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On some sets with even partition functions

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Abstract: Let $P \in \mathbb{F}_2[z]$ with $P(0) = 1$ and $\mathcal{A} = \mathcal{A}(P)$ be (cf. [12]) the unique subset of $\mathbb{N}$ such that $\sum_{n=0}^{\infty} p(\mathcal{A}, n) z^n \equiv P(z) \pmod 2$, where $p(\mathcal{A}, n)$ is the number of partitions of n with parts in $\mathcal{A}$. In [10], the elements of the set $\mathcal{A}_0 = \mathcal{A}(P^{(0)})$, where $P^{(0)} = 1 + z + z^3 + z^4 + z^5$, are determined and an asymptotic to the counting function of this set is given. In $\mathbb{F}_2[z]$, one has $\frac{1 - z^{31}}{1 - z} = \prod_{i=0}^{5} P^{(i)}$, where the $P^{(i)}\ 's$ are all irreducible of degree 5 and order 31. In this paper, we'll make general the results of [10] for all $\mathcal{A}_i\ 's = \mathcal{A}(P^{(i)})\ 's$, $0 \le i \le 5$. Surprisingly, we obtain for all i, $0 \le i \le 5$, the counting function $A(P^{(i)}, x)$ of the set $\mathcal{A}_i$, when x tends to infinity, satisfies $A(P^{(i)}, x) \sim \kappa \frac{x}{(\log x)^{\frac{1}{4}}}$, where $\kappa = 1.469696766\ldots$. Moreover, we'll give estimates to the counting functions of all $\mathcal{A} = \mathcal{A}(P)$ when $P \in \mathbb{F}_2[z]$ is any squarefree polynomial of order 31.

Keywords: Sets with even partition functions, 2-adic numbers, Selberg-Delange formula, counting functions

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1. Introduction

Let $\mathbb{N}$ be the set of positive integers, $\mathcal{A} = \{a_1, a_2, \ldots\} \subset \mathbb{N}$ and $p(\mathcal{A}, n)$ be the number of partitions of n with parts in $\mathcal{A}$, i.e. the number of solutions of the diophantine equation

$$a_1x_1 + a_2x_2 + ... = n,$$

in non negative integers $x_1, x_2, ...$, with $p(\mathcal{A}, 0) = 1$. Let $\mathbb{F}_2$ be the field with two elements, $f \in \mathbb{F}_2[[z]]$ with $f(0) = 1$ and $\mathcal{A} = \mathcal{A}(f)$ be the unique set of positive integers satisfying (cf. [12] p. 299, [4] p. 67),

$$F(z) = F_{\mathcal{A}}(z) = \prod_{a\in\mathcal{A}} \frac{1}{1-z^a} = \sum_{n=0}^{\infty} p(\mathcal{A}, n)z^n \equiv f(z) \pmod 2. \tag{1.1}$$

Let us recall the construction of these sets. We write

$$f(z) = \sum_{n=0}^{\infty} \epsilon_n z^n \text{ with } \epsilon_n \in \{0, 1\} \text{ and } \epsilon_0 = 1,$$

so that the set $\mathcal{A}(f)$ will satisfy (1.1) if and only if

$$p(\mathcal{A}(f), n) \equiv \epsilon_n \pmod 2, \;\; n = 1, 2, 3, ... \tag{1.2}$$

For $n = 1$,

$$p(\mathcal{A}(f), 1) = \chi(\mathcal{A}(f), 1) = \begin{cases} 1 \text{ if } 1 \in \mathcal{A}(f), \\ 0 \text{ if } 1 \notin \mathcal{A}(f) \end{cases}$$

and therefore, by (1.2),

$$1 \in \mathcal{A}(f) \Leftrightarrow \epsilon_1 = 1.$$

By assuming that the elements of $\mathcal{A}(f)$ up to $n-1$ are known, we set $(\mathcal{A}(f))_{n-1} = \mathcal{A}(f) \cap \{1, 2, ..., n-1\}$ and use the fact that there is exactly one partition of n with the part n to obtain

$$p(\mathcal{A}(f), n) = p((\mathcal{A}(f))_{n-1}, n) + \chi(\mathcal{A}(f), n),$$

so that, from (1.2),

$$n \in \mathcal{A}(f) \Leftrightarrow \chi(\mathcal{A}(f), n) = 1 \Leftrightarrow \epsilon_n \equiv p((\mathcal{A}(f))_{n-1}, n) + 1 \pmod 2.$$

Of course, if $f = P$ is a polynomial in $\mathbb{F}_2[z]$ with $P(0) = 1$, then we can consider P as a formal power series with all coefficients zero from a certain rank and obtain a unique set $\mathcal{A}(P)$ which satisfies (1.1).

Let $A(P,x)$ be the counting function of the set $\mathcal{A}(P)$,

$$A(P,x) = | \{n : 1 \le n \le x,\ n \in \mathcal{A}(P)\} | . \tag{1.3}$$

In 1998 when J.-L. Nicolas, I. Z. Rusza and A. Sárközy introduced the sets $\mathcal{A}(P)$, they hoped obtain some one satisfying $A(P,x) \sim \frac{x}{2},\ x \to \infty$ and gave $\mathcal{A}(1+z+z^3)$ and $\mathcal{A}(1+z+z^3+z^4+z^5)$ as candidates. Their claim turned to be not true, when F. Ben Saïd et al. proved (cf. [5] p. 1118) that for all P, $A(P,x) = o(x)$. Moreover, Ben Saïd et al. determined (cf. [9] and [10]) the elements of $\mathcal{A}(1+z+z^3)$ and $\mathcal{A}(1+z+z^3+z^4+z^5)$.

Let $\sigma(\mathcal{A}(P),n)$ be the sum of the divisors of n belonging to $\mathcal{A}(P)$, i.e.

$$\sigma(\mathcal{A}(P),n) = \sum_{d|n,\ d\in\mathcal{A}(P)} d = \sum_{d|n} d\chi(\mathcal{A}(P),d) \tag{1.4}$$

where $\chi(\mathcal{A}(P),d) = 1$ if $d \in \mathcal{A}(P)$ and $\chi(\mathcal{A}(P),d) = 0$ if $d \notin \mathcal{A}(P)$. By making the logarithmic derivative of (1.1), one obtains

$$\sum_{n=1}^{\infty} \sigma(\mathcal{A}(P),n) z^n = z\frac{F'(z)}{F(z)} \equiv z\frac{P'(z)}{P(z)} \pmod 2. \tag{1.5}$$

This formula first allowed to prove that the sequence $(\sigma(\mathcal{A}(P),n) \mod 2)_{n\ge 1}$ is periodic, then after being generalized to

$$\sum_{n=1}^{\infty} \sigma(\mathcal{A}(P),2^k n) z^n = z\frac{F'_{(k)}(z)}{F_{(k)}(z)} \equiv z\frac{P'_{(k)}(z)}{P_{(k)}(z)} \pmod{2^{k+1}},$$

where $P_{(k)}$ is the Graeffe transformation of P of order k and $P'_{(k)}$ is the derivative of $P_{(k)}$ (cf. [8] p. 190), gave that for all $k \ge 0$, $(\sigma(\mathcal{A}(P),2^k n) \mod 2^{k+1})_{n\ge 1}$ is periodic. F. Ben Saïd and J.-L. Nicolas gave first (cf. [8] p. 192) the smallest period of $(\sigma(\mathcal{A}(P),2^k n) \mod 2^{k+1})_{n\ge 1}$ for all $k \ge 0$.

Let β be an odd integer ≥ 1, $P \in \mathbb{F}_2[z]$ with $P(0) = 1$ and of order β, i.e. β is the smallest positive integer such that P divides $1 + z^\beta$ in $\mathbb{F}_2[z]$. In [3] p. 28, N. Baccar et al. showed that β is the smallest period of all the sequences $(\sigma(\mathcal{A}(P),2^k n) \mod 2^{k+1})_{n\ge 1}$ and that if $m_2 \equiv 2^a m_1 \pmod \beta$ for some $a \ge 0$, then

$$\sigma(\mathcal{A}(P),2^k m_2) \equiv \sigma(\mathcal{A}(P),2^k m_1) \pmod{2^{k+1}}. \tag{1.6}$$

Let $(\mathbb{Z}/\beta\mathbb{Z})^*$ be the group of invertible elements modulo β and $< 2 >$ be its subgroup generated by 2. By setting $r = \dfrac{\varphi(\beta)}{s}$, where s is the order of 2 modulo β, that is the smallest positive integer s such that $2^s \equiv 1 \pmod{\beta}$, one has

$$\mathbb{Z}/\beta\mathbb{Z} = \{x_0 = 1, ..., x_{r-1}, y_1, ..., y_q\},$$

where the $x_i's$ are the invertible orbits and the $y_i's$ are the non invertible ones. The orbit $Orb(n)$ of some $n \in \mathbb{Z}/\beta\mathbb{Z}$ is given by

$$Orb(n) = \{2^k n, \quad 0 \leq k \leq s-1\}. \tag{1.7}$$

Let ϕ_β be the cyclotomic polynomial of index β,

$$\phi_\beta = \prod_{1\leq i\leq\beta,\ gcd(i,\beta)=1} (z - \zeta^i),$$

where ζ is some primitive β^{th} root of unity over $\mathbb{F}_2$. If ϕ_β is written as

$$\phi_\beta = P^{(0)} P^{(1)} \cdots P^{(r-1)}$$

where for all j, $0 \leq j \leq r-1$,

$$P^{(j)} = \prod_{i=0}^{s-1} (z - \zeta^{2^i x_j})$$

and if $\mathcal{A}_j = \mathcal{A}(P^{(j)})$ is the set obtained from (1.1) then (cf. [3] p. 30), for all $k \geq 0$, one has

$$\sigma(\mathcal{A}_j, 2^k\beta) \equiv -s \pmod{2^{k+1}}, \tag{1.8}$$

$$\sigma(\mathcal{A}_j, 2^k n) \equiv \sigma(\mathcal{A}_0, 2^k n x_j) \pmod{2^{k+1}}. \tag{1.9}$$

