$ and $\gamma = 1/(n+1-k)$. For all $q>0$, let*

$$P_1(q)=\cdots=P_{k-1}(q)=A(q), \quad P_k(q)=B(q) \quad \textit{and} \quad P_{k+1}(q)=\cdots=P_{n+1}(q)=C(q).$$

Then the function $\mathbf{P} : (0,\infty) \longrightarrow \mathbb{R}^{n+1}$ defined by

$$\mathbf{P}(q) := (P_1(q),\dots,P_{n+1}(q)) \qquad (q>0)$$

is an generalized $(n+1)$-system *on* $(0,\infty)$. *Moreover, we have*

$$\lim_{q\to\infty} P_1(q)=\infty, \quad \limsup_{q\to\infty}\frac{P_{k-1}(q)}{q}=0 \quad \textit{and} \quad \liminf_{q\to\infty}\frac{P_{k+1}(q)}{q}=\frac{1}{n-k+2}.$$

PROOF. The components $P_1,\dots,P_{n+1}$ of $\mathbf{P}$ are continuous and piecewise linear on $(0,\infty)$. They satisfy

$$0 \leqslant P_1(q) \leqslant \cdots \leqslant P_{n+1}(q) \quad \text{and} \quad P_1(q)+\cdots+P_{n+1}(q)=q \qquad (q>0).$$

The function $\mathbf{P}$ is differentiable on $(0, \infty)$ except at the points r_m, s_m, t_m given by (6) and (8). On each of the interval $[r_m, s_m]$, $[s_m, t_m]$, $[t_m, r_{m+1}]$, the components $P_1, \dots, P_{n+1}$ are constant except for few, say h of them, which coincide on the interval and which have slope $1/h$. At the point r_m, the slopes of $P_1, \dots, P_{k-1}$ go from $1/(k-1)$ to 0, while the slope of P_k goes from 0 to 1, and all these functions take the same value, i. e.

$$P_1(r_m) = \cdots = P_k(r_m) \qquad (m \in \mathbb{Z}).$$

At the point s_m, the function P_k goes from slope 1 to slope 0, while the slopes of $P_{k+1}, \dots, P_{n+1}$ go from 0 to $1/(n-k+1)$, and similary.

$$P_k(s_m) = P_{k+1}(s_m) = \cdots = P_{n+1}(s_m) \qquad (m \in \mathbb{Z}).$$

Finally, at the point t_m, the slopes of $P_{k+1}, \dots, P_{n+1}$ go from $1/(n-k+1)$ to 0, while the slopes of $P_1, \dots, P_{k-1}$ go from 0 to $1/(k-1)$, and we have

$$P_1(t_m) = \cdots = P_{k-1}(t_m) < P_k(t_m) < P_{k+1}(t_m) = \cdots = P_n(t_m) \qquad (m \in \mathbb{Z}).$$

Therefore, the function $\mathbf{P}$ is an *generalized* $(n+1)$*-system* on $(0, \infty)$. The second assertion of the proposition follows from (7). □

In [4, §4], D. Roy shows that for each *generalized* $(n+1)$*-system* $\mathbf{P}$ on $[q_0, \infty)$ with $q_0 \geqslant 0$, there exists a non-zero point ξ of $\mathbb{R}^{n+1}$ such that the difference $\mathbf{L}_\xi - \mathbf{P}$ is bounded. Then, we have

$$\limsup_{q\to\infty} \frac{L_{\xi,j}(q)}{q} = \limsup_{q\to\infty} \frac{P_j(q)}{q} \quad \text{and} \quad \liminf_{q\to\infty} \frac{L_{\xi,j}(q)}{q} = \liminf_{q\to\infty} \frac{P_j(q)}{q} \qquad (1 \leqslant j \leqslant n+1).$$

In the context of Proposition 4, this guarantees the existence of a point $\xi \in \mathbb{R}^{n+1}$ with

$$\limsup_{q\to\infty} \frac{L_{\xi,k-1}(q)}{q} = 0 \quad \text{and} \quad \liminf_{q\to\infty} \frac{L_{\xi,k+1}(q)}{q} = \frac{1}{n-k+2}. \tag{11}$$

Moreover, since $\lim_{q\to\infty} P_1(q) = \infty$, the function $L_{\xi,1}$ is unbounded. It follows that ξ is a point whose coordinates are linearly independent over $\mathbb{Q}$.

