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Tiling the integer lattice with translated sublattices

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Abstract: When $\mathbb{Z}^d$ is represented as a finite disjoint union of translated integer sublattices, the translated sublattices must possess some special properties. Such a representation is called a *lattice tiling*. We develop a theoretical framework, based on multiple residues and dual groups, to provide a set of necessary and sufficient conditions for such a lattice tiling to exist. We also investigate the question of when a lattice tiling must possess at least two translated sublattices which are translates of one another.

Keywords: integer lattice, sublattice, tiling, generating function, Pontryagin dual group, disjoint covering system

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1. Introduction

1.1. Overview

A *sublattice* is a full-rank subgroup of $\mathbb{Z}^d$. Suppose we decompose $\mathbb{Z}^d$ into a finite, disjoint union of integer translates of sublattices. We call such a decomposition of the integer lattice $\mathbb{Z}^d$ a *lattice tiling*. Given a lattice tiling, what can be said about the structure of the translated sublattices?

A lattice tiling has the *translational property* if at least two of its translated sublattices are different cosets of the same sublattice. To motivate the results of this paper, we focus on the following three questions:

QUESTION 1.1. What are some natural and general *necessary* and *sufficient* conditions for the existence of a lattice tiling?

QUESTION 1.2. In any general dimension d, are there some nice sufficient conditions for a lattice tiling to have the translational property?

QUESTION 1.3. For $d = 2$, is there a lattice tiling which does not have the translational property?

In the process of trying to answer these questions, we develop some analytic tools which may be of independent interest. These tools involve generating functions associated to sublattices of $\mathbb{Z}^d$, residue calculus of holomorphic functions of several variables [14], and some elementary considerations.

1.2. Terminologies and notations

Before stating the main results we layout some concise definitions of notions that will be used throughout the paper.

The index of a sublattice $\mathcal{L}$ in $\mathbb{Z}^d$ is called the *determinant* of $\mathcal{L}$, denoted by $\det \mathcal{L}$. Throughout the article, we write $d-$dimensional vectors in bold, to distinguish them from scalars. Thus any vector $\mathbf{v} \in \mathbb{Z}^d$ has coordinates $(v_1, \dots, v_d)$. Furthermore, we write $\mathbf{v} \geqslant 0$ if $v_1, \dots, v_d \geqslant 0$. Define $\mathbf{0} := (0, \dots, 0)$ and $\mathbf{1} := (1, \dots, 1)$.

For any sublattice $\mathcal{L} \subseteq \mathbb{Z}^d$ and integer vector $\mathbf{v} \in \mathbb{Z}^d$, we call the discrete set of vectors

$$\mathbf{v} + \mathcal{L} := \{\mathbf{v} + \mathbf{w} \mid \mathbf{w} \in \mathcal{L}\}$$

a *translate* of $\mathcal{L}$. The vector $\mathbf{v}$ will be referred to as a *translate vector*. Thus, a more formal description of a lattice tiling is the existence of a collection of ($d-$dimensional) sublattices $\mathcal{L}_1, \dots, \mathcal{L}_n$ and translate vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{Z}^d$ such that

$$\bigcup_{j=1}^{n} \{\mathbf{v}_j + \mathcal{L}_j\} = \mathbb{Z}^d,$$

and such that $\{\mathbf{v}_i + \mathcal{L}_i\} \cap \{\mathbf{v}_j + \mathcal{L}_j\} = \varnothing$ for all $i \neq j$. In other words, for any $\mathbf{w} \in \mathbb{Z}^d$ there exists a unique $j \in \{1, \ldots, n\}$ such that $\mathbf{w} - \mathbf{v}_j \in \mathcal{L}_j$.

Let $\mathbb{T}^d = \{(z_1, \ldots, z_d) \in \mathbb{C}^d \colon |z_1| = \ldots = |z_d| = 1\}$. For any $\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{T}^d$ and $\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{Z}^d$, we define

$$\mathcal{X}_{\mathbf{z}}(\mathbf{v}) := z_1^{v_1} \ldots z_d^{v_d} \in \mathbb{T}.$$

Such a homomorphism $\mathcal{X}_{\mathbf{z}} : \mathbb{Z}^d \to \mathbb{T}$ is refered to as a *character*. We also say that a character $\mathcal{X}_{\mathbf{z}}$ has *finite order* if each z_j is additionally assumed to be a root of unity, a condition tantamount to saying that the image of $\mathcal{X}_{\mathbf{z}}$ is finite.

For a sublattice $\mathcal{L} \subseteq \mathbb{Z}^d$, we call a complex point $\mathbf{z} \in \mathbb{T}^d$ a *dual point* of $\mathcal{L}$ if $\mathcal{X}_{\mathbf{z}}(\mathbf{v}) = 1$ for all $\mathbf{v} \in \mathcal{L}$. Since a point $\mathbf{z} \in \mathbb{T}^d$ is a dual point of $\mathcal{L}$ if and only if the homomorphism $\mathcal{X}_{\mathbf{z}}$ restricted to $\mathcal{L}$ is trivial, the dual points can be regarded as characters on the finite abelian group

$$G_{\mathcal{L}} := \mathbb{Z}^d/\mathcal{L},$$

which we call the *group of the sublattice*. If $G_{\mathcal{L}}$ is cyclic then $\mathcal{L}$ is called a *cyclic sublattice*. It is a standard fact that the dual points form a group, known as the Pontryagin dual to $G_{\mathcal{L}}$, and it is particularly useful that this group is isomorphic to $G_{\mathcal{L}}$. We denote the dual group by $\widehat{G}_{\mathcal{L}}$. We clearly have

$$|\widehat{G}_{\mathcal{L}}| = |G_{\mathcal{L}}| = \det \mathcal{L}.$$

1.3. Statement of results

Many necessary conditions can be deduced about the structure of a lattice tiling by elementary arguments. One notable condition among such various results that we will prove in Subsection 2.1 is:

THEOREM 1.4. *In a lattice tiling, if there is a prime p such that p^k divides the determinant of one of the sublattice translates, then p^k divides the determinant of another sublattice translate.*

To give an answer to Question 1.2, we have the following result, stated in terms of cyclic sublattices.

THEOREM 1.5. *If we have a lattice tiling in which the sublattice translate with largest determinant is cyclic, then our lattice tiling has the translational property.*

This will be proved after Corollary 4.4. In dimension 2 we can prove a stronger condition, made precise in Theorem 5.5. Finally, the following necessary and sufficient conditions answer question (1.1), which we call the character formulas. This result came with the help of generating functions.

THEOREM 1.6. *We have a lattice tiling with* $\mathbf{v}_1 + \mathcal{L}_1, \dots, \mathbf{v}_n + \mathcal{L}_n$ *if and only if for any character of finite order* $\chi_{\mathbf{z}}$, *the following holds*

$$\sum_{j:\ \mathbf{z}\in\widehat{G}_{\mathcal{L}_j}} \frac{\chi_{\mathbf{z}}(\mathbf{v}_j)}{\det \mathcal{L}_j} = \begin{cases} 1, & \text{if } \mathbf{z} = (1,\dots,1) \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The 'only if' part of Theorem 1.6 is stated and proved in Proposition 4.3. The 'if' part is Theorem 4.7. The proof will also illustrate that (1) is indeed a finite set of conditions.

The layout of the following sections are as follows. Section 2 displays some basic properties of lattice tilings and also the definition of our generating functions. Section 3 examines dual points and characters in connection with these generating functions. Section 4 brings in the main analytic ingredients to prove Theorem 1.6. Although Question 1.3 remains open, Section 5 will illustrate the restrictive nature of 2-dimensional lattice tilings. Lastly, we will pose two open questions in Section 6.