In this paper, we are interested on the case $\beta = 31$. The order of 2 modulo 31 is $s = 5$. The group $(\mathbb{Z}/31\mathbb{Z})^*$ is cyclic with $g = 3$ as a generator. The orbits of $\mathbb{Z}/31\mathbb{Z}$ are $x_i = 3^i$, $0 \leq i \leq 5$ and $y_1 = 31$. In $\mathbb{F}_2[z]$, we will write

$$\frac{1 - z^{31}}{1 - z} = P^{(0)} P^{(1)} ... P^{(5)},$$

where

$$P^{(0)} = 1 + z + z^3 + z^4 + z^5,\ P^{(1)} = 1 + z^2 + z^5,\ P^{(2)} = 1 + z^2 + z^3 + z^4 + z^5,$$
$$P^{(3)} = 1 + z + z^2 + z^4 + z^5,\ P^{(4)} = 1 + z^3 + z^5,\ P^{(5)} = 1 + z + z^2 + z^3 + z^5 \tag{1.10}$$

in a way that (cf. (1.9)) for all i, $0 \le i \le 5$,

$$\sigma(\mathcal{A}_i, 2^k n) \equiv \sigma(\mathcal{A}_0, 2^k 3^i n) \pmod{2^{k+1}}. \tag{1.11}$$

This choice of indexing the $P^{(i)}$'s is very important. It is the basic idea which leads to the results of this paper.

From now on, for $0 \le i \le 5$, $\mathcal{A}_i = \mathcal{A}(P^{(i)})$ is the set defined by (1.1). To determine the elements of the sets $\mathcal{A}_i$, $0 \le i \le 5$, of the form $2^k m$, where $k \ge 0$ and m is odd, we should consider the sum

$$S_{\mathcal{A}_i}(m, k) = \sum_{h=0}^{k} 2^h \chi(\mathcal{A}_i, 2^h m). \tag{1.12}$$

as well as the 2-adic number

$$S_i(m) = \sum_{j=0}^{\infty} 2^j \chi(\mathcal{A}_i, 2^j m), \tag{1.13}$$

which satisfies

$$S_i(m) \equiv S_{\mathcal{A}_i}(m, k) \pmod{2^{k+1}},\ \forall\ k \ge 0. \tag{1.14}$$

Note here that for all $k \ge 0$, we have

$$0 \le S_{\mathcal{A}_i}(m, k) < 2^{k+1},$$

so that

$$S_{\mathcal{A}_i}(m, k) \equiv 0 \pmod{2^{k+1}} \Rightarrow S_{\mathcal{A}_i}(m, k) = 0 \Rightarrow 2^h m \notin \mathcal{A},\ \forall\, h,\ 0 \le h \le k. \tag{1.15}$$

For $n = 2^k m$ with m odd, we have

$$\sigma(\mathcal{A}_i, n) = \sum_{\delta | n} \delta\chi(\mathcal{A}_i, \delta) = \sum_{d|m} \sum_{h=0}^{k} 2^h d\chi(\mathcal{A}_i, 2^h d) = \sum_{d|m} dS_{\mathcal{A}_i}(d, k)$$

and by Möbius inversion formula,

$$mS_{\mathcal{A}_i}(m, k) = \sum_{d|m} \mu(d)\sigma(\mathcal{A}_i, \frac{n}{d}) = \sum_{d|\overline{m}} \mu(d)\sigma(\mathcal{A}_i, \frac{n}{d}), \tag{1.16}$$

where $\overline{m} = \prod_{p|m} p$ is the radical of m, with $\overline{1} = 1$.
Since a divisor of $2^{k+1}m$ is a divisor of $2^k m$ or a multiple of 2^{k+1}, from (1.4), we have

$$\sigma(\mathcal{A}_i, 2^{k+1}m) \equiv \sigma(\mathcal{A}_i, 2^k m) \pmod{2^{k+1}}, \ \forall \ k \geq 0$$

and the sequence $\left(\sigma(\mathcal{A}_i, 2^k m)\right)_{k\geq 0}$ defines a 2-adic integer $\rho_i(m)$ which satisfies

$$\rho_i(m) \equiv \sigma(\mathcal{A}_i, 2^k m) \pmod{2^{k+1}}, \ \forall \ k \geq 0. \tag{1.17}$$

This with (1.14) and (1.16) give

$$mS_i(m) = \sum_{d\,|\,m} \mu(d)\rho_i(\frac{m}{d}) = \sum_{d\,|\,\overline{m}} \mu(d)\rho_i(\frac{m}{d}). \tag{1.18}$$

In Section 2, for $0 \leq i \leq 5$, we evaluate the 2-adic integer $S_i(m)$, from which we obtain results of section 3 that give the elements of the sets $\mathcal{A}_i = \mathcal{A}(P^{(i)})$, $0 \leq i \leq 5$. The last section is devoted to the determination of asymptotic estimates to the counting functions $A(P^{(i)}, x)$, $0 \leq i \leq 5$, as well as $A(P, x)$ when P is any squarefree polynomial in $\mathbb{F}_2[z]$ of order 31 . Our paper is inspired from and based on [10].

2. The 2-adic integers $S_i(m)$, $0 \leq i \leq 5$

We begin by the following:

Remark 2.1. [3] p. 28. No element of the set $\mathcal{A}_i$, $0 \leq i \leq 5$, has an odd prime factor in the orbit $Orb(1)$ and no integer divisible by 31^2 belongs to $\mathcal{A}_i$.

For $k \geq 0$ and $0 \leq j \leq 5$, let

$$u_{k,j} = \sigma(\mathcal{A}_0, 2^k 3^j) \pmod{2^{k+1}}. \tag{2.1}$$

Since a divisor of $2^{k+1}3^j$ is either a divisor of $2^k 3^j$ or a multiple of 2^{k+1}, then from (1.4) and (2.1), $u_{k+1,j} \equiv u_{k,j} \pmod{2^{k+1}}$, $\forall\ k \geq 0$, holds and the sequence $(u_{k,j})_{k \geq 0}$ defines a 2-adic integer U_j satisfying for all $k \geq 0$,

$$U_j \equiv u_{k,j} \pmod{2^{k+1}}. \tag{2.2}$$

In [1] Theorem 1, it is shown that the $U_j's$ are the roots of the polynomial

$$R(y) = y^6 - y^5 + 3y^4 - 11y^3 + 44y^2 - 36y + 32.$$

Moreover, it is given in [10] p. 55 that the Galois group of $R(y)$ is cyclic of order 6 and that the roots U_1, ..., U_5 are polynomials in $\theta = U_0$. We have (cf. [10] p. 55)

Z		$Z \bmod 2^{11}$
$U_0 =$	θ	1183
$U_1 =$	$\frac{1}{32}(3\theta^5 + 5\theta^3 - 36\theta^2 + 84\theta)$	1598
$U_2 =$	$\frac{1}{32}(-3\theta^5 - 5\theta^3 + 20\theta^2 - 100\theta)$	1554
$U_3 =$	$\frac{1}{32}(-\theta^5 - 7\theta^3 + 12\theta^2 - 44\theta + 32)$	845
$U_4 =$	$\frac{1}{32}(-\theta^5 + 4\theta^4 + \theta^3 + 24\theta^2 - 68\theta + 96)$	264
$U_5 =$	$\frac{1}{16}(\theta^5 - 2\theta^4 + 3\theta^3 - 10\theta^2 + 48\theta - 48)$	701

TABLE 1

Since the $U_j's$ are periodic with period 6, we write for all $j \in \mathbb{Z}$,

$$U_j = U_{j \bmod 6}. \tag{2.3}$$

Let $(v_i)_{i\in\mathbb{Z}}$ be the periodic sequence of period 12 defined by

$$v_i = \begin{cases} \dfrac{2}{\sqrt{3}} \cos\left(i\dfrac{\pi}{6}\right) & \text{if i is odd} \\ 2\cos\left(i\dfrac{\pi}{6}\right) & \text{if i is even.} \end{cases}$$

We also consider the 2-adic integers

$$E_i = \sum_{j=0}^{5} v_{i+2j} U_j, \;\; F_i = \sum_{j=0}^{5} v_{i+4j} U_j, \;\; i \in \mathbb{Z}, \quad G = \sum_{j=0}^{5} (-1)^j U_j. \tag{2.4}$$

The values of these integers are given (cf. [10] p. 60) in the following table:

Z		$Z \bmod 2^{11}$
$E_0 =$	$\frac{1}{32}(11\theta^5 - 8\theta^4 + 29\theta^3 - 124\theta^2 + 500\theta - 256)$	1157
$E_1 =$	$\frac{1}{16}(3\theta^5 - 2\theta^4 + 9\theta^3 - 26\theta^2 + 136\theta - 64)$	1533
$E_2 =$	$3E_1 - E_0$	1394
$E_3 =$	$2E_1 - E_0$	1909
$E_4 =$	$3E_1 - 2E_0$	237
$E_5 =$	$E_1 - E_0$	376
$F_0 =$	$\frac{1}{32}(-3\theta^5 - 21\theta^3 + 36\theta^2 - 36\theta + 64)$	1987
$F_1 =$	$\frac{1}{32}(-3\theta^5 - 4\theta^4 - 13\theta^3 + 24\theta^2 - 28\theta - 64)$	166
$F_2 =$	$3F_1 - F_0$	559
$F_3 =$	$2F_1 - F_0$	393
$F_4 =$	$3F_1 - 2F_0$	620
$F_5 =$	$F_1 - F_0$	227
$G =$	$\frac{1}{4}(-\theta^5 + \theta^4 - \theta^3 + 11\theta^2 - 34\theta + 20)$	1905

TABLE 2

Note here that we have,

$$E_{i+12} = E_i,\ E_{i+6} = -E_i,\quad F_{i+12} = F_i, F_{i+6} = -F_i. \tag{2.5}$$

We also have (cf [10] p. 60),

LEMMA 2.1. *The polynomials $(U_j)_{0\le j\le 5}$ form a basis of $\mathbb{Q}[\theta]$. The polynomials E_0, E_1, F_0, F_1, G, U_0 form another basis of $\mathbb{Q}[\theta]$. For all $i's$, E_i and F_i are linear combinations of respectively E_0 and E_1 and F_0 and F_1.*

Some other properties of these 2-adic integers and the sequence $(v_i)_{i\in\mathbb{Z}}$ can be found in [10] p. 59.

