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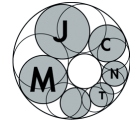
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New inequalities for cross-intersecting families

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Abstract: Two families \mathcal{F} and \mathcal{G} are called cross-intersecting if for all $F \in \mathcal{F}$, $G \in \mathcal{G}$ the intersection $F \cap G$ is non-empty. Under some additional conditions we prove best possible bounds on $|\mathcal{F}| + |\mathcal{G}|$. These results were recently applied to obtain a stability result for a classical theorem of Katona [4].

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1. Introduction

Let $[n] = \{1, \dots, n\}$ be the standard n -element set. For an integer k , $0 \leq k \leq n$, $\binom{[n]}{k}$ is the collection of k -subsets of $[n]$. A family $\mathcal{F} \subset \binom{[n]}{k}$ is called t -intersecting if $|F \cap F'| \geq t$ for all $F, F' \in \mathcal{F}$. Also, if $\mathcal{F} \subset \binom{[n]}{k}$ and $\mathcal{G} \subset \binom{[n]}{\ell}$ are two families satisfying $F \cap G \neq \emptyset$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$ then they are called *cross-intersecting*.

The study of possible maximum sizes of t -intersecting families and pairs of cross-intersecting families is one of the central problems of extremal set theory. It goes back to the papers of Erdős–Ko–Rado [1], Katona [4] and Hilton–Milner [3].

Since this is a short note we are unable to give an overview of the known results. We restrict ourselves to mentioning some results that are closely related to our results and/or which are used in the proof. First of all let us state our results.

THEOREM. Let n, k, t be non-negative integers, $n \geq k + 2t$. Suppose that $\mathcal{F} \subset \binom{[n]}{k+t}$, $\mathcal{G} \subset \binom{[n]}{k}$ are cross-intersecting. Then (i) and (ii) hold.

(i) If \mathcal{F} is t -intersecting then

$$|\mathcal{F}| + |\mathcal{G}| \leq \binom{n}{k}. \tag{1}$$

(ii) If \mathcal{F} is $(t + 1)$ -intersecting and non-empty then

$$|\mathcal{F}| + |\mathcal{G}| \leq 1 + \binom{n}{k} - \binom{n-t-k}{k}. \tag{2}$$

2. Tools and proofs

Our main tool is the following classical result.

For a family \mathcal{H} and an integer ℓ let $\Delta_\ell(\mathcal{H}) \subset \binom{[n]}{\ell}$ be the ℓ -shadow of \mathcal{H} :

$$\Delta_\ell(\mathcal{H}) = \left\{ D \in \binom{[n]}{\ell} : \exists H \in \mathcal{H} \text{ satisfying } D \subset H \right\}.$$

KATONA INTERSECTING SHADOW THEOREM 1 [4]. Suppose that $\mathcal{H} \subset \binom{[n]}{h}$ is r -intersecting. Then

$$|\Delta_{h-r}(\mathcal{H})| \geq |\mathcal{H}| \text{ holds.} \tag{3}$$

First let us prove (i). Define $\mathcal{H} = \{[n] - F : F \in \mathcal{F}\} \subset \binom{[n]}{n-k-t}$ and note that $n \geq 2k + t$ implies $n - k - t \geq k$.

For $F, F' \in \mathcal{F}$ the t -intersecting property implies $|F \cup F'| = |F| + |F'| - |F \cap F'| \leq 2(k + t) - t = 2k + t$. Equivalently, $|([n] - F) \cap ([n] - F')| = n - |F \cup F'| \geq n - 2k - t$, i. e., \mathcal{H} is $(n - 2k - t)$ -intersecting.

Let us apply (3) with $h = n - k - t$, $r = n - 2k - t$ to obtain

$$|\Delta_k(\mathcal{H})| \geq |\mathcal{H}|.$$

On the other hand $\Delta_k(\mathcal{H})$ and \mathcal{G} are disjoint by the cross-intersecting property. Indeed $G \in \mathcal{G} \cap \Delta_k(\mathcal{H})$ would mean $G \subset ([n] - F)$ for some $F \in \mathcal{F}$, i. e., $G \cap F = \emptyset$.

Since both \mathcal{G} and \mathcal{H} are subsets of $\binom{[n]}{k}$, $|\mathcal{G}| + |\mathcal{H}| = |\mathcal{G}| + |\mathcal{F}| \leq \binom{n}{k}$ follows. □

To prove (2) is more difficult. Let us recall that a family $\mathcal{F} \subset 2^{[n]}$ is called *shifted* if for all $1 \leq i < j \leq n$, $F \cap \{i, j\} = \{j\}$ implies for $F \in \mathcal{F}$ that $(F - \{j\}) \cup \{i\}$ is also in \mathcal{F} .