1.4. Background review

Note that equation (1) is a basic relation between roots of unity, with rational coefficients. Indeed, 1-dimensional lattice tilings have been extensively studied using vanishing sums of roots of unity over the rationals. In dimension 1, a lattice tiling is also known in the literature as a Disjoint Covering System (DCS). Paul Erdős initiated the study of covering systems in general (which means that the arithmetic progressions may not necessarily be disjoint, see [5]), and Erdős credits the beautiful proof of the translational property, in dimension 1, to an unpublished paper by Mirsky and Newman, and independently to an unpublished paper of Davenport and Rado. Many interesting papers have since been written about the 1-dimensional case of lattice tilings, and for more background, including some fascinating results on vanishing sums of roots of unity, the reader may refer to [2], [3], [4], [9], [10], [12], [15].

There is a related question in the context of a general group G. Suppose that G is partitioned into some cosets of some of its subgroups. Then the conjecture, known as the Herzog—Schönheim conjecture, says that at least two of the cosets must have the same index. The Herzog—Schönheim conjecture was solved for finite nilpotent groups in 1986, in the paper [1]. Since then stronger results have been proved (see [13]), but the general case still remains open.

Since a lattice tiling may also be thought of as a covering of the group $\mathbb{Z}^d$ by a finite disjoint union of cosets of subgroups of $\mathbb{Z}^d$, it then follows from [1], in our abelian group setting, that any lattice tiling must contain at least two sublattice translates of the same determinant. In other words, there are two sublattice translates that must have the same volume for their fundamental domain. But almost any other question about these fundamental domains remains open.

In the recent paper [7], the translational property for higher dimensional lattice tilings was considered from a discrete Fourier perspective. The translational property for dimension $d > 1$ was apparently first considered in another unpublished manuscript, this time an MIT Master's thesis by A. Schwartz [11], using purely combinatorial methods. We note that in general, the translational property does not hold for dimension $d > 2$ (see Example 4.5).

Finally, we mention that Paul Erdős [6] himself has been quoted as saying in 1995 that "Perhaps my favorite problem of all concerns covering systems".

2. Sublattices, generating functions, and characters

2.1. The tiling condition

Definition 2.1. *A tiling* $\mathbf{v}_1 + \mathcal{L}_1, \ldots, \mathbf{v}_n + \mathcal{L}_n$ *of* $\mathbb{Z}^d$ splits *if* $\{1, \ldots, n\}$ *can be partitioned into* $I_1 \cup \ldots \cup I_k$ *with* $k > 1$, $|I_j| > 0$ *for* $j = 1, \ldots, k$, $|I_1| > 1$, *so that for any* $j = 1, \ldots, k$ *the union*

$$\bigcup_{i \in I_j} \{\mathbf{v}_i + \mathcal{L}_i\}$$

is another sublattice translate. Otherwise a tiling is called primitive. *This definition simply captures the intuition that one can generate more complex lattice tilings by starting with a simple one, and splitting one of the existing sublattice translates into new "coarser" sublattice translates.*

LEMMA 2.2. *Assume that* $\mathcal{L} \subset \mathbb{Z}^d$ *is a sublattice with* $\det \mathcal{L} = p$, *for* p *a prime. Fix any integer vector* $\mathbf{v} \notin \mathcal{L}$. *Then we have* $\operatorname{span}\{\mathcal{L}, \mathbf{v}\} = \mathbb{Z}^d$.

PROOF. Let $\mathcal{T}$ be the sublattice generated by $\mathcal{L}$ and $\mathbf{v}$. Since $\mathbf{v} \notin \mathcal{L}$, we have $\mathcal{L} \subsetneq \mathcal{T} \subseteq \mathbb{Z}^d$ and the index of $\mathcal{T}$ in $\mathbb{Z}^d$ is therefore a proper divisor of p. We conclude that $\det \mathcal{T} = 1$ and hence $\mathcal{T} = \mathbb{Z}^d$. □

PROPOSITION 2.3. *If we have* $\det \mathcal{L}_k = p$ *with* p *a prime for some* k *then the tiling either splits or all the sublattices in the tiling are equal to* $\mathcal{L}_k$.

PROOF. Assume that $\det \mathcal{L}_1 = p$ and translate the tiling if necessaray so that $\mathbf{v}_1 = 0$. By this assumption, we have $\mathcal{L}_1 \cap (\mathbf{v}_i + \mathcal{L}_i) = \varnothing$ for all $i > 1$. So $\mathbf{v}_i \notin \operatorname{span}\{\mathcal{L}_1, \mathcal{L}_i\}$ and hence $\operatorname{span}\{\mathcal{L}_1, \mathcal{L}_i\} \subsetneq \mathbb{Z}^d$. From Lemma 2.2, we conclude that $\mathcal{L}_i \subseteq \mathcal{L}_1$ for all $i > 1$ and this implies $\mathbf{v}_i + \mathcal{L}_i$ lies in the translate of $\mathcal{L}_1$ under $\mathbf{v}_i$.

Since $\mathbb{Z}^d/\mathcal{L}_1$ is a group with p elements, it is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Let us fix an isomorphism $\mathbb{Z}^d/\mathcal{L}_1 \to \mathbb{Z}/p\mathbb{Z}$ and define $\pi\colon\ \mathbb{Z}^d \to \mathbb{Z}^d/\mathcal{L}_1 \xrightarrow{\cong} \mathbb{Z}/p\mathbb{Z}$ as a composition of the projection map with this isomorphism. For any $k \in \{0, \dots, p-1\}$, we define the subset of indices

$$I_k = \{i \in \{1, \dots, n\}\colon\ \pi(\mathbf{v}_i) = k\}.$$

The union

$$\bigcup_{i \in I_k} \{\mathbf{v}_i + \mathcal{L}_i\}$$

is equal to $\pi^{-1}(k)$. Therefore, it is equal to a translate of $\mathcal{L}_1$. If $|I_k| > 1$ for somke k, we have a split tiling. If for all k we have $|I_k| = 1$, then all sublattices are equal to $\mathcal{L}_1$. □

LEMMA 2.4. *Assume that the sublattices* $\mathcal{L}_1, \dots, \mathcal{L}_n$ *have coprime determinants, i. e.*

$$\gcd(\det \mathcal{L}_1, \dots, \det \mathcal{L}_n) = 1.$$

Then the sublattice $\mathcal{L}$ *generated by* $\bigcup \mathcal{L}_i$, *over the integers, is isomorphic to* $\mathbb{Z}^d$.

PROOF. Let $\mathcal{L}$ be the sublattice generated by $\mathcal{L}_1, \dots, \mathcal{L}_n$. Repeating the argument in Lemma 2.2, we see that $\det \mathcal{L} \mid \det \mathcal{L}_i$ for $1 \le i \le n$. Thus we have $\det \mathcal{L} \mid \gcd(\det \mathcal{L}_1, \dots, \det \mathcal{L}_n) = 1$ and consequently $\mathcal{L} = \mathbb{Z}^d$. □

PROPOSITION 2.5. *Let* $\mathbf{v}_1 + \mathcal{L}_1, \ldots, \mathbf{v}_n + \mathcal{L}_n$ *be a lattice tiling. Then for any* i *and* j *we have* $\gcd(\det \mathcal{L}_i, \det \mathcal{L}_j) > 1$.