We make (cf. [10] p. 56) the following definitions:

- ℓ is the completely additive function from $\mathbb{Z}\setminus 31\mathbb{Z}$ to $\mathbb{Z}/6\mathbb{Z}$ given by

$$\ell(n) = j \text{ if } n \in Orb(3^j). \tag{2.6}$$

From (1.6) and (1.17), we have for $0 \le i \le 5$, $k \ge 0$ and $31 \nmid n_1 n_2$,

$$\ell(n_1) = \ell(n_2) \Rightarrow \sigma(\mathcal{A}_i, 2^k n_1) \equiv \sigma(\mathcal{A}_i, 2^k n_2) \pmod{2^{k+1}} \Rightarrow \rho_i(n_1) = \rho_i(n_2). \tag{2.7}$$

- The set of odd primes different from 31 is split into six classes $(\mathcal{P}_j)_{0\le j\le 5}$ according to the values of ℓ. More precisely, for $0 \le j \le 5$,

$$p \in \mathcal{P}_j \Leftrightarrow \ell(p) = j \Leftrightarrow p \equiv 2^k 3^j \pmod{31},\ k = 0, 1, ..., 4. \tag{2.8}$$

- For $0 \le j \le 5$, $\omega_j : \mathbb{N} \to \mathbb{N} \cup \{0\}$ is the additive function given by

$$\omega_j(n) = \sum_{p|n,\ p\in\mathcal{P}_j} 1 = \sum_{p|n,\ \ell(p)=j} 1 \tag{2.9}$$

and $\omega(n) = \omega_0(n) + ... + \omega_5(n) = \sum_{p|n} 1$, $(n, 31) = 1$.

The radical $\overline{m}$ of an odd integer $m \ne 1$, relatively prime with 31 and free of prime factors belonging to $\mathcal{P}_0$ will be written

$$\overline{m} = p_1 \cdots p_{\omega_1} p_{\omega_1+1} \cdots p_{\omega_1+\omega_2} p_{\omega_1+\omega_2+1} \cdots p_{\omega_1+\omega_2+\omega_3+\omega_4+1} \cdots p_{\omega}, \tag{2.10}$$

where $\ell(p_i) = j$ for $\omega_1 + \cdots + \omega_{j-1} + 1 \le i \le \omega_1 + \cdots + \omega_j$, $\omega_j = \omega_j(m) = \omega_j(\overline{m})$ and $\omega = \omega(m) = \omega(\overline{m}) \ge 1$. We also define, for $0 \le i \le 5$, the arithmetical functions from $\mathbb{Z} \setminus 31\mathbb{Z}$ into $\mathbb{Z}/12\mathbb{Z}$:

$$\alpha_i = \alpha_i(m) = 2\omega_5 - 2\omega_1 + \omega_4 - \omega_2 - 2i \mod 12, \tag{2.11}$$

$$a_i = a_i(m) = \omega_5 - \omega_1 + \omega_2 - \omega_4 - 4i \mod 12 \tag{2.12}$$

and make the convention

$$0^n = \begin{cases} 1 \text{ if } n = 0 \\ 0 \text{ if } n > 0. \end{cases} \tag{2.13}$$

Lemma 2.2. [10] *p.* 61. *Let m be an odd integer not divisible by* 31, *with $\overline{m}$ of the form* (2.10) *and*

$$T(m,j) = T(\overline{m},j) = \sum_{d|\overline{m},\ \ell(d)\equiv j \pmod 6} \mu(d), \tag{2.14}$$

where μ is the Möbius function and ℓ is the function defined by (2.6). *Under the convention* (2.13), *we have*

$$T(m,j) = 2^{\omega_3-1}3^{\lceil\frac{\omega_2+\omega_4}{2}-1\rceil}v_{\alpha_j} + \frac{0^{\omega_3}}{2}3^{\lceil\frac{\omega}{2}-1\rceil}v_{a_j} + 0^{\omega_2+\omega_4}\frac{(-1)^j}{3}2^{\omega-1}, \tag{2.15}$$

where

$$\lceil x \rceil = inf\{n \in \mathbb{Z},\ n \ge x\}$$

is the ceiling of the real number x.

Theorem 2.1. *Let $m \ne 1$ be an odd integer relatively prime with* 31 *and with $\overline{m}$ of the form* (2.10). *Under the above notation and the convention* (2.13), *for all i, $0 \le i \le 5$, we have*:

(1) *The* 2*-adic integer $S_i(m)$ defined by* (1.13) *satisfies*

$$mS_i(m) = 2^{\omega_3-1}3^{\lceil\frac{\omega_2+\omega_4}{2}-1\rceil}E_{\alpha_i-2\ell(m)} + \frac{0^{\omega_3}}{2}3^{\lceil\frac{\omega}{2}-1\rceil}F_{a_i-4\ell(m)} + \frac{0^{\omega_2+\omega_4}}{3}2^{\omega-1}(-1)^{\ell(m)+i}G. \tag{2.16}$$

(2) *The* 2*-adic integer $S_i(31m)$ satisfies*

$$S_i(31m) = -31^{-1}S_i(m), \tag{2.17}$$

where 31^{-1} is the inverse of 31 *in $\mathbb{Z}_2$.*

PROOF OF THEOREM 2.1 (1). From (1.18), we have

$$mS_i(m) = \sum_{d\,|\,\overline{m}} \mu(d)\rho_i(\frac{m}{d}),$$

which, by (1.17) and (1.11), gives

$$mS_i(m) = \sum_{d\,|\,\overline{m}} \mu(d)\rho_0(3^i\frac{m}{d}).$$

Since $gcd(m,31)=1$, $3^i\dfrac{m}{d}$ and $3^{\ell(3^i\frac{m}{d})}$ are in the same orbit so that

$$mS_i(m) = \sum_{d\,|\,\overline{m}} \mu(d)\rho_0(3^{\ell(3^i\frac{m}{d})}).$$

This with (1.17) and (2.2) give

$$mS_i(m) = \sum_{d\,|\,\overline{m}} \mu(d)U_{\ell(3^i\frac{m}{d})}.$$

Further, from the additivity of ℓ and (2.14), we get

$$mS_i(m) = \sum_{d|\overline{m}} \mu(d)U_{\ell(m)+i-\ell(d)} = \sum_{j=0}^{5} T(m,j)U_{\ell(m)+i-j} = \sum_{j=0}^{5} T(m,\ell(m)-j)U_{i+j}.$$

So, by using Lemma 2.2 and (2.4), we obtain

$$\begin{aligned} mS_i(m) &= 2^{\omega_3-1}3^{\lceil\frac{\omega_2+\omega_4}{2}-1\rceil}\sum_{j=0}^{5} v_{\alpha_{\ell(m)-j}}U_{i+j} + \frac{0^{\omega_3}}{2}3^{\lceil\frac{\omega}{2}-1\rceil}\sum_{j=0}^{5} v_{a_{\ell(m)-j}}U_{i+j} \\ &\quad + 0^{\omega_2+\omega_4}\frac{(-1)^{\ell(m)}}{3}2^{\omega-1}\sum_{j=0}^{5}(-1)^jU_{i+j} \\ &= 2^{\omega_3-1}3^{\lceil\frac{\omega_2+\omega_4}{2}-1\rceil}\sum_{j=0}^{5} v_{\alpha_i-2\ell(m)+2j}U_j + \frac{0^{\omega_3}}{2}3^{\lceil\frac{\omega}{2}-1\rceil}\sum_{j=0}^{5} v_{a_i-4\ell(m)+4j}U_j \\ &\quad + 0^{\omega_2+\omega_4}\frac{(-1)^{\ell(m)}}{3}2^{\omega-1}\sum_{j=0}^{5}(-1)^i(-1)^jU_j \\ &= 2^{\omega_3-1}3^{\lceil\frac{\omega_2+\omega_4}{2}-1\rceil}E_{\alpha_i-2\ell(m)} + \frac{0^{\omega_3}}{2}3^{\lceil\frac{\omega}{2}-1\rceil}F_{a_i-4\ell(m)} + 0^{\omega_2+\omega_4}\frac{(-1)^{\ell(m)+i}}{3}2^{\omega-1}G. \quad \square \end{aligned}$$

Proof of Theorem 2.1 (2). By using (1.18), we get

$$31mS_i(31m) = \sum_{d|31m} \mu(d)\rho_i(31\frac{m}{d}) = \sum_{d|31\overline{m}} \mu(d)\rho_i(31\frac{m}{d})$$

$$= \sum_{d|\overline{m}} \mu(d)\rho_i(31\frac{m}{d}) - \sum_{d|\overline{m}} \mu(d)\rho_i(\frac{m}{d})$$

$$= \sum_{d|\overline{m}} \mu(d)\rho_i(31\frac{m}{d}) - mS_i(m). \tag{2.18}$$

For all d dividing $\overline{m}$, $31\dfrac{m}{d} \in Orb(31)$. Hence, from (1.8), $\rho_i(31\frac{m}{d}) = \rho_i(31) = -5$, so that from (2.18), we get

$$31mS_i(31m) + mS_i(m) = -5\sum_{d|\overline{m}} \mu(d). \tag{2.19}$$

Since $m \neq 1$, the sum $\sum_{d|\overline{m}} \mu(d)$ vanishes and therefore $31mS_i(31m) + mS_i(m) = 0$, so that $31S_i(31m) + S_i(m) = 0$ and $S_i(31m) = -31^{-1}S_i(m)$, which ends the proof. □