To finish the proof, it remains to show that one can construct uncountably many such points. For each $\theta \in (0, \infty)$, we define

$$a_m^{(\theta)} = \theta 2^{m^3} \qquad (m \in \mathbb{Z}).$$

Then, the sequence $\left(a_m^{(\theta)}\right)_{m\in\mathbb{Z}}$ belongs to Δ, and Propositions 3 and 4 associate to it an *generalized* $(n+1)$*-system* $\mathbf{P}^{(\theta)}$ on $(0, \infty)$, and a point $\xi^{(\theta)} \in \mathbb{R}^{n+1}$. Extending the notation in an obvious manner gives

$$\begin{aligned} r_m^{(\theta)} &= k a_m^{(\theta)} + (n-k+1) a_{m+1}^{(\theta)} < (n+1) a_{m+1}^{(\theta)} \\ t_m^{(\theta)} &= (k-1) a_m^{(\theta)} + a_{m+1}^{(\theta)} + (n-k+1) a_{m+2}^{(\theta)} > a_{m+2}^{(\theta)} \end{aligned}$$

for all $m \in \mathbb{Z}$, and $t_m^{(\theta)}/r_m^{(\theta)}$ tends to infinity with m. Thus, if $\theta, \theta' \in (0, \infty)$ with $\theta < \theta'$, then

$$r_m^{(\theta)} < r_m^{(\theta')} = (\theta'/\theta) r_m^{(\theta)} < t_m^{(\theta)},$$

for all sufficiently large $m \in \mathbb{Z}$, and so

$$\|\mathbf{P}^{(\theta')}(r_m^{(\theta')}) - \mathbf{P}^{(\theta)}(r_m^{(\theta')})\| \geqslant |P_1^{(\theta')}(r_m^{(\theta')}) - P_1^{(\theta)}(r_m^{(\theta')})| = |a_m^{(\theta')} - a_m^{(\theta)}| = (\theta' - \theta) 2^{m^3}.$$

This means that the difference $\mathbf{P}^{(\theta')} - \mathbf{P}^{(\theta)}$ is unbounded. Thus, the points $\xi^{(\theta')}$ and $\xi^{(\theta)}$ are distinct, and consequently, the map $\theta \mapsto \xi^{(\theta)}$ is injective on $(0, \infty)$. Its image is therefore uncountable.

4. Proof of Theorem 3

Let ξ be a point in $\mathbb{R}^{n+1}$ whose coordinates are linearly independent over $\mathbb{Q}$. On the model of Schmidt and Summerer in [6, §1], we define

$$\underline{\varphi}_j(\xi) = \liminf_{q\to\infty} \frac{L_{\xi,j}(q)}{q} \quad \text{and} \quad \overline{\varphi}_j(\xi) = \limsup_{q\to\infty} \frac{L_{\xi,j}(q)}{q} \qquad (1 \leqslant j \leqslant n+1).$$

In [6, §1], Schmidt and Summerer show that these quantities satisfy

$$\underline{\varphi}_{j+1}(\xi) \leqslant \overline{\varphi}_j(\xi) \qquad (1 \leqslant j \leqslant n). \tag{12}$$

Now, suppose that $\overline{\varphi}_{k-1}(\xi) = 0$ for some integer k with $2 \leqslant k \leqslant n$. Since $q - \sum_{j=1}^{n+1} L_{\xi,j}(q)$ is a bounded function in q on $(0, \infty)$, we have that

$$(n-k+2)\overline{\varphi}_k(\xi) \leqslant \limsup_{q\to\infty} \frac{1}{q}\sum_{j=k}^{n+1} L_{\xi,j}(q) = \limsup_{q\to\infty} \frac{1}{q}\left(q - \sum_{j=1}^{k-1} L_{\xi,j}(q)\right) = 1,$$

and so $\overline{\varphi}_k(\xi) \leqslant 1/(n-k+2)$. This yields $\underline{\varphi}_{k+1}(\xi) \leqslant 1/(n-k+2)$.

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Bibliography

1. ***J. W. S. Cassels***, *An introduction to the Geometry of Numbers*, Classics in Mathematics, Springer-Verlag, Berlin, Corrected reprint of the 1971 edition, 1997.
2. ***N. G. Moshchevitin,*** *Proof of W. M. Schmidt's conjecture concerning successive minima of a lattice*, J. London Math. Soc. (2) **86** (2012), 129–151.
3. ***D. Roy,*** *On Schmidt and Summerer parametric geometry of numbers*, Ann. of Math., **182** (2015), 739–786.
4. ***D. Roy,*** *Spectrum of the exponents of best rational approximation*, Math. Z. to appear.
5. ***W. M. Schmidt,*** *Open problems in Diophatine approximation*, Approximations Diophantiennes et nombres transcendants, Luminy, 1982, Progress in Mathematics (Birkhäuser, Basel, 1983) 271–289.
6. ***W. M. Schmidt, L. Summerer,*** *Parametric geometry of numbers and applications*, Acta Arith. **140** (2009), 67–91.
7. ***W. M. Schmidt, L. Summerer,*** *Diophantine approximation and parametric geometry of numbers*, Monatsh. Math. **169** (2013), 51–104.

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