PROOF. Assume that $\gcd(\det \mathcal{L}_1, \det \mathcal{L}_2) = 1$. By Lemma 2.4, $\mathcal{L}_1 \cup \mathcal{L}_2$ generates $\mathbb{Z}^d$ and so any integer vector $\mathbf{v} \in \mathbb{Z}^d$ can be written as $\mathbf{w}_1 - \mathbf{w}_2$ for $\mathbf{w}_1 \in \mathcal{L}_1$ and $\mathbf{w}_2 \in \mathcal{L}_2$. Representing $\mathbf{v}_2 - \mathbf{v}_1$ in this form, we have $\mathbf{v}_1 + \mathbf{w}_1 = \mathbf{v}_2 + \mathbf{w}_2 \in (\mathbf{v}_1 + \mathcal{L}_1) \cap (\mathbf{v}_2 + \mathcal{L}_2) \neq \varnothing$. We obtain a contradiction. □

LEMMA 2.6. *In a lattice tiling, we have the following relation*

$$\sum_{i=1}^{n} \frac{1}{\det \mathcal{L}_i} = 1. \tag{2}$$

PROOF. We observe that the number of integer points of $\mathcal{L}_i$ in a cube $[-N, N]^d$ is equal to $(2N)^d / \det \mathcal{L}_i + O(N^{d-1})$. □

This "density" results will turn out to be a subcase of (14) with $\mathbf{z} = \mathbf{1}$. Nevertheless, this already allows us to prove Theorem 1.4 from the introduction.

PROOF OF THEOREM 1.4. Assume $p^k \mid \det \mathcal{L}_i$ and $p^k \nmid \det \mathcal{L}_j$ for $j \neq i$. Equation (2) is now

$$1 = \frac{1}{\det \mathcal{L}_i} + \sum_{j \neq i} \frac{1}{\det \mathcal{L}_j} = \frac{1}{p^k a} + \frac{c}{p^k b}$$

where $p \nmid b$ and $p \mid c$. Rewrite this as $p^k ab = b + ac$ and contradition follows from divisibility by p. □

2.2. Generating functions

We define one of our main objects of study, namely a generating function that is attached to each sublattice translate of a lattice tiling.

DEFINITION 2.7. *Let* $\mathbf{v} + \mathcal{L}$ *be a sublattice translate, with* $\mathcal{L}$ *any integer sublattice of* $\mathbb{Z}^d$ *and* $\mathbf{v}$ *any integer vector. We define its* generating function *by*

$$\Theta_{\mathcal{L}+\mathbf{v}}(\mathbf{z}) := \sum_{\substack{\mathbf{w} \in \mathcal{L}+\mathbf{v} \\ \mathbf{w} \geq 0}} \mathbf{z}^{\mathbf{w}} = \sum_{\substack{\mathbf{w} \in \mathcal{L}+\mathbf{v} \\ \mathbf{w} \geq 0}} z_1^{w_1} \ldots z_d^{w_d}.$$

Note the important fact that we are restricting our summation to the positive orthant. This makes the series absolutely convergent for all $\mathbf{z} \in \mathbb{C}^d$ such that $|z_j| < 1$ for all $1 \leq j \leq d$. Our next step is to give an algorithm for computing $\Theta_{\mathcal{L}+\mathbf{v}}$ and to show that it is in fact a rational function on $\mathbb{C}^d$. To this end we introduce another definition.

DEFINITION 2.8. *Let $\mathcal{L}$ be a sublattice. We define $t_1, \ldots, t_d$ as the minimal positive integers such that $(0, \ldots, 0, t_j, 0, \ldots) \in \mathcal{L}$. These integers are called the* polar values *of $\mathcal{L}$.*

EXAMPLE 2.9. Assume that $\mathcal{L}$ is a sublattice in dimension 2 spanned by the vectors (a, b) and (c, d). Let us define $\widetilde{e} = \gcd(a, c)$ and $\widetilde{f} = \gcd(b, d)$. Then it is routine to see that

$$t_1 = \frac{|ad - bc|}{\widetilde{f}}, \ t_2 = \frac{|ad - bc|}{\widetilde{e}}.$$

It easy to see that if (a, b) and (c, d) span $\mathcal{L}$, then the determinant of $\mathcal{L}$ is $|ad - bc|$. Hence the polar values divide the determinant, a fact that we can prove in any dimension.

LEMMA 2.10. *The polar values divide the determinant.*

PROOF. Let $e_1 = (1,0, \ldots, 0)$. The fact that t_1 is a polar value means that $t_1 e_1 \in \mathcal{L}$ and for any $k = 1, \ldots, t_1 - 1$, $ke_1 \notin \mathcal{L}$. Consider the subgroup of $G_{\mathcal{L}} = \mathbb{Z}^d/\mathcal{L}$ spanned by the image of e_1. We see that its order is t_1, so it divides the order of $G_{\mathcal{L}}$. □

LEMMA 2.11. *Let S be the half open cube*

$$S = \{(x_1, \ldots, x_d)\colon\ 0 \leqslant x_i < t_i\}. \tag{3}$$

Then we have

$$\#(S \cap \mathcal{L}) = \frac{t_1 t_2 \ldots t_d}{\det \mathcal{L}}. \tag{4}$$

PROOF. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ be a basis of $\mathcal{L} \subseteq \mathbb{Z}^d$, and consider the map $J\colon\ \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$J(x_1, \ldots, x_d) = x_1\mathbf{v}_1 + \ldots + x_d\mathbf{v}_d. \tag{5}$$

Then J is a linear map and $J(\mathbb{Z}^d) = \mathcal{L}$. Let $T = J^{-1}(S)$. Then T is a half-open parallelepiped with integral corners. We have

$$\#(S \cap \mathcal{L}) = \#(T \cap \mathbb{Z}^d) = \mathrm{vol}(T) = \frac{\mathrm{vol}\, S}{\det J} = \frac{t_1 \dots t_d}{\det \mathcal{L}}. \qquad \square$$

PROPOSITION 2.12. *Let* $\mathbf{v}+\mathcal{L}$ *be a sublattice translate and* $t_1,\dots, t_d$ *be the polar values of* $\mathcal{L}$. *Then*

$$\Theta_{\mathcal{L}+\mathbf{v}}(\mathbf{z}) = \frac{R^{\mathbf{v}}(\mathbf{z})}{(1-z_1^{t_1})\dots(1-z_d^{t_d})},$$

where

$$R^{\mathbf{v}}(\mathbf{z}) = \sum_{\mathbf{w}\in S\cap(\mathbf{v}+\mathcal{L})} \mathbf{z}^{\mathbf{w}}. \tag{6}$$

PROOF. Let $\mathcal{L}_S$ be the sublattice spanned by vectors $(t_1,0,\dots,0),\dots,(0,\dots,t_d)$. It is clear that

$$\Theta_{\mathcal{L}_S}(\mathbf{z}) = \frac{1}{(1-z_1^{t_1})\dots(1-z_d^{t_d})}.$$

Now we have

$$\mathbf{v}+\mathcal{L} = \bigcup_{\mathbf{w}\in S\cap(\mathbf{v}+\mathcal{L})} \{\mathbf{w}+\mathcal{L}_S\}.$$

Hence

$$\Theta_{\mathcal{L}+\mathbf{v}} = \sum_{\mathbf{w}\in S} \mathbf{z}^{\mathbf{w}}\Theta_{\mathcal{L}_S}(\mathbf{z}) = \frac{R^{\mathbf{v}}(\mathbf{z})}{(1-z_1^{t_1})\dots(1-z_d^{t_d})}. \qquad \square$$

Remark 2.13. If the sublattice translate is in fact a sublattice, i. e. if $\mathbf{v} = \mathbf{0}$, then we write $R(\mathbf{z})$ instead of $R^{\mathbf{v}}(\mathbf{z})$. Similarly, we write $\Theta_{\mathcal{L}}$ instead of $\Theta_{\mathcal{L}+\mathbf{v}}$ if $\mathbf{v} = 0$. These notations will be used later in Lemma 3.4.