3. Elements of the sets $\mathcal{A}_i = \mathcal{A}(P^{(i)})$, $0 \leq i \leq 5$

As mentioned in Remark 2.1, no element of the set $\mathcal{A}_i$ has a prime factor in the orbit $Orb(1)$ and no integer divisible by 31^2 belongs to $\mathcal{A}_i$. So, here, we consider integers of the form $n = 2^k 31^\tau m$, where $\overline{m}$ satisfies (2.10) and $\tau \in \{0, 1\}$. In [1] Theorem 3.1, it is shown that the elements of the set $\mathcal{A}_0$ of the form 2^k, $k \geq 0$, are given by the 2-adic solution

$$\sum_{k\geq 0} \chi(\mathcal{A}_0, 2^k)2^k = S_0(1) = U_0 = 1 + 2 + 2^2 + 2^3 + 2^4 + 2^7 + 2^{10} + 2^{11} + ...$$

of the equation

$$y^6 - y^5 + 3y^4 - 11y^3 + 44y^2 - 36y + 32 = 0.$$

Hence, by using (1.9), we may conclude that the elements of $\mathcal{A}_i$, $0 \le i \le 5$, with this form, are given by the 2-adic solution U_i of the same equation. For instance, those of $\mathcal{A}_1$ are given by

$$U_1 = 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^9 + 2^{10} + \dots$$

From (2.19), it follows that the elements of the set $\mathcal{A}_i$ of the form $31 \cdot 2^k$, $k \ge 0$, are given by

$$\sum_{k \ge 0} \chi(\mathcal{A}_i, 31 \cdot 2^k) 2^k = S_i(31) = \frac{5 + S_i(1)}{1 - 32}$$

For example, those of the set $\mathcal{A}_1$ are given by

$$S_1(31) = \frac{5 + U_1}{1 - 32} = (1 + 4 + U_1)(1 + 2^5 + 2^{10} + \dots) = 1 + 2 + 2^5 + 2^7 + 2^9 + \dots$$

Now, let us look for the other elements.

THEOREM 3.1. *Let $m \ne 1$ be an odd integer not divisible by any prime in Orb(1) neither by 31. Then the sum $S_i(m)$, $0 \le i \le 5$, given by (1.13) does not vanish. Let*

$$\gamma_i = \gamma_i(m) = v_2(S_i(m)) \tag{3.1}$$

be the 2-adic valuation of $S_i(m)$. We recall the quantities $\omega_i = \omega_i(m)$ given by (2.9), $\ell(m)$ defined by (2.6), $\alpha_i = \alpha_i(m)$, $a_i = a_i(m)$ from (2.11) and (2.12). We also introduce, for $0 \le i \le 5$, the quantities

$$\alpha'_i = \alpha'_i(m) = \alpha_i - 2\ell(m) \bmod 12 = 2\omega_5 - 2\omega_1 + \omega_4 - \omega_2 - 2i - 2\ell(m) \bmod 12, \tag{3.2}$$

$$a'_i = a'_i(m) = a_i - 4\ell(m) \bmod 12 = \omega_5 - \omega_1 + \omega_2 - \omega_4 - 4i - 4\ell(m) \bmod 12, \tag{3.3}$$

$$t = t(m) = \lceil \frac{\omega_1 + \omega_5 + \omega_2 + \omega_4}{2} - 1 \rceil - \lceil \frac{\omega_2 + \omega_4}{2} - 1 \rceil$$

$$= \begin{cases} \lceil \frac{\omega_1 + \omega_5}{2} \rceil & \text{if } \omega_1 + \omega_5 \equiv \omega_2 + \omega_4 \equiv 1 \pmod 2 \\ \lceil \frac{\omega_1 + \omega_5}{2} - 1 \rceil & \text{if not.} \end{cases} \tag{3.4}$$

We have

(*i*) *If* $\omega_3 \neq 0$ *and* $\omega_2 + \omega_4 \neq 0$, *the value of* $\gamma_i = \gamma_i(m)$ *is given by*

$$\gamma_i = \begin{cases} \omega_3 - 1 \text{ if } \alpha_i' \equiv 0,\ 1,\ 3,\ 4 \pmod 6 \\ \omega_3 \quad \text{if } \alpha_i' \equiv 2 \pmod 6 \\ \omega_3 + 2 \text{ if } \alpha_i' \equiv 5 \pmod 6). \end{cases} \tag{3.5}$$

(*ii*) *If* $\omega_2 + \omega_4 = 0$ *and* $\omega_3 \neq 0$, *we set*

$$\alpha_i" = \alpha_i' + 6(\ell(m) + i) \bmod 12 \text{ and } \delta_j = v_2(E_j + 2^{v_2(E_j)}G) \tag{3.6}$$

and we have

$$if\ \omega_1 + \omega_5 < v_2(E_{\alpha_i"}),\ \text{then } \gamma_i = \omega_3 - 1 + \omega_1 + \omega_5,$$

$$if\ \omega_1 + \omega_5 = v_2(E_{\alpha_i"}),\ \text{then } \gamma_i = \omega_3 - 1 + \delta(\alpha_i"),$$

$$if\ \omega_1 + \omega_5 > v_2(E_{\alpha_i"}),\ \text{then } \gamma_i = \omega_3 - 1 + v_2(E_{\alpha_i"}).$$

(*iii*) *If* $\omega_3 = 0$ *and* $\omega_2 + \omega_4 \neq 0$, *we have*

$$\gamma_i = -1 + v_2(E_{\alpha_i'} + 3^t F_{a_i'}).$$

(*iv*) *If* $\omega_2 = \omega_3 = \omega_4 = 0$ *and* $\omega_1 + \omega_5 \neq 0$, *we have*

$$\gamma_i = -1 + v_2(E_{\alpha_i'} + 3^t F_{a_i'} + 2^{\omega_1+\omega_5}(-1)^{\ell(m)+i}G).$$

Proof of Theorem 3.1 (i). From (2.16), we get

$$mS_i(m) = 2^{\omega_3-1}3^{\lceil \frac{\omega_2+\omega_4}{2} - 1 \rceil} E_{\alpha_i - 2\ell(m)} = 2^{\omega_3-1}3^{\lceil \frac{\omega_2+\omega_4}{2} - 1 \rceil} E_{\alpha_i'}.$$

Since, from TABLE 2 and (2.5), $E_{\alpha_i'} \neq 0$, then $S_i(m)$ does not vanish and

$$\gamma_i = v_2(S_i(m)) = \omega_3 - 1 + v_2(E_{\alpha_i'}).$$

To obtain (3.5), it suffices to use the values of $E_{\alpha_i'}$ mod 2^{11} given in TABLE 2. □

Proof of Theorem 3.1 (ii). In this case, by use of (3.6) and (2.5), formula (2.16) becomes

$$mS_i(m) = \frac{2^{\omega_3 - 1}}{3}(E_{\alpha_i'} + 2^{\omega_1+\omega_5}(-1)^{\ell(m)+i}G) = \frac{2^{\omega_3-1}}{3}(-1)^{\ell(m)+i}(E_{\alpha_i''} + 2^{\omega_1+\omega_5}G).$$

This with Lemma 2.1 show that $S_i(m)$ does not vanish and that

$$\gamma_i = \omega_3 - 1 + v_2(E_{\alpha_i''} + 2^{\omega_1+\omega_5}G). \qquad \square$$

Proof of Theorem 3.1 (iii). If $\omega_3 = 0$ and $\omega_2 + \omega_4 \neq 0$, formula (2.16) gives

$$mS_i(m) = \frac{3^{\lceil \frac{\omega_2+\omega_4}{2} - 1 \rceil}}{2} E_{\alpha_i - 2\ell(m)} + \frac{3^{\lceil \frac{\omega}{2} - 1 \rceil}}{2} F_{a_i - 4\ell(m)} = \frac{3^{\lceil \frac{\omega_2+\omega_4}{2} - 1 \rceil}}{2}(E_{\alpha_i'} + 3^t F_{a_i'}).$$

Since, from Lemma 2.1, E_i and F_i are non-zero linear combinations of respectively E_0 and E_1 and F_0 and F_1, then $E_{\alpha_i'} + 3^t F_{a_i'}$ does not vanish and

$$\gamma_i = -1 + v_2(E_{\alpha_i'} + 3^t F_{a_i'}). \qquad \square$$

Proof of Theorem 3.1 (iv). If $\omega_2 = \omega_3 = \omega_4 = 0$, formula (2.16) gives

$$mS_i(m) = \frac{1}{6}(E_{\alpha_i'} + 3^t F_{a_i'} + 2^{\omega_1+\omega_5}(-1)^{\ell(m)+i}G).$$

Once again, from Lemma 2.1, we have

$$E_{\alpha_i'} + 3^t F_{a_i'} + 2^{\omega_1+\omega_5}(-1)^{\ell(m)+i}G \neq 0,$$

so that $S_i(m) \neq 0$ and

$$\gamma_i = -1 + v_2(E_{\alpha_i'} + 3^t F_{a_i'} + 2^{\omega_1+\omega_5}(-1)^{\ell(m)+i}G). \qquad \square$$

Remark 3.1. Under the same hypothesis made on m in Theorem 3.1, by using (2.17), we obtain for all $i \in \{0, ..., 5\}$, $S_i(31m) \neq 0$ and

$$v_2(S_i(31m)) = v_2(S_i(m)).$$

THEOREM 3.2. *Let m be an odd integer satisfying $m \neq 1$, $\gcd(m, 31) = 1$ and with $\overline{m}$ of the form (2.10). Let $\gamma_i = \gamma_i(m) = v_2(mS_i(m))$ be the 2-adic valuation of $mS_i(m)$ and $\widehat{mS_i(m)}$ be the odd part of the right hand side of (2.16), so that*

$$mS_i(m) = 2^{\gamma_i(m)}\widehat{mS_i(m)} \tag{3.7}$$

(*i*) *If $k < \gamma_i$, then $2^h m \notin \mathcal{A}_i$ and $2^h 31m \notin \mathcal{A}_i$, $\forall\ h \leq k$.*