The following statement which is implicitly contained in [1] and explicitly used in [4] and [3] is very useful in obtaining inequalities concerning t -intersecting families and pairs of cross-intersecting families.

LEMMA. Let $\mathcal{F} \subset \binom{[n]}{\ell}$ and $\mathcal{G} \subset \binom{[n]}{k}$ be cross-intersecting families and let \mathcal{F} be t -intersecting as well. Then there exist shifted families $\tilde{\mathcal{F}} \subset \binom{[n]}{\ell}$, $\tilde{\mathcal{G}} \subset \binom{[n]}{k}$ such that $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{G}}$ are cross-intersecting, $\tilde{\mathcal{F}}$ is t -intersecting and $|\tilde{\mathcal{F}}| = |\mathcal{F}|$, $|\tilde{\mathcal{G}}| = |\mathcal{G}|$ hold.

In view of the Lemma, upon proving (ii) we may assume that both \mathcal{F} and \mathcal{G} are shifted.

PROOF OF (ii). We want to apply double induction on n and k . Therefore we first check the validity of (2) in the cases $k = 0$ and 1 and also for $n = 2k + t$. If $k = 0$ or 1 then $\mathcal{F} \neq \emptyset$ implies by shiftedness $\mathcal{F} = \{\{1, 2, \dots, k + t\}\}$. Since \mathcal{F} and \mathcal{G} are cross-intersecting $\mathcal{G} \cap \binom{[n] - \{1, 2, \dots, k + t\}}{k} = \emptyset$ must hold. Thus

$$|\mathcal{G}| \leq \binom{n}{k} - \binom{n - k - t}{k}, \text{ proving (2).}$$

In the case $n = 2k + t$ the formula on the RHS of (2) reduces simply to $\binom{n}{k}$ and it readily follows from (1). Or also from $\mathcal{G} \cap \{[n] - F : F \in \mathcal{F}\} = \emptyset$, using the cross-intersecting property.

From now on $k \geq 2$, $n > 2k + t$. For $\mathcal{H} \subset 2^{[n]}$ define the following two families:

$$\begin{aligned} \mathcal{H}(n) &= \{H - \{n\} : n \in H \in \mathcal{H}\}, \\ \mathcal{H}(\bar{n}) &= \{H : n \notin H \in \mathcal{H}\}. \end{aligned}$$

Note that

$$|\mathcal{H}| = |\mathcal{H}(n)| + |\mathcal{H}(\bar{n})| \text{ holds.}$$

CLAIM 1. $\mathcal{F}(n) \subset \binom{[n-1]}{k+t-1}$ is $t + 1$ -intersecting.

PROOF. If $F, F' \in \mathcal{F}$ then $|F \cap F'| \geq t + 1$ implies $|F \cup F'| = |F| + |F'| - |F \cap F'| \leq 2(k + t) - (t + 1) = 2k + t - 1 \leq n - 2$. Therefore one can find $i \in [n - 1]$ with

$i \notin F \cup F'$. Suppose now $n \in F \cap F'$. We need to prove

$$|(F - \{n\}) \cap (F' - \{n\})| \geq t + 1. \quad (4)$$

Since $|F \cap F'| \geq t + 1$, the only way that (4) can fail is

$$|(F - \{n\}) \cap (F' - \{n\})| = t.$$

Suppose for contradiction that this is the case. By shiftedness

$$F'' \stackrel{\text{def}}{=} (F - \{n\}) \cup \{i\} \text{ is in } \mathcal{F}.$$

However,

$$F'' \cap F' = F \cap F' - \{n\}, \text{ i.e., } |F'' \cap F'| = t, \text{ a contradiction. } \square$$

CLAIM 2. $\mathcal{F}(n)$ and $\mathcal{G}(n)$ are cross-intersecting.

PROOF. The proof is similar to Claim 1, therefore we shall be somewhat sketchy. The only way that Claim 2 can be false is if there are $F \in \mathcal{F}$, $G \in \mathcal{G}$ such that $F \cap G = \{n\}$ holds.