EXAMPLE 2.14. Consider a sublattice $\mathcal{L} = (2\mathbb{Z}\times 2Z) \cup [(1,1) + (2\mathbb{Z}\times 2\mathbb{Z})] = \{(x,y)\colon\ x+y = 0 \bmod 2\}$. The generating function of the sublattice $2\mathbb{Z}\times 2\mathbb{Z}$ is clearly $\dfrac{1}{(1-x^2)(1-y^2)}$. We get therefore

$$\Theta_{\mathcal{L}}(x,y) = \frac{1+xy}{(1-x^2)(1-y^2)}.$$

Example 2.15. Assume that $\mathcal{L}$ is spanned by $\mathbf{v}_1 = (4,1)$ and $\mathbf{v}_2 = (2,3)$. Then $t_1 = 10$ and $t_2 = 5$. In this example we see that $S \cap \mathcal{L} = \{(0,0),(2,3),(4,1),(6,4),(8,2)\}$ (the marked points on Figure 1). Hence $R(x,y) = 1 + x^2y^3 + x^4y^1 + x^6y^4 + x^8y^2$.

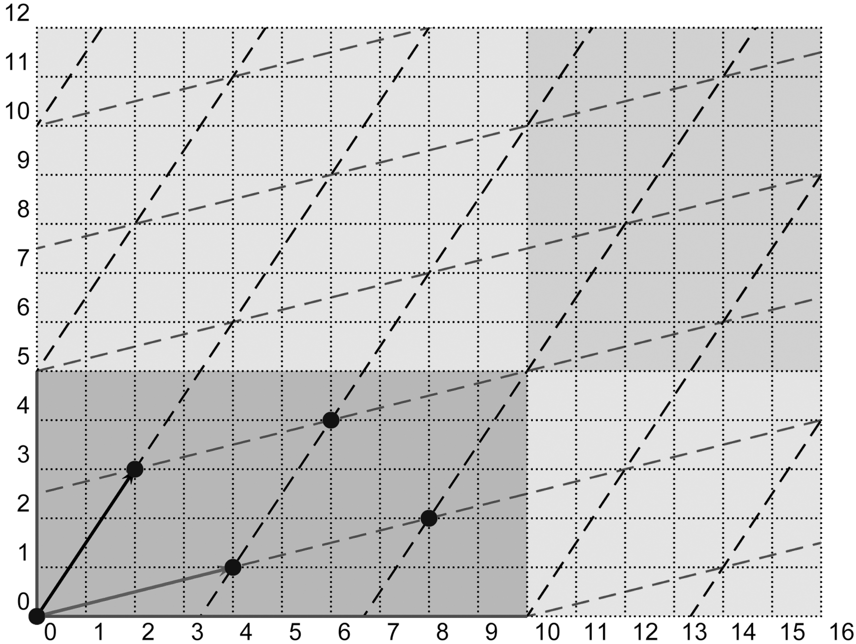

Fig. 1. A sublattice spanned by $(4,1)$ and $(2,3)$. Here the polar values are $t_1 = 10$ and $t_2 = 5$. See Example 2.15

3. More on dual points and characters

Let us recall that $\mathbf{z} \in \mathbb{T}^d$ is called a dual point of $\mathcal{L}$ if $\chi_{\mathbf{z}}(\mathbf{w}) = \mathbf{z}^{\mathbf{w}} = 1$ for all $\mathbf{w} \in \mathcal{L}$. Here $\mathbb{T}^d = \{\mathbf{z} \in \mathbb{C}^d : |z_i| = 1\}$. The following "orthogonality relation" for the characters $\chi_{\mathbf{z}}$ is well-known. But as it is crucial to our proofs, we include the proof for the sake of completeness.

Lemma 3.1. *Let $\mathcal{L} \subseteq \mathbb{Z}^d$ be a sublattice with a basis $\mathbf{v}_1, \ldots, \mathbf{v}_d$ and the fundamental parallelepiped $\mathcal{P} = \{\lambda_1\mathbf{v}_1 + \ldots + \lambda_d\mathbf{v}_d : 0 \le \lambda_i < 1\} \subset \mathbb{R}^d$. If $\mathbf{z} = (z_1, \ldots, z_d)$ is a dual point different from $\mathbf{1} = (1, \ldots, 1)$, then*

$$\sum_{\mathbf{w} \in \mathbb{Z}^d \cap \mathcal{P}} \chi_{\mathbf{z}}(\mathbf{w}) = 0.$$

Proof. Observe that the elements of $\mathcal{P} \cap \mathbb{Z}^d$ are in one-to-one correspondence with elements of the quotient group $\mathbb{Z}^d/\mathcal{L}$. The character $\mathcal{X}_{\mathbf{z}}$ can now be regarded as a character on the group $\mathbb{Z}^d/\mathcal{L}$. This character is non-trivial because $\mathbf{z} \neq \mathbf{1}$. Now we use the standard fact that the average of a non-trivial character over a compact group (in particular over a finite group) is zero, and we are done. □

Example 3.2. Consider again $\mathcal{L}$ generated by $\mathbf{v}_1 = (4,1)$ and $\mathbf{v}_2 = (2,3)$. The points in $\mathcal{P}$ are $(0,0)$, $(1,1)$, $(2,1)$, $(3,1)$, $(2,2)$, $(3,2)$, $(4,2)$, $(3,3)$, $(4,3)$ and $(5,3)$; see Figure 2. We consider $\mathbf{z} = (\varepsilon, \varepsilon)$, where $\varepsilon = e^{2\pi i/5}$. Then $\mathcal{X}_{\mathbf{z}}$ has the following values at these points: $\varepsilon^0, \varepsilon^2, \varepsilon^3, \varepsilon^4, \varepsilon^4, \varepsilon^0, \varepsilon^1, \varepsilon^1, \varepsilon^2$ and ε^3. The sum is clearly 0.

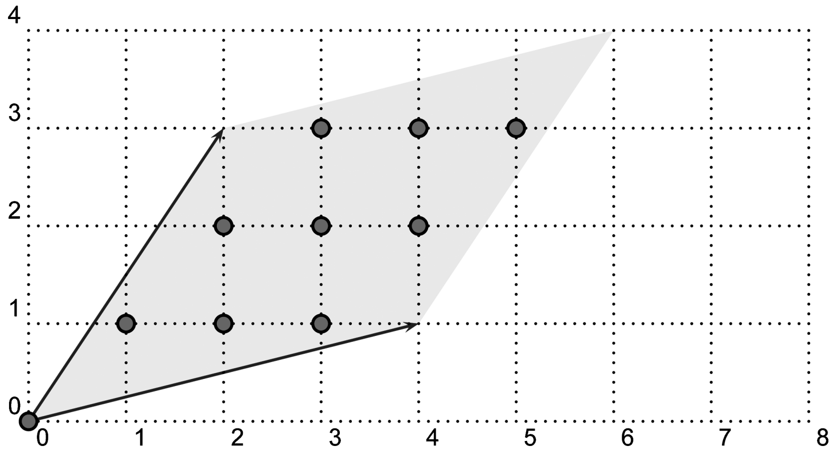

Fig. 2. The points in the parallelepiped P for the sublattice spanned by $(4,1)$ and $(2,3)$

The next two results will also be very useful for us. We first prove it for sublattices, namely when the translate vector is $(0,\dots,0)$.