(*ii*) *If $k = \gamma_i$, then $2^k m \in \mathcal{A}_i$ and $2^k 31m \in \mathcal{A}_i$.*

(*iii*) *If $k = \gamma_i + r$, $r \geq 1$, then we set $\mathcal{S}_r = \{2^r + 1, 2^r + 3, ..., 2^{r+1} - 1\}$ and we have*

$$2^{\gamma_i + r} m \in \mathcal{A} \Leftrightarrow \exists\ l \in \mathcal{S}_r,\ m \equiv l^{-1}\widehat{mS_i(m)} \pmod{2^{r+1}},$$

$$2^{\gamma_i + r} 31m \in \mathcal{A} \Leftrightarrow \exists\ l \in \mathcal{S}_r,\ m \equiv -(31l)^{-1}\widehat{mS_i(m)} \pmod{2^{r+1}}.$$

PROOF OF THEOREM 3.2 (i). From (1.14), we have

$$S_i(m) \equiv S_{\mathcal{A}_i}(m, k) \pmod{2^{k+1}}. \tag{3.8}$$

Since m is odd then, from (3.7), we get for $\gamma_i > k$, $S_{\mathcal{A}_i}(m, k) \equiv 0 \pmod{2^{k+1}}$. So that from (1.15), $S_{\mathcal{A}_i}(m, k) = 0$ and $2^h m \notin \mathcal{A}_i$, for all h, $0 \leq h \leq k$. This with (2.17) also give $2^h 31m \notin \mathcal{A}_i$, for all h, $0 \leq h \leq k$. □

PROOF OF THEOREM 3.2 (ii). When $\gamma_i = k$, proceeding as above gives,

$$mS_{\mathcal{A}_i}(m, k) \equiv 2^k \widehat{mS_i(m)} \pmod{2^{k+1}}.$$

So that, from Theorem 3.2 (i) and (1.12), we get

$$2^k m\chi(\mathcal{A}_i, 2^k m) \equiv 2^k \widehat{mS_i(m)} \pmod{2^{k+1}}.$$

Since m and $\widehat{mS_i(m)}$ are odd, we obtain

$$\chi(\mathcal{A}_i, 2^k m) \equiv 1 \pmod{2},$$

which shows that $2^k m \in \mathcal{A}_i$. This with (2.17) give $2^k 31m \in \mathcal{A}_i$. □

PROOF OF THEOREM 3.2 (iii). In this case, (3.7) and (1.14) give,

$$mS_{\mathcal{A}_i}(m,k) \equiv 2^{\gamma_i} m\widehat{S_i(m)} \pmod{2^{\gamma_i+r+1}}.$$

So that , from Theorem 3.2 (i) and (ii), we get

$$m\left(2^{\gamma_i} + 2^{\gamma_i+1}\chi(\mathcal{A}_i, 2^{\gamma_i+1}m) + \ldots + 2^{\gamma_i+r}\chi(\mathcal{A}_i, 2^{\gamma_i+r}m)\right) \equiv 2^{\gamma_i} m\widehat{S_i(m)} \pmod{2^{\gamma_i+r+1}},$$

which reduces to,

$$m\left(1 + 2\chi(\mathcal{A}_i, 2^{\gamma_i+1}m) + \ldots + 2^{r}\chi(\mathcal{A}_i, 2^{\gamma_i+r}m)\right) \equiv m\widehat{S_i(m)} \pmod{2^{r+1}}.$$

Here, we can observe that $2^{\gamma_i+r}m \in \mathcal{A}$ if and only if

$$l = 1 + 2\chi(\mathcal{A}_i, 2^{\gamma_i+1}m) + \ldots + 2^{\gamma_i+r}\chi(\mathcal{A}_i, 2^{\gamma_i+r}m)$$

is an odd integer in $\mathcal{S}_r = \{2^r+1, 2^r+3, \ldots, 2^{r+1}-1\}$ and we get

$$2^{\gamma_i+r}m \in \mathcal{A} \Leftrightarrow m \equiv l^{-1} m\widehat{S_i(m)} \pmod{2^{r+1}},\ l \in \mathcal{S}_r.$$

This with (2.17) give the result for $2^{\gamma_i+r}31m$. □

4. The counting function of the set $\mathcal{A}(P)$ when P is any polynomial of order 31

We begin by some results from [10]. In the sequel $\mathcal{O}$ denotes the Landau's symbol.

LEMMA 4.1. [10, *Lemma* 4.3]. *For* $i \in \{2,3,4\}$, *let*

$$K_i = K_i(x) = \prod_{p\le x,\ \ell(p)\in\{0,i\}} p = \prod_{p\le x,\ p\in\mathcal{P}_0\cup\mathcal{P}_i} p,$$

where ℓ, $\mathcal{P}_0$ *and* $\mathcal{P}_i$ *are defined by* (2.6)-(2.8). *Then for* x *large enough*,

$$|\{n : 1 \le n \le x,\ \ \gcd(n, K_i) = 1\}| = \mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right).$$

Lemma 4.2. [10, *Lemma* 4.5]. *We keep the above notation and we let $\mathcal{G}$ be the set of integers of the form $n = 2^{\omega_3(m)}m$ with the following conditions*:

(*i*) m *odd and* $\gcd(m, 31) = 1$.

(*ii*) $m = m_1m_2m_3m_4m_5$, *where all prime factors of m_i satisfy $\ell(p) = i$.*

If $G(x)$ is the counting function of the set $\mathcal{G}$, when x tends to infinity, we have

$$G(x) = C\frac{x}{(\log x)^{\frac{1}{4}}}\left(1 + \mathcal{O}\left(\frac{\log\log x}{\log x}\right)\right),$$

where

$$C = \frac{C_1C_2}{C_3} = 0.61568378... \tag{4.1}$$

with

$$C_1 = \prod_{p\in\mathcal{P}_3}\left(1 + \frac{1}{2(p-1)}\right)\left(1 - \frac{1}{p}\right)^{\frac{1}{2}}$$
$$= \prod_{p\in\mathcal{P}_3}\left(1 - \frac{1}{2p}\right)\left(1 - \frac{1}{p}\right)^{-\frac{1}{2}} \approx 1.000479390466.$$

$$C_2 = \lim_{x\to\infty}\prod_{p\in\mathcal{P}_1\cup\mathcal{P}_2\cup\mathcal{P}_4\cup\mathcal{P}_5,\ p\le x}\left(1 - \frac{1}{p}\right)^{-1}\prod_{p\in\mathcal{P}_3,\ p\le x}\left(1 - \frac{1}{p}\right)^{-\frac{1}{2}}\prod_{p\le x}\left(1 - \frac{1}{p}\right)^{\frac{3}{4}}$$
$$\approx 0.75410767606$$

$$C_3 = \Gamma(\frac{3}{4}) \approx 1.225416702465...$$

Lemma 4.3. [10, *Lemma* 4.7]. *Let $\mathcal{G}$ be the set defined in Lemma* 4.2, *ω_j and α_0' be the functions given by* (2.9) *and* (3.2). *For $0 \le j \le 11$, r, u, λ, $t \in \mathbb{N}\cup\{0\}$ such that t is odd, we let $\mathcal{G}_{j,r,u,\lambda,t}$ be the set of integers $n = 2^{\omega_3(m)}m$ in $\mathcal{G}$ with the following conditions*:

(*i*) $\alpha_0'(m) \equiv j \pmod{12}$,

(*ii*) $\omega_2(m) + \omega_4(m) \equiv \lambda \pmod{2^u}$,

(*iii*) $m \equiv t \pmod{2^{r+1}}$.

If ρ is the completely multiplicative function defined on primes by

$$\rho(p) = \begin{cases} 0 \ if \ p \in \mathcal{P}_0 \ or \ p = 31 \\ 1 \ if \ not, \end{cases} \tag{4.2}$$

the counting function $G_{j,r,u,\lambda,t}(x)$ of the set $\mathcal{G}_{j,r,u,\lambda,t}$ is equal to

$$G_{j,r,u,\lambda,t}(x) = \sum_{\substack{2^{\omega_3(m)} m \le x, \ m \equiv t \pmod{2^{r+1}} \\ \alpha'_0(m) \equiv j \pmod{12}, \ \omega_2(m)+\omega_4(m) \equiv \lambda \pmod{2^u}}} \rho(m).$$

If $u \geq 1$ and $\lambda \not\equiv j \pmod 2$, $\mathcal{G}_{j,r,u,\lambda,t}$ is empty while, if $\lambda \equiv j \pmod 2$, when x tends to infinity, we have

$$G_{j,r,u,\lambda,t}(x) = \frac{C}{6 \cdot 2^{r+u}} \frac{x}{(\log x)^{\frac{1}{4}}} \left(1 + \mathcal{O}\left(\frac{1}{(\log x)^{2^{-2u-3}}}\right)\right),$$

where C is the constant given by (4.1).
If $u = 0$, then

$$G_{j,r,0,0,t}(x) = \frac{C}{12 \cdot 2^r} \frac{x}{(\log x)^{\frac{1}{4}}} \left(1 + \mathcal{O}\left(\frac{1}{(\log x)^{\frac{1}{8}}}\right)\right).$$

THEOREM 4.1. *Let $\mathcal{A}$ be a set of positive integers such that no element of $\mathcal{A}$ has an odd prime factor in the orbit $Orb(1)$ and no element of $\mathcal{A}$ is a multiple of 31^2. We assume that for any odd integer m relatively prime with 31 and such that $\omega_3(m) > 0$ and $\omega_2(m) + \omega_4(m) > 0$, the 2-adic integer $S_{\mathcal{A}}(m) = \sum_{h=0}^{\infty} 2^h \chi(\mathcal{A}, 2^h m)$ satisfies*

$$m S_{\mathcal{A}}(m) = 2^{\beta + \omega_3(m) - 1} \delta E_{h+\alpha'_i}, \tag{4.3}$$

for some integers h, $i \in \{0, 1, ..., 5\}$, where β is a non-negative integer and δ is an odd integer. If $A(x)$ is the counting function of the set $\mathcal{A}$, when x tends to infinity, we have

$$A(x) = \frac{\kappa}{2^\beta} \frac{x}{(\log x)^{\frac{1}{4}}} + \mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right), \tag{4.4}$$

with $\kappa = \dfrac{74}{31} C = 1.469696766...$ and C is the constant given by (4.1).