$|F| + |G| = 2k + t < n$ implies the existence of $i \in [n - 1]$, $i \notin F \cup G$. By shiftedness, $F' \stackrel{\text{def}}{=} (F - \{n\}) \cup \{i\}$ is in \mathcal{F} . However, $F' \cap G = \emptyset$, a contradiction. \square

Now we are ready to do the induction step.

Since $\mathcal{F} \neq \emptyset$ is shifted, $\{1, 2, \dots, k + t\} \in \mathcal{F}(\bar{n})$ holds. Let us apply the induction hypothesis to $\mathcal{F}(\bar{n}) \subset \mathcal{F}$ and $\mathcal{G}(\bar{n}) \subset \mathcal{G}$.

We infer

$$|\mathcal{F}(\bar{n})| + |\mathcal{G}(\bar{n})| \leq 1 + \binom{n-1}{k} + \binom{n-1-k-t}{k}. \quad (5)$$

Note that $\{1, \dots, k + t\} \in \mathcal{F}$ and the cross-intersecting property imply

$$|\mathcal{G}(n)| \leq \binom{n-1}{k-1} - \binom{n-1-k-t}{k-1}. \quad (6)$$

If $\mathcal{F}(n) = \emptyset$, then adding (5) and (6) yields (2). Suppose $\mathcal{F}(n) \neq \emptyset$ then by the induction hypothesis we have

$$\begin{aligned}
 |\mathcal{F}(n)| + |\mathcal{G}(n)| &\leq 1 + \binom{n-1}{k-1} - \binom{n-1-(k+t-1)}{k-1} \\
 &= \binom{n-1}{k-1} - \binom{n-1-k-t}{k-1} + \left(1 - \binom{n-1-k-t}{k-2}\right). \tag{7}
 \end{aligned}$$

Since $k \geq 2$ and $n > 2k + t$ the term in the bracket is non-positive, adding (5) and (7) concludes the proof. \square

To see that the bound (2) is best possible one takes $\mathcal{F} = \{[k+t]\}$ and $\mathcal{G} = \left\{G \subset \binom{[n]}{k} : G \cap [k+t] \neq \emptyset\right\}$. In the case of (1) equality holds for $\mathcal{F} = \emptyset$ and $\mathcal{G} = \binom{[n]}{k}$ and if $n = 2k + t$ then also for $\mathcal{F} = \binom{[2k+t]}{k+t}$, $\mathcal{G} = \emptyset$. One can show that in all other cases the inequalities are strict.

3. Concluding remarks

Let us mention that Hilton and Milner [3] proved (2) in the case $t = 0$, with the much weaker assumption that both \mathcal{F} and \mathcal{G} are non-empty. However, one can easily show that for $t \geq 1$ it is not sufficient to assume the t -intersecting property instead of \mathcal{F} being $(t + 1)$ -intersecting.

Let us close this paper by a conjecture. Let k, t, s be positive integers, $k \geq s + 1$. Consider the two families

$$\mathcal{F} = \binom{[k+t+s]}{k+t}, \quad \mathcal{G} = \left\{G \in \binom{[n]}{k} : |G \cap [k+t+s]| \geq s+1\right\}.$$

Note that \mathcal{F} is $(t + 1)$ -intersecting, \mathcal{F} and \mathcal{G} are cross-intersecting and $|\mathcal{G}| = \binom{n}{k} - \sum_{0 \leq i \leq s} \binom{k+t+s}{i} \binom{n-k-t-s}{k-i}$ hold.

CONJECTURE. Let k, t, s be positive integers, $k \geq s + 1$. Suppose that $\mathcal{F} \subset \binom{[n]}{k+t}$ is $(t + 1)$ -intersecting and $\binom{[k+t+s]}{t} \subset \mathcal{F}$. Let $\mathcal{G} \subset \binom{[n]}{k}$ and suppose that \mathcal{F} and \mathcal{G}

are cross-intersecting. Then for $n \geq 2k + t$,

$$|\mathcal{F}| + |\mathcal{G}| \leq \binom{k+t+s}{k+t} + \binom{n}{k} - \sum_{0 \leq i \leq s} \binom{k+t+s}{i} \binom{n-k-t-s}{k-i} \quad (8)$$

holds.

Note that setting $s = 0$ we get back (2). We can prove (8) for some small values of k and also for $n > ck^2$, however to prove it in the full range appears to be difficult. Let us mention that it is shown in [2] that in the case $s = t = 1$ assuming $|\mathcal{F}| \geq k + 2$ instead of $\binom{k+2}{k+1} \subset \mathcal{F}$ is not sufficient to guarantee (8).

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