Lemma 3.3. *Suppose that* $\mathbf{z} = (z_1,\dots,z_d) \in \mathbb{T}^d$ *satisfies* $z_1^{t_1} = \dots = z_d^{t_d} = 1$ *and* $\mathbf{z}$ *is* not *a dual point of* $\mathcal{L}$ (*the* t_i *are the polar values*). *Then* $R(\mathbf{z}) = 0$.

Proof. By definition $\mathbf{z}$ is a dual point of the sublattice $\mathcal{L}_S$, defined to be the integral span of the vectors $(t_1, 0, \dots, 0), \dots, (0, \dots, 0, t_d)$. The map J defined in (5) induces a dual map $J^*\colon\ \mathbb{T}^d \to \mathbb{T}^d$ by the formula

$$\mathcal{X}_{J^*\mathbf{z}}(\mathbf{w}) = \mathcal{X}_{\mathbf{z}}(J\mathbf{w}).$$

Let $\mathbf{y} = J^*\mathbf{z}$, $\mathcal{T} = J^{-1}(\mathcal{L}_S)$ and $\mathcal{P}(\mathcal{T})$ the fundamental parallelepiped of $\mathcal{T}$. By definition, J^* maps dual points of $\mathcal{L}_S$ to dual points of $J^{-1}(\mathcal{T})$. We have

$$R(z_1, \dots, z_d) = \sum_{\mathbf{t} \in S \cap \mathcal{L}} \chi_{\mathbf{z}}(\mathbf{t}) = \sum_{\mathbf{w} \in \mathcal{P}(\mathcal{T})} \chi_{\mathbf{z}}(J\mathbf{w}) = \sum_{\mathbf{w} \in \mathcal{P}(\mathcal{T})} \chi_{\mathbf{y}}(\mathbf{w}).$$

Now $\mathbf{y}$ is a dual point of $\mathcal{T}$ and $y \neq \mathbf{1}$, for otherwise $\mathbf{z}$ would be a dual point of $\mathcal{L}$. The result follows by applying Lemma 3.1. □

The version for a translated sublattice is:

LEMMA 3.4. *Let $\mathcal{L}$ be a sublattice with polar values $(t_1, \dots, t_d)$. Suppose that $\mathbf{z} = (z_1, \dots, z_d)$ satisfies $z_1^{t_1} = \dots = z_d^{t_d} = 1$. Then for any translate vector $\mathbf{v}$ we have*

$$R^{\mathbf{v}}(\mathbf{z}) = \chi_{\mathbf{z}}(\mathbf{v}) R(\mathbf{z}).$$

PROOF. Let S be the cube as defined in (3). We have:

$$R^{\mathbf{v}}(\mathbf{z}) = \sum_{\substack{\mathbf{w} \in \mathcal{L} \\ \mathbf{w}+\mathbf{v} \in S}} \chi_{\mathbf{z}}(\mathbf{w} + \mathbf{v}) = \chi_{\mathbf{z}}(\mathbf{v}) \sum_{\substack{\mathbf{w} \in \mathcal{L} \\ \mathbf{w}+\mathbf{v} \in S}} \chi_{\mathbf{z}}(\mathbf{w}) = \chi_{\mathbf{z}}(\mathbf{v}) \sum_{\mathbf{w} \in \mathcal{L} \cap (S - \mathbf{v})} \chi_{\mathbf{z}}(\mathbf{w}).$$

Observer that for any $\mathbf{w} \in \mathcal{L} \cap (S - v)$, there is a unique $\mathbf{t}_{\mathbf{w}} \in \mathcal{L}_S$ such that $\mathbf{w} + \mathbf{t}_{\mathbf{w}} \in S$. The points in $\mathcal{L} \cap S$ and $\mathcal{L} \cap (S - v)$ are actually in one-to-one correspondence. Because $\mathbf{z}$ is a dual point of $\mathcal{L}_s$, we have $\chi_{\mathbf{z}}(\mathbf{w} + \mathbf{t}_{\mathbf{w}}) = \chi_{\mathbf{z}}(\mathbf{w}) \chi_{\mathbf{z}}(\mathbf{t}_{\mathbf{w}}) = \chi_{\mathbf{z}}(\mathbf{w})$, and hence

$$\sum_{\mathbf{w} \in \mathcal{L} \cap (S - \mathbf{v})} \chi_{\mathbf{z}}(\mathbf{w}) = \sum_{\mathbf{w} \in \mathcal{L} \cap (S - \mathbf{v})} \chi_{\mathbf{z}}(\mathbf{w} + \mathbf{t}_{\mathbf{w}}) = \sum_{\mathbf{w} \in \mathcal{L} \cap S} \chi_{\mathbf{z}}(\mathbf{w}) = R(\mathbf{z}).$$ □

We finish this section with another important result, whose proof follows immediately from Pontryagin duality.

LEMMA 3.5. *Assume that $\mathcal{L} \subset \mathbb{Z}^d$ is a sublattice and $\mathbf{v} \in \mathbb{Z}^d \setminus \mathcal{L}$. Then there exists a dual point $\mathbf{z} \in \widehat{G}_{\mathcal{L}}$ such that $\chi_{\mathbf{z}}(\mathbf{v}) \neq 1$.*

COROLLARY 3.6. *If we are given two sublattices $\mathcal{L}_1$ and $\mathcal{L}_2$ and each dual point of $\mathcal{L}_1$ is a dual point of $\mathcal{L}_2$ then $\mathcal{L}_2 \subseteq \mathcal{L}_1$. In particular, if $\widehat{G}_{\mathcal{L}_1} = \widehat{G}_{\mathcal{L}_2}$ as subgroups of $\mathbb{T}^d$, then $\mathcal{L}_1 = \mathcal{L}_2$.*

Proof. Assume that $\mathbf{v} \in \mathcal{L}_2 \setminus \mathcal{L}_1$. Then there exist a dual point $\mathbf{z}$ of $\mathcal{L}_1$ such that $\chi_{\mathbf{z}}(\mathbf{v}) \neq 1$. But as $\mathbf{z}$ is a dual point of $\mathcal{L}_2$ by assumption, we get $\chi_{\mathbf{z}}(\mathbf{v}) = 1$, a contradiction. □

4. Residue calculus

4.1. Residues of generating functions

Assume now that we have sublattices $\mathcal{L}_1, \ldots, \mathcal{L}_n$ in $\mathbb{Z}^d$. Denote the polar values of the sublattice $\mathcal{L}_i$ by $t_{i,1}, \ldots, t_{i,d}$. Assume that there exist non-negative vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ such that we have the lattice tiling

$$\begin{cases} \{\mathbf{v}_1 + \mathcal{L}_1\} \cup \{\mathbf{v}_2 + \mathcal{L}_2\} \cup \ldots \cup \{\mathbf{v}_n + \mathcal{L}_n\} = \mathbb{Z}^d, & \\ \{\mathbf{v}_i + \mathcal{L}_i\} \cap \{\mathbf{v}_j + \mathcal{L}_j\} = \varnothing & \text{for } i \neq j. \end{cases} \tag{7}$$

Let Θ_i be the generating function for $\mathbf{v}_i + \mathcal{L}_i$.