PROOF. From Lemma 4.1, we can deduce that

$$A(x) = A'(x) + \mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right), \tag{4.5}$$

where $A'(x)$ is the counting function of the set

$$\mathcal{A}' = \{n : 1 \le n = 2^k m \le x,\ n \in \mathcal{A},\ \omega_3(m) \neq 0 \text{ and } \omega_2(m) + \omega_4(m) \neq 0\}. \tag{4.6}$$

Let us write

$$\mathcal{A}' = \mathcal{B}_0 \cup \mathcal{B}_1, \tag{4.7}$$

where

$$\mathcal{B}_j = \{n : n = 2^k 31^j m \in \mathcal{A}',\ \ \gcd(m, 31) = 1\},\ 0 \le j \le 1.$$

If $\widehat{E}_i$ is the odd part of E_i then

$$\widehat{E}_i = 2^{-1-\nu_i} E_i, \tag{4.8}$$

where (cf. TABLE 2),

$$\nu_i = v_2(E_i) - 1 = \begin{cases} -1 \text{ if } i \equiv 0,\ 1,\ 3,\ 4 \pmod 6 \\ \ \ 0 \text{ if } i \equiv 2 \pmod 6 \\ \ \ 2 \text{ if } i \equiv 5 \pmod 6). \end{cases} \tag{4.9}$$

If $\gamma(m) = v_2(S_{\mathcal{A}}(m)$ then from (4.3),

$$\gamma(m) = \beta + \omega_3(m) - 1 + v_2(E_{h+\alpha'_i}),$$

so that if $h + \alpha'_i \equiv j \pmod{12}$ then

$$\gamma(m) - \omega_3(m) = \beta + \nu_j. \tag{4.10}$$

Let us consider the integers of the form $2^k m$ in $\mathcal{B}_0$ with $k = \gamma + r$, $r \in \mathbb{Z}$. From Theorem 3.2, if $r < 0$ then $2^k m \notin \mathcal{A}_i$. Hence if $B_j(x)$ is the counting function of $\mathcal{B}_j$, $0 \le j \le 1$, we have

$$\mathcal{B}_0(x) = \sum_{r\ge 0} \sum_{j=0}^{11} T_r^{(j)}(x), \tag{4.11}$$

where

$$T_r^{(j)}(x) = \sum_{\substack{n=2^{\gamma(m)+r} m\in\mathcal{A}',\ n\le x \\ h+\alpha_i'(m)\equiv j \pmod{12}}} \rho(m) \tag{4.12}$$

and ρ is the function given by (4.2). Let us begin by estimating $T_0^{(j)}(x)$, We have

$$T_0^{(j)}(x) = \sum_{\substack{n=2^{\gamma(m)} m\in\mathcal{A}',\ n\le x \\ h+\alpha_i'(m)\equiv j \pmod{12}}} \rho(m) = \sum_{\substack{n=2^{\gamma(m)} m\in\mathcal{A},\ n\le x,\ \omega_3(m)\ne 0,\ \omega_2(m)+\omega_4(m)\ne 0 \\ h+\alpha_i'(m)\equiv j \pmod{12}}} \rho(m).$$

So, by Theorem 3.2 (ii),

$$T_0^{(j)}(x) = \sum_{\substack{n=2^{\gamma(m)} m\le x,\ \omega_3(m)\ne 0,\ \omega_2(m)+\omega_4(m)\ne 0 \\ h+\alpha_i'(m)\equiv j \pmod{12}}} \rho(m),$$

which, via (4.10), gives

$$T_0^{(j)}(x) = \sum_{\substack{2^{\omega_3(m)} m\le 2^{-\beta-\nu_j} x,\ \omega_3(m)\ne 0,\ \omega_2(m)+\omega_4(m)\ne 0 \\ h+\alpha_i'(m)\equiv j \pmod{12}}} \rho(m).$$

By Lemma 4.1, we get

$$T_0^{(j)}(x) = \sum_{\substack{2^{\omega_3(m)} m\le 2^{-\beta-\nu_j} x \\ h+\alpha_i'(m)\equiv j \pmod{12}}} \rho(m) + \mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right).$$

So that, from Lemma 4.3 and (3.2), we obtain

$$\begin{aligned} T_0^{(j)}(x) &= G_{j+2i-h,0,0,0,1}\left(\frac{x}{2^{\beta+\nu_j}}\right) + \mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right) \\ &= \frac{C}{12}\frac{x}{2^{\beta+\nu_j}(\log x)^{\frac{1}{4}}}\left(1+\mathcal{O}\left(\frac{1}{(\log x)^{\frac{1}{12}}}\right)\right). \end{aligned} \tag{4.13}$$

Now, let us consider r positif. With the conditions $\omega_3(m) \neq 0$ and $\omega_2(m)+\omega_4(m) \neq 0$, by using (4.3), (4.10) and (4.8), we get when $h+\alpha'_i \equiv j \pmod{12}$,

$$m\widehat{S_{\mathcal{A}}(m)} = 2^{-\gamma(m)} m S_{\mathcal{A}}(m) = 2^{\beta+\omega_3(m)-1-\gamma(m)}\delta E_j$$

$$= 2^{-\nu_j-1}\delta E_j = \delta \widehat{E}_j. \tag{4.14}$$

From (4.12) and (4.6), we obtain

$$T_r^{(j)}(x) = \sum_{\substack{n=2^{\gamma(m)+r}m\in\mathcal{A},\ n\leq x,\ \omega_3(m)\neq 0,\ \omega_2(m)+\omega_4(m)\neq 0 \\ h+\alpha'_i(m)\equiv j \pmod{12}}} \rho(m)$$

and, by using Theorem 3.2 (iii) and (4.14), we get

$$T_r^{(j)}(x) = \sum_{l\in\mathcal{S}_r} \sum_{\substack{2^{\omega_3(m)}m\leq 2^{-\beta-\nu_j-r}x,\ \omega_3(m)\neq 0,\ \omega_2(m)+\omega_4(m)\neq 0 \\ h+\alpha'_i(m)\equiv j \pmod{12},\ m\equiv l^{-1}\delta\widehat{E}_j \pmod{2^{r+1}}}} \rho(m),$$

where $\mathcal{S}_r$ is the set defined in Theorem 3.2 (iii). From Lemma 4.1, we can remove the conditions $\omega_3(m) \neq 0$ and $\omega_2(m)+\omega_4(m) \neq 0$ with an error term $\mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right)$. This with Lemma 4.3 give

$$T_r^{(j)}(x) = \sum_{l\in\mathcal{S}_r} G_{j+2i-h,r,0,0,l^{-1}\delta\widehat{E}_j}\left(\frac{x}{2^{\beta+\nu_j+r}}\right) + \mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right)$$

$$= \frac{C}{24}\frac{x}{2^{\beta+\nu_j+r}(\log x)^{\frac{1}{4}}}\left(1+\mathcal{O}\left(\frac{1}{(\log x)^{\frac{1}{8}}}\right)\right) + \mathcal{O}\left(\frac{1}{(\log x)^{\frac{1}{3}}}\right) =$$

$$= \frac{C}{24}\frac{x}{2^{\beta+\nu_j+r}(\log x)^{\frac{1}{4}}}\left(1+\mathcal{O}\left(\frac{1}{(\log x)^{\frac{1}{12}}}\right)\right) \tag{4.15}$$

So that, by use of (4.11), (4.13) and (4.15), we get

$$B_0(x) = \frac{C}{12\cdot 2^{\beta}}\frac{x}{(\log x)^{\frac{1}{4}}}\left(\left(\sum_{j=0}^{11}\frac{1}{2^{\nu_j}}\right)\left(1+\frac{1}{2}\sum_{r=1}^{\infty}\frac{1}{2^r}\right)\right)\left(1+\mathcal{O}\left(\frac{1}{(\log x)^{\frac{1}{12}}}\right)\right)$$

This with (4.9) give when x tends to infinity

$$B_0(x) = \frac{C}{12\cdot 2^\beta}\frac{x}{(\log x)^{\frac14}}\binom{37}{2}\binom{3}{2}\left(1+\mathcal{O}\left(\frac{1}{(\log x)^{\frac{1}{12}}}\right)\right)$$
$$= \frac{37}{16\cdot 2^\beta}\frac{Cx}{(\log x)^{\frac14}} + \mathcal{O}\left(\frac{x}{(\log x)^{\frac13}}\right).$$

Proceeding as above and using Remark 3.1, we also obtain

$$B_1(x) = \frac{1}{31}\frac{37}{16\cdot 2^\beta}\frac{x}{(\log x)^{\frac14}} + \mathcal{O}\left(\frac{x}{(\log x)^{\frac13}}\right),$$

so that, from (4.7) and (4.5), we get

$$A(x) = \frac{\kappa}{2^\beta}\frac{x}{(\log x)^{\frac14}} + \mathcal{O}\left(\frac{x}{(\log x)^{\frac13}}\right),$$

where $\kappa = \frac{37}{16}\left(1+\frac{1}{31}\right) = \frac{74}{31}C = 1.469696766...$ □

COROLLARY 4.1. *For* $0 \le i \le 5$, *let* $P^{(i)}$ *be the polynomial defined by* (1.10), $\mathcal{A}_i = \mathcal{A}(P^{(i)})$ *be the set obtained from* (1.1) *and* $A_i(x) = A(P^{(i)}, x)$ *be its counting function. When* x *tends to infinity*, *we have*

$$A_i(x) = \kappa\frac{x}{(\log x)^{\frac14}} + \mathcal{O}\left(\frac{x}{(\log x)^{\frac13}}\right),$$

where $\kappa = \frac{74}{31}C = 1.469696766...$ *and* C *is the constant given by* (4.1).