Lemma 4.1. *We have the following equality of rational functions, with* $\mathbf{z} = (z_1, \ldots, z_d)$:

$$\sum_{i=1}^{n} \Theta_i(\mathbf{z}) = \frac{1}{(1 - z_1) \cdots (1 - z_d)}. \tag{8}$$

Moreover, the identity (8) *is equivalent to the condition that* $\mathbf{v}_1 + \mathcal{L}_1$, ..., $\mathbf{v}_n + \mathcal{L}_n$ *provide a lattice tiling of* $\mathbb{Z}^d$.

Proof. The right-hand side of (8) is the generating function for the tiling consisting of a single sublattice $\mathbb{Z}^d$. Let us choose $\mathbf{z}$ such that $|z_j| < 1$ for all j. By the tiling condition we have

$$\sum_{\substack{\mathbf{w} \in \mathbb{Z}^d \\ \mathbf{w} \geq 0}} \mathbf{z}^{\mathbf{w}} = \sum_{j=1}^{n} \sum_{\substack{\mathbf{w} \in \mathbf{v}_j + \mathcal{L}_j \\ \mathbf{w} \geq 0}} \mathbf{z}^{\mathbf{w}}, \tag{9}$$

where we use the fact that we can change the summation order of absolutely convergent power series. Equation (9) is equivalent to (8) for $\mathbf{z}$ such that $|z_j| < 1$ (see Definition 2.7). Both sides of (8) are rational functions agreeing on an open subset of $\mathbb{C}^d$, so they are equal. It is also clear that (9) implies the tiling condition. □

We now fix some $\mathbf{z} \in \mathbb{C}^d$. Consider a torus $T = \{\mathbf{u} \in \mathbb{C}^d\colon\ |z_1 - u_1| = \ldots = |z_d - u_d| = \varepsilon\}$ for $\varepsilon > 0$ sufficiently small. Equation (8) implies that

$$\sum_{j=1}^{n} \int_T \Theta_j(u) d\mathbf{u} = \int_T \frac{d\mathbf{u}}{(1-u_1)\ldots(1-u_d)}. \tag{10}$$

Here $d\mathbf{u} = du_1 \wedge \ldots \wedge du_d$ is the volume form on T (note that (10) can be regarded as comparison of multidimensional residues, see [14]). We want to study the integrals appearing on the left-hand side of (10). To this end recall that by Proposition 2.12 that we can write

$$\Theta_i(\mathbf{z}) = \frac{R_i(\mathbf{z})}{(1-z_1^{t_{i,1}})\cdots(1-z_d^{t_{i,d}})},$$

where

$$R_i(\mathbf{z}) = \sum_{\mathbf{w} \in S_i \cap (\mathbf{v}_i + \mathcal{L}_i)} \chi_{\mathbf{z}}(\mathbf{w}).$$

LEMMA 4.2. *For any $j = 1, \ldots, n$, the integral*

$$\int_T \frac{R_j(\mathbf{u})}{(1-u_1^{t_{j,1}})\ldots(1-u_d^{t_{j,d}})} d\mathbf{u}$$

is zero unless $\mathbf{z} \in \widehat{G}_{\mathcal{L}_j}$. In the latter case it is equal to $\dfrac{(-2\pi i)^d}{\det \mathcal{L}_j} \chi_{\mathbf{z}}(\mathbf{v}_j + \mathbf{1})$.

PROOF. As R_j is analytic at $\mathbf{z}$, we have

$$\int_T \frac{R_j(\mathbf{u})}{(1-u_1^{t_{j,1}})\ldots(1-u_d^{t_{j,d}})} d\mathbf{u} = R_j(\mathbf{z}) \int_T \frac{d\mathbf{u}}{(1-u_1^{t_{j,1}})\ldots(1-u_d^{t_{j,d}})}. \tag{11}$$

But

$$\int_T \frac{d\mathbf{u}}{(1-u_1^{t_{j,1}})\ldots(1-u_d^{t_{j,d}})} = \prod_{k=1}^{d} \int_{|u_k - z_k| = \varepsilon} \frac{du_k}{1-u_k^{t_{j,k}}}.$$

By Goursat's lemma the integrals on the right-hand side vanish unless $z_1^{t_{j,1}} = \ldots = z_d^{t_{j,d}} = 1$. So assume that $z_1^{t_{j,1}} = \ldots = z_d^{t_{j,d}} = 1$. Since for any $x_0 \in \mathbb{C}$ and any integer $m > 0$

$$\int\limits_{|x-x_0|=\varepsilon} \frac{dx}{1-x^m} \overset{x=zx_0}{=} x_0 \int\limits_{|z-1|=\varepsilon} \frac{dz}{1-z^m} = -2\pi i \frac{x_0}{m},$$

we have

$$\prod_{k=1}^{d} \int\limits_{|u_k-z_k|=\varepsilon} \frac{du_k}{1-u_k^{t_{j,k}}} = \frac{(-2\pi i)^d}{t_{j,1}\dots t_{j,d}} z_1 \dots z_d. \tag{12}$$

Given that $z_1^{t_{j,1}} = \dots = z_d^{t_{j,d}} = 1$, to compute $R_j(\mathbf{z})$ in this case we use Lemma 3.3 and Lemma 3.4. We get $R_j(\mathbf{z}) = 0$ unless $\mathbf{z}$ is a dual point of L_j. In the latter case, by Lemma 3.4 and (4), we have

$$R_j(\mathbf{z}) = \frac{t_{j,1}\dots t_{j,d}}{\det \mathcal{L}_j} \chi_{\mathbf{z}}(\mathbf{v}_j). \tag{13}$$

Substituting the R_j from (13) and the integral from (12) into (11) we conclude the proof. □

We now combine Lemma 4.2 with Lemma 4.1 to obtain our main technical result.

PROPOSITION 4.3. *Let $\mathbf{z} \in \mathbb{T}^d$ be arbitrary. In a lattice tiling, we have*

$$\sum_{j:\ \mathbf{z}\in\widehat{G}_{\mathcal{L}_j}} \frac{\chi_{\mathbf{z}}(\mathbf{v}_j)}{\det \mathcal{L}_j} = \begin{cases} 1, & \text{if } \mathbf{z} = \mathbf{1} \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

PROOF. The case $\mathbf{z} = \mathbf{1}$ is Lemma 2.6. If $\mathbf{z} \neq \mathbf{1}$, consider (10). The integrals on the left-hand side can be computed with Lemma 4.2. The integral on the right-hand side is 0. The proposition follows immediately. □

From Proposition 4.3 we can deduce an immediate corollary.

COROLLARY 4.4. *Given a lattice tiling, assume that $\mathbf{z}$ is a dual point of $\mathcal{L}_i$ and $\mathbf{z} \neq \mathbf{1}$. Then there exists at least one other $\mathcal{L}_j$ such that $\mathbf{z}$ is also a dual point of $\mathcal{L}_j$.*

PROOF. If $\mathbf{z}$ belongs only to $\widehat{G}_{\mathcal{L}_i}$, then the left-hand side of (14) is $\dfrac{\chi_{\mathbf{z}}(\mathbf{v}_i)}{\det \mathcal{L}_i} \neq 0$, and we obtain a contradiction. □

PROOF OF THEOREM 1.5. Assume $\mathcal{L}_1$ is cyclic and has largest determinant. Let $\mathbf{z}$ be an element in $\widehat{G}_{\mathcal{L}_1}$ of order $\det \mathcal{L}_1$. By Corollary 4.4 there exists $j > 1$ such that $\mathbf{z} \in \widehat{G}_{\mathcal{L}_j}$. But then the whole group generated by $\mathbf{z}$ lies in $\widehat{G}_{\mathcal{L}_j}$, and hence $\widehat{G}_{\mathcal{L}_1} \subseteq \widehat{G}_{\mathcal{L}_j}$. By maximality of $\det \mathcal{L}_1$ we have $\widehat{G}_{\mathcal{L}_1} = \widehat{G}_{\mathcal{L}_j}$. Now Corollary 3.6 implies that $\mathcal{L}_1 = \mathcal{L}_j$. □