PROOF. From Remark 2.1, for all i, $0 \le i \le 5$, no element of the set $\mathcal{A}_i = \mathcal{A}(P^{(i)})$ has an odd prime factor in the orbit $Orb(1)$ and no element of $\mathcal{A}_i$ is a multiple of 31^2. Moreover, for all odd integer m relatively prime with 31 and such that $\omega_3(m) > 0$ and $\omega_2(m)+\omega_4(m) > 0$, the 2-adic integer $S_i(m) = \sum_{h=0}^{\infty} 2^h\chi(\mathcal{A}_i, 2^h m)$ satisfies (cf. (2.16)),

$$mS_i(m) = 2^{\beta+\omega_3(m)-1}\delta E_{h+\alpha_i'},$$

with $\beta = 0$, $\delta = 3^{\lceil \frac{\omega_2+\omega_4}{2}-1\rceil}$ and $h = 0$. Hence, the result follows immediately from Theorem 4.1. □

Corollary 4.2. *Let $P \neq P^{(i)}$ be a polynomial of order* 31 *in* $\mathbb{F}_2[z]$, *i.e.* $P = \prod_{i\in I_j} P^{(i)}$ *is the product of* j *distinct polynomials from* (1.10) *with* $j = Card(I_j) \in \{2,3,...,6\}$ *or* $P = 1 - z^{31}$. *Let* $\mathcal{A} = \mathcal{A}(P)$ *be the set obtained from* (1.1) *and* $A(P,x)$ *be its counting function. When* x *tends to infinity, we have*:

(1) i. *If* $P = P^{(i)}P^{(j)}$ *with* $j - i \not\equiv 0 \pmod 3$ *then* $A(P,x) \sim \kappa \dfrac{x}{(\log x)^{\frac{1}{4}}}$.

ii. *If* $P = P^{(i)}P^{(i+3)}$ *then* $A(P,x) = \mathcal{O}\left(\dfrac{x}{(\log x)^{\frac{1}{3}}}\right)$.

(2) i. *If* $P = P^{(i)}P^{(i+1)}P^{(i+2)}$ *or* $P = P^{(i)}P^{(i+1)}P^{(i+5)}$ *or* $P = P^{(i)}P^{(i+4)}P^{(i+5)}$ *then* $A(P,x) \sim \dfrac{\kappa}{2}\dfrac{x}{(\log x)^{\frac{1}{4}}}$.

(*ii*) *If* $P = P^{(i)}P^{(i+2)}P^{(i+4)}$ *then* $A(P,x) = \mathcal{O}\left(\dfrac{x}{(\log x)^{\frac{1}{3}}}\right)$.

(*iii*) *If* $P = P^{(i)}P^{(j)}P^{(k)}$ *is not of the last two forms then* $A(P,x) \sim \kappa \dfrac{x}{(\log x)^{\frac{1}{4}}}$.

(3) i. *If* $P = P^{(i)}P^{(i+1)}P^{(i+3)}P^{(i+4)}$ *then* $A(P,x) = \mathcal{O}\left(\dfrac{x}{(\log x)^{\frac{1}{3}}}\right)$.

(*ii*) *If* P *is the product of four distinct polynomials* $P^{(i)}$, $0 \le i \le 5$, *and not of the last form then* $A(P,x) \sim \kappa \dfrac{x}{(\log x)^{\frac{1}{4}}}$.

(4) *If* P *is the product of five distinct polynomials* $P^{(i)}$, $0 \le i \le 5$, *then* $A(P,x) \sim \kappa \dfrac{x}{(\log x)^{\frac{1}{4}}}$.

(5) *If* $P = \prod_{i=0}^{5} P^{(i)}$ *then* $\mathcal{A} = \mathcal{A}(P) = \{1\} \cup \{2^k \cdot 31,\ k \ge 0\}$ *and* $A(P,x) = \dfrac{\log x}{\log 2} + \mathcal{O}(1)$.

(6) *If* $P = 1 - z^{31}$ *then* $\mathcal{A} = \mathcal{A}(P) = \{2^k \cdot 31,\ k \ge 0\}$ *and* $A(P,x) = \dfrac{\log x}{\log 2} + \mathcal{O}(1)$.

κ *is the constant of Theorem* 4.1. *We remind that* $P^{(i)} = P^{(i \mod 6)}$.

Proof. For m odd ≥ 1, let (cf. (1.13)),

$$S_{\mathcal{A}(P)}(m) = \sum_{j=0}^{\infty} 2^j \chi(\mathcal{A}(P), 2^j m). \tag{4.16}$$

From (1.5), (1.16) and (1.14), we can conclude that

$$S_{\mathcal{A}(P)}(m) = \sum_{i \in I_j} S_{\mathcal{A}(P^{(i)})}(m) = \sum_{i \in I_j} S_i(m). \tag{4.17}$$

From Remark 2.1 and (4.17), we deduce that no element of the set $\mathcal{A}(P)$ has an odd prime factor in the orbit $Orb(1)$ and no integer divisible by 31^2 belongs to $\mathcal{A}(P)$. Let m^* stands for those odd positive integers which have no prime divisors in the orbit $Orb(1)$, $31^2 \nmid m^*$ and such that $\omega_3(m^*) > 0$ and $\omega_2(m^*) + \omega_4(m^*) > 0$. From Lemma 4.1, we obtain

$$A(P, x) = \mid \{n = 2^k m^* \in \mathcal{A}(P),\ n \le x\} \mid + \mathcal{O}\left(\frac{x}{(\log x)^{\frac{1}{3}}}\right). \tag{4.18}$$

Corollary 4.2 will follow immediately from Theorem 4.1, (2.16), (4.17) and (4.18). Indeed, we have

$E_0 + E_2 = 3E_1$ $E_0 + E_4 = E_2$ $E_0 + E_6 = 0$ $E_0 + E_8 = -E_4$ $E_0 + E_{10} = -3E_5$	$E_1 + E_3 = E_2$ $E_1 + E_5 = E_3$ $E_1 + E_7 = 0$ $E_1 + E_9 = -E_5$ $E_1 + E_{11} = E_0$	$E_2 + E_4 = 3E_3$ $E_2 + E_6 = E_4$ $E_2 + E_8 = 0$ $E_2 + E_{10} = E_0$	$E_3 + E_5 = E_4$ $E_3 + E_7 = E_5$ $E_3 + E_9 = 0$ $E_3 + E_{11} = E_1$
$E_4 + E_6 = 3E_5$ $E_4 + E_8 = -E_0$ $E_4 + E_{10} = 0$	$E_5 + E_7 = -E_0$ $E_5 + E_9 = -E_1$ $E_5 + E_{11} = 0$	$E_6 + E_8 = -3E_1$ $E_6 + E_{10} = -E_2$	$E_7 + E_9 = -E_2$ $E_7 + E_{11} = -E_3$
$E_8 + E_{10} = -3E_3$	$E_9 + E_{11} = -E_4$		

TABLE 3

From this table, (4.17) and (2.16), we obtain

- If $P = P^{(i)} P^{(i+1)}$ then

$$m^* S_{\mathcal{A}(P)}(m^*) = m^* \left(S_i(m^*) + S_{i+1}(m^*)\right)$$

$$= 2^{\omega_3(m^*)-1} 3^{\lceil \frac{\omega_2(m^*)+\omega_4(m^*)}{2} - 1 \rceil} \times \begin{cases} 3E_{\alpha_i'-1} & \text{if } \alpha_0 \text{ is even} \\ E_{\alpha_i'-1} & \text{if } \alpha_0 \text{ is odd.} \end{cases} \tag{4.19}$$

($\alpha_0 = \alpha_0(m^*)$ is the quantity given by (2.11) with $i = 0$ and α'_i is given by (3.2).)

- If $P = P^{(i)}P^{(i+2)}$ then

$$m^* S_{\mathcal{A}(P)}(m^*) = m^* S_{i+1}(m^*) = 2^{\omega_3(m^*)-1} 3^{\lceil \frac{\omega_2(m^*)+\omega_4(m^*)}{2} - 1\rceil} E_{\alpha'_{i+1}}. \tag{4.20}$$

- If $P = P^{(i)}P^{(i+3)}$ then

$$m^* S_{\mathcal{A}(P)}(m^*) = m^* \left(S_i(m^*) + S_{i+3}(m^*)\right) = 0. \tag{4.21}$$

For the product of three polynomials, we use the following table:

$E_0 + E_2 + E_4 = 2E_2$	$E_1 + E_3 + E_5 = 2E_3$	$E_2 + E_4 + E_6 = 2E_4$	$E_3 + E_5 + E_7 = 2E_5$
$E_0 + E_2 + E_6 = E_2$	$E_1 + E_3 + E_7 = E_3$	$E_2 + E_4 + E_8 = E_4$	$E_3 + E_5 + E_9 = E_5$
$E_0 + E_2 + E_8 = E_0$	$E_1 + E_3 + E_9 = E_1$	$E_2 + E_4 + E_{10} = E_2$	$E_3 + E_5 + E_{11} = E_3$
$E_0 + E_2 + E_{10} = 2E_0$	$E_1 + E_3 + E_{11} = 2E_1$	$E_2 + E_6 + E_8 = -E_0$	$E_3 + E_7 + E_9 = -E_1$
$E_0 + E_4 + E_6 = E_4$	$E_1 + E_5 + E_7 = E_5$	$E_2 + E_6 + E_{10} = 0$	$E_3 + E_7 + E_{11} = 0$
$E_0 + E_4 + E_8 = 0$	$E_1 + E_5 + E_9 = 0$	$E_2 + E_8 + E_{10} = -E_4$	$E_3 + E_9 + E_{11} = -E_5$
$E_0 + E_4 + E_{10} = E_0$	$E_1 + E_5 + E_{11} = E_1$		
$E_0 + E_6 + E_8 = -E_2$	$E_1 + E_7 + E_9 = -E_3$		
$E_0 + E_6 + E_{10} = -E_4$	$E_1 + E_7 + E_{11} = -E_5$		
$E_0 + E_8 + E_{10} = -2E_4$	$E_1 + E_9 + E_{11} = -2E_5$		
$E_4 + E_6 + E_8 = -2E_0$	$E_5 + E_7 + E_9 = -2E_1$	$E_6 + E_8 + E_{10} = -2E_2$	$E_7 + E_9 + E_{11} = -2E_3$
$E_4 + E_6 + E_{10} = -E_0$	$E_5 + E_7 + E_{11} = -E_1$		
$E_4 + E_8 + E_{10} = -E_2$	$E_5 + E_9 + E_{11} = -E_3$		

TABLE 4

We obtain:

- If $P = P^{(i)}P^{(j)}P^{(k)}$ with $j - i \equiv 0 \pmod 3$ then

$$m^* S_{\mathcal{A}(P)}(m^*) = m^* S_k(m^*) = 2^{\omega_3(m^*)-1} 3^{\lceil \frac{\omega_2(m^*)+\omega_4(m^*)}{2} - 1\rceil} E_{\alpha'_k}(m^*). \tag{4.22}$$

- If $P = P^{(i)}P^{(i+1)}P^{(i+2)}$ then

$$m^* S_{\mathcal{A}(P)}(m^*) = m^* \left(S_i(m^*) + S_{i+1}(m^*) + S_{i+2}(m^*)\right)$$

$$= 2m^* S_{i+1}(m^*) = 2^{\omega_3(m^*)} 3^{\lceil \frac{\omega_2(m^*)+\omega_4(m^*)}{2} - 1 \rceil} E_{\alpha'_{i+1}}(m^*). \tag{4.23}$$

- If $P = P^{(i)} P^{(i+2)} P^{(i+4)}$ then

$$m^* S_{\mathcal{A}(P)}(m^*) = m^* \left(S_i(m^*) + S_{i+2}(m^*) + S_{i+4}(m^*) \right) = 0. \tag{4.24}$$

For the product of four polynomials, we will use

$E_0 + E_2 + E_4 + E_6 = 3E_3$	$E_1 + E_3 + E_5 + E_7 = E_4$	$E_2 + E_4 + E_6 + E_8 = 3E_5$
$E_0 + E_2 + E_4 + E_8 = E_2$	$E_1 + E_3 + E_5 + E_9 = E_3$	$E_2 + E_4 + E_6 + E_{10} = E_4$
$E_0 + E_2 + E_4 + E_{10} = 3E_1$	$E_1 + E_3 + E_5 + E_{11} = E_2$	$E_2 + E_4 + E_8 + E_{10} = 0$
$E_0 + E_2 + E_6 + E_8 = 0$	$E_1 + E_3 + E_7 + E_9 = 0$	$E_2 + E_6 + E_8 + E_{10} = -E_2$
$E_0 + E_2 + E_6 + E_{10} = E_0$	$E_1 + E_3 + E_7 + E_{11} = E_1$	
$E_0 + E_2 + E_8 + E_{10} = -3E_5$	$E_1 + E_3 + E_9 + E_{11} = E_0$	
$E_0 + E_4 + E_6 + E_8 = -E_0$	$E_1 + E_5 + E_7 + E_9 = -E_1$	
$E_0 + E_4 + E_6 + E_{10} = 0$	$E_1 + E_5 + E_7 + E_{11} = 0$	
$E_0 + E_4 + E_8 + E_{10} = -E_4$	$E_1 + E_5 + E_9 + E_{11} = -E_5$	
$E_0 + E_6 + E_8 + E_{10} = -3E_3$	$E_1 + E_7 + E_9 + E_{11} = -E_4$	
$E_3 + E_5 + E_7 + E_9 = -E_0$	$E_4 + E_6 + E_8 + E_{10} = -3E_1$	$E_5 + E_7 + E_9 + E_{11} = -E_2$
$E_3 + E_5 + E_7 + E_{11} = E_5$		
$E_3 + E_5 + E_9 + E_{11} = 0$		
$E_3 + E_7 + E_9 + E_{11} = -E_3$		

TABLE 5

We obtain:

- If $P = P^{(i)} P^{(i+1)} P^{(i+2)} P^{(i+3)}$ then

$$m^* S_{\mathcal{A}(P)}(m^*) = m^* \left(S_i(m^*) + S_{i+1}(m^*) + S_{i+2}(m^*) + S_{i+3}(m^*) \right)$$

$$= 2^{\omega_3(m^*)-1} 3^{\lceil \frac{\omega_2(m^*)+\omega_4(m^*)}{2} - 1 \rceil} \times \begin{cases} 3E_{\alpha'_i - 3} & \text{if } \alpha_0 \text{ is even} \\ E_{\alpha'_i - 3} & \text{if } \alpha_0 \text{ is odd.} \end{cases} \tag{4.25}$$

• If $P = P^{(i)}P^{(i+1)}P^{(i+2)}P^{(i+4)}$ then

$$m^*S_{\mathcal{A}(P)}(m^*) = m^*\left(S_i(m^*) + S_{i+1}(m^*) + S_{i+2}(m^*) + S_{i+4}(m^*)\right) = m^*S_{i+1}(m^*)$$

$$= 2^{\omega_3(m^*)-1}3^{\lceil \frac{\omega_2(m^*)+\omega_4(m^*)}{2}-1\rceil}E_{\alpha'_{i+1}}. \tag{4.26}$$

• If $P = P^{(i)}P^{(i+1)}P^{(i+3)}P^{(i+4)}$ then

$$m^*S_{\mathcal{A}(P)}(m^*) = m^*\left(S_i(m^*) + S_{i+1}(m^*) + S_{i+3}(m^*) + S_{i+4}(m^*)\right) = 0. \tag{4.27}$$

For the product of five polynomials, we have

$E_0 + E_2 + E_4 + E_6 + E_8 = E_4$	$E_1 + E_3 + E_5 + E_7 + E_9 = E_5$
$E_0 + E_2 + E_4 + E_6 + E_{10} = E_2$	$E_1 + E_3 + E_5 + E_7 + E_{11} = E_3$
$E_0 + E_2 + E_4 + E_8 + E_{10} = E_0$	$E_1 + E_3 + E_5 + E_9 + E_{11} = E_1$
$E_0 + E_2 + E_6 + E_8 + E_{10} = -E_4$	$E_1 + E_5 + E_7 + E_9 + E_{11} = -E_3$
$E_0 + E_4 + E_6 + E_8 + E_{10} = -E_2$	

TABLE 6

We obtain:

• If $P = P^{(i)}P^{(i+1)}P^{(i+2)}P^{(i+3)}P^{(i+4)}$ then

$$m^*S_{\mathcal{A}(P)}(m^*) = m^*\left(S_i(m^*) + S_{i+1}(m^*) + S_{i+2}(m^*) + S_{i+3}(m^*) + S_{i+4}(m^*)\right)$$

$$= m^*S_{i+2}(m^*) = 2^{\omega_3(m^*)-1}3^{\lceil \frac{\omega_2(m^*)+\omega_4(m^*)}{2}-1\rceil}E_{\alpha'_{i+2}}. \tag{4.28}$$

For the product of six polynomials, we have

$E_0 + E_2 + E_4 + E_6 + E_8 + E_{10} = 0$	$E_1 + E_3 + E_5 + E_7 + E_9 + E_{11} = 0$

TABLE 7

So that

$$m^*S_{\mathcal{A}(P)}(m^*) = m^*\left(S_0(m^*) + S_1(m^*) + S_2(m^*) + S_3(m^*) + S_4(m^*) + S_5(m^*)\right) = 0. \tag{4.29}$$

In fact, here we can say more. Since $P = \prod_{i=0}^{5} P^{(i)}$ then in $\mathbb{F}_2[z]$,

$$P(z) = \frac{1 - z^{31}}{1 - z} \equiv \frac{1}{1 - z} \prod_{k \geq 0} \frac{1}{1 - z^{2^k \cdot 31}} \pmod 2.$$

So that the set $\mathcal{A}(P)$ obtained from (1.1) is given by

$$\mathcal{A}(P) = \{1\} \cup \{2^k \cdot 31,\ k \geq 0\}$$

and $A(P, x) = \dfrac{\log x}{\log 2} + \mathcal{O}(1)$. □

Lastly, if $P = 1 - z^{31}$ then $P \equiv \prod_{k \geq 0} \dfrac{1}{(1 - z^{2^k \cdot 31})} \pmod 2$ so that $\mathcal{A}(P) = \{2^k \cdot 31,\ k \geq 0\}$ and $A(P, x) = \dfrac{\log x}{\log 2} + \mathcal{O}(1)$.

Remark 4.1. In corollary 4.2, we can improve on the results where there is a big $\mathcal{O}$. By using Selberg-Delange's formula and a series of lemmas which we found too long to reproduce here, we obtain: Under the same hypothesis of Corollary 4.2, there are absolute positive constants α and β such that when x tends to infinity, we have

(1) If $P = P^{(i)} P^{(i+3)}$ or $P = P^{(i)} P^{(i+1)} P^{(i+3)} P^{(i+4)}$ then $A(P, x) \sim \alpha \dfrac{x}{(\log x)^{\frac{1}{3}}}$.

(2) If $P = P^{(i)} P^{(i+2)} P^{(i+4)}$ then $A(P, x) \sim \beta \dfrac{x}{(\log x)^{\frac{1}{3}}}$.

It seems difficult to find a general formula for an asymptotic equivalence of $A(P, x)$ when P is any product of $P^{(i)}$, distinct or not.

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