EXAMPLE 4.5. Consider the four sublattices $\mathcal{L}_1 = (2\mathbb{Z}\times 2\mathbb{Z}\times\mathbb{Z})$, $\mathcal{L}_2 = (2\mathbb{Z}\times\mathbb{Z}\times 2\mathbb{Z})$, $\mathcal{L}_3 = (\mathbb{Z}\times 2\mathbb{Z}\times 2\mathbb{Z})$ and $\mathcal{L}_4 = (2\mathbb{Z}\times 2\mathbb{Z}\times 2\mathbb{Z}) \cup [(1,1,1) + (2\mathbb{Z}\times 2\mathbb{Z}\times 2\mathbb{Z})]$. It is known that $(1,0,0)+\mathcal{L}_1$, $(0,0,1)+\mathcal{L}_2$, $(0,1,0)+\mathcal{L}_3$ and $\mathcal{L}_4$ tile $\mathbb{Z}^3$. We have

$$\begin{aligned} &\frac{z_1}{(1-z_1^2)(1-z_2^2)(1-z_3)} + \frac{z_3}{(1-z_1^2)(1-z_2)(1-z_3^2)} + \\ &\qquad + \frac{z_2}{(1-z_1)(1-z_2^2)(1-z_3^2)} + \frac{1+z_1z_2z_3}{(1-z_1^2)(1-z_2^2)(1-z_3^2)} = \\ &\qquad\qquad = \frac{1}{(1-z_1)(1-z_2)(1-z_3)}. \end{aligned}$$

The nontrivial dual points are

$$\begin{aligned} \widehat{G}_{\mathcal{L}_1}&:\ (1,-1,1),\ (-1,1,1),\ (-1,-1,1)\\ \widehat{G}_{\mathcal{L}_2}&:\ (1,1,-1),\ (-1,1,1),\ (-1,1,-1)\\ \widehat{G}_{\mathcal{L}_3}&:\ (1,1,-1),\ (1,-1,1),\ (1,-1,-1)\\ \widehat{G}_{\mathcal{L}_4}&:\ (1,-1,-1),\ (-1,1,-1),\ (-1,-1,1). \end{aligned}$$

We see that each dual point occurs precisely twice. This is a lattice tiling in dimension 3 without the translational property.

COROLLARY 4.6. *Assume that the point* $\mathbf{z} \neq \mathbf{1}$ *is a dual point of the sublattices* $\mathcal{L}_i$ *and* $\mathcal{L}_j$ *and no other sublattices. Then* $\det \mathcal{L}_i = \det \mathcal{L}_j$.

PROOF. By (14) we get

$$\frac{\chi_{\mathbf{z}}(\mathbf{v}_i)}{\det \mathcal{L}_i} + \frac{\chi_{\mathbf{z}}(\mathbf{v}_j)}{\det \mathcal{L}_j} = 0.$$

But the numerators are roots of unity, so the denominators, being both positive integers, must also agree. □

4.2. Sufficiency of the residue condition

We will show now that the conditions given by Proposition 4.3 are sufficient for the tiling. More precisely, we have the following result.

THEOREM 4.7. *Let $\mathbf{v}_1 + \mathcal{L}_1, \ldots, \mathbf{v}_n + \mathcal{L}_n$ be sublattice translates in $\mathbb{Z}^d$ with generating functions $\Theta_1, \ldots, \Theta_n$. Suppose that for any $\mathbf{z} \in \mathbb{T}^d$ we have the relation* (14), *then the sublattice translates tile $\mathbb{Z}^d$.*

Remark 4.8. If $\mathbf{z}$ is not a dual point of any of the sublattices, then (14) is an empty relation. Therefore it is enough to check (14) for finitely many cases.

PROOF. We would like to show that $\Theta_1, \ldots, \Theta_n$ satisfy (8). In the following, for a polynomial P in variables $z_1, \ldots, z_d$, we write $\deg_k P$ as the degree in variable z_k.

Observe that each generating function Θ_j vanishes at infinity. More precisely, if we fix $z_1, \ldots, \widehat{z}_k, \ldots, z_d$ such that $z_m^{t_{j,m}} \neq 1$ for any $m \neq k$ (where $t_{j,m}$ denotes the m-th polar value of $\mathcal{L}_j$), then we have $\lim_{z_k \to \infty} \Theta_j(z_1, \ldots, z_d) = 0$. This is a direct consequence of the fact that $\deg_k R_j < t_{j,k}$. In particular if we define

$$\Theta(\mathbf{z}) = \Theta_1(\mathbf{z}) + \ldots + \Theta_n(\mathbf{z}) - \frac{1}{(1 - z_1) \ldots (1 - z_d)},$$

then Θ also vanishes at infinity. We can write Θ in the following way.

$$\Theta(\mathbf{z}) = \frac{R(\mathbf{z})}{Q_1(z_1) \ldots Q_d(z_d)},$$

where R is a polynomial in $z_1, \ldots, z_d$ and Q_m is the least common multiple of $1 - z_m^{t_{1,m}}, \ldots, 1 - z_m^{t_{n,m}}$. Notice that each Q_m is square free. The asymptotics of Θ implies that

$$\deg_k R < \deg Q_k \quad \text{for } 1 \leqslant k \leqslant d. \tag{15}$$

We first claim that if $u_1, \ldots, u_d$ are such that $Q_1(u_1) = \ldots = Q_d(u_d) = 0$, then we have $R(u_1, \ldots, u_d) = 0$. Indeed, if $\mathbf{u} = (u_1, \ldots, u_d)$ is not a dual point of any $\mathcal{L}_j$, then the residue of any $\Theta_j(\mathbf{z})d\mathbf{z}$ at $\mathbf{u}$ is zero (see proof of Lemma 4.2). If it is a dual point of some sublattices, then the second case of (14) applies. Therefore, $R(u_1, \ldots, u_d)$ is proportional to the total residue of the form $\Theta(\mathbf{z})d\mathbf{z}$ at $u_1, \ldots, u_d$, which is zero.

Now by induction we shall show that for any k, and any $u_{k+1}, \dots, u_d$ satisfying $Q_{k+1}(u_{k+1}) = \dots = Q_d(u_d) = 0$, we have

$$R(z_1, \dots, z_k, u_{k+1}, \dots, u_d) \equiv 0 \text{ as a polynomial in } z_1, \dots, z_k.$$

The induction assumption (for $k = 0$) is done. Now suppose we have proved it for $k - 1$. Thus, the polynomial

$$P_k(z_1, \dots, z_{k-1}, z_k) := R(z_1, \dots, z_k, u_{k+1}, \dots, u_d)$$

vanishes at any $z_k = u_k$ that satisfy $Q_k(u_k) = Q_{k+1}(u_{k+1}) = \dots = Q_d(u_d) = 0$. Hence, P_k is divisible by $(z_k - w_1) \dots (z_k - w_l)$, where $w_1, \dots, w_l$ are roots of Q_k. Since Q_k is square free, we have $l = \deg Q_k$. But $\deg_k P_k < \deg Q_k$. The only possibility is that $P_k \equiv 0$. The induction step is done.

The statement for $k = d$ implies that R is identically zero. This is equivalent to (8), and the proof is finished. □

Remark 4.9. The above proof is a generalization of the fact that if a rational function on $\mathbb{C}$ has only simple poles, vanishes at infinity and has residue 0 at each pole, then it is equal to zero everywhere. One could express the above proof in the language of multidimensional residues, but we wanted the proofs to be accessible to non-experts.

5. The translational property

5.1. The translational property in $\mathbb{Z}^d$

A lattice tiling has the translational property if at least two of its translated sublattices are different cosets of the same sublattice.

The translational property is known in $\mathbb{Z}^1$ (see Section 1.4). In fact, for any expression of $\mathbb{Z}^1$ as a nontrivial *rational linear combination* of arithmetic progressions, there are two progressions having same difference.

In [7], a counterexample to the translational property in $\mathbb{Z}^3$ was given. We reproduced this in Example 4.5. This construction directly generalizes to higher dimensions. We now focus on the translational property in lattice tilings of $\mathbb{Z}^2$. But first, we want to point out a counterexample for Question 1.3 in a more general setting, in which sublattice translates carry rational weights and are allowed to overlap.

EXAMPLE 5.1. Consider $\mathcal{L}_1 = \operatorname{span}((2,0),(0,1))$, $\mathcal{L}_2 = \operatorname{span}((1,0),(0,2))$, $\mathcal{L}_3 = \operatorname{span}((2,0),(0,2))$, $\mathcal{L}_4 = \operatorname{span}((1,1),(1,-1))$. These sublattices of $\mathbb{Z}^2$ are obviously distinct. However, we have a tiling:

$$\mathbb{Z}^2 = \frac{1}{2}(\mathcal{L}_1 + (0,0)) + \frac{1}{2}(\mathcal{L}_2 + (0,0)) + 1(\mathcal{L}_3 + (1,1)) + \frac{1}{2}(\mathcal{L}_4 + (1,0)).$$

Here the rational coefficient in front of each translate $(\mathcal{L}_i + \mathbf{v}_i)$ tells us how many "times" a point in that translate is counted.

This implies that relaxing the disjointness condition will destroy the combinatorial rigidity of lattice tilings. Before investigating the translational property in $\mathbb{Z}^2$, we look at some properties of 2-dimensional dual groups.

5.2. Characterisation of dual groups of sublattices in $\mathbb{Z}^2$

It is easy to show that a 2-dimensional sublattice $\mathcal{L}$ generated by $v_1 = (a,b)$ and $v_2 = (c,d)$ is cyclic if and only if $\gcd(a,b,c,d) = 1$. The quantity $e = \gcd(a,b,c,d)$ does not depend on the choice of basis and we will call it the *multiplicity* of $\mathcal{L}$.

It follows from the definition of Smith Normal Form that we have an isomorphism, namely $\widehat{G}_{\mathcal{L}} \cong \mathbb{Z}_e \oplus \mathbb{Z}_{\frac{\det \mathcal{L}}{e}}$ and furthermore $e^2 | \det \mathcal{L}$. In particular, if $\det \mathcal{L}$ is square free, the sublattice $\mathcal{L}$ is necessarily cyclic. In dimensions higher than 2, cyclicity is more subtle and the group $\widehat{G}_{\mathcal{L}}$ might be more complicated.

To stress the difference between the 2-dimensional case and the higher dimensional ones, we first prove a simple result.

LEMMA 5.2. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two sublattices of $\mathbb{Z}^2$ with equal multiplicities $e_1 = e_2 = e$. Assume that there is a common vector $w = (a,b) \in \mathcal{L}_1 \cap \mathcal{L}_2$ such that* $\gcd(a,b) = e$. *Assume also that* $\det \mathcal{L}_1 = \det \mathcal{L}_2$, *then we have* $\mathcal{L}_1 = \mathcal{L}_2$.

PROOF. Rescaling the two sublattices by the factor $1/e$ we can assume that $e = 1$. Let $\mathbf{w} = (a,b)$ and $\mathbf{v}_1 = (c_1,d_1)$, $\mathbf{v}_2 = (c_2,d_2)$ be two vectors such that $(\mathbf{w},\mathbf{v}_i)$ spans $\mathcal{L}_i$ and $ad_i - bc_i = \det \mathcal{L}_i$, $i = 1,2$. The equality of determinants implies that

$$a(d_1 - d_2) = b(c_1 - c_2).$$

As $\gcd(a,b) = 1$, we infer that there exists $k \in \mathbb{Z}$ such that $c_1 - c_2 = ka$, $d_1 - d_2 = kb$. This means that $\mathbf{v}_2 = \mathbf{v}_1 + k\mathbf{w}$. In particular $\mathbf{v}_2 \in \mathcal{L}_1$, so $\mathcal{L}_2 \subseteq \mathcal{L}_1$. Similarly $\mathcal{L}_1 \subseteq \mathcal{L}_2$ and we are done. □

Remark 5.3. The proof does not work if we do not assume that $\gcd(a,b)=e$. For example consider $\mathcal{L}_1=2\mathbb{Z}\times\mathbb{Z}$ and $\mathcal{L}_2=\mathbb{Z}\times 2\mathbb{Z}$. Then $e=1$, $(2,2)\in\mathcal{L}_1\cap\mathcal{L}_2$ and $\det\mathcal{L}_1=\det\mathcal{L}_2=2$, but $\mathcal{L}_1$ and $\mathcal{L}_2$ are different. In the language of dual points and dual groups we can reformulate Lemma 5.2 as follows.

COROLLARY 5.4. *Given two sublattices $\mathcal{L}_1$ and $\mathcal{L}_2$ with the same multiplicity e and determinant Δ. Assume that there exists $g_1\in\widehat{G}_{\mathcal{L}_1}$ and $g_2\cap\widehat{G}_{\mathcal{L}_2}$, both of order Δ/e, such that $g_1^e=g_2^e$. Then $\mathcal{L}_1=\mathcal{L}_2$.*

PROOF. Let $g=g_1^e=g_2^e=(z_1,z_2)\in\mathbb{T}^2$. Consider the two integer sublattices $\frac{1}{e}\mathcal{L}_1$ and $\frac{1}{e}\mathcal{L}_2$. They both are cyclic with determinant Δ/e^2 and admit g as a common dual point. Now g also has order Δ/e^2, hence it is a generator for both $\widehat{G}_{\frac{1}{e}\mathcal{L}_1}$ and $\widehat{G}_{\frac{1}{e}\mathcal{L}_2}$. So we must have $\widehat{G}_{\frac{1}{e}\mathcal{L}_1}=\widehat{G}_{\frac{1}{e}\mathcal{L}_2}$. The result follows easily by applying Corollary 3.6 and then rescaling to the original sublattices. □

5.3. The translational property in $\mathbb{Z}^2$

In this subsection, we prove some sufficient conditions for the translational property in $\mathbb{Z}^2$ to hold. Assume that we are given sublattice translates $\mathbf{v}_1+\mathcal{L}_1,\dots,\mathbf{v}_n+\mathcal{L}_n$ that together tile $\mathbb{Z}^2$. Let us reorder the sublattices in the following way.

(a) If $i<j$, then $\frac{1}{e_i}\det\mathcal{L}_i\geqslant\frac{1}{e_j}\det\mathcal{L}_j$. In other words, the maximal cyclic subgroup of $\widehat{G}_{\mathcal{L}_i}$ has at least the same order as the maximal cyclic subgroup of $\widehat{G}_{\mathcal{L}_j}$.

(b) If $i<j$, but $\frac{1}{e_i}\det\mathcal{L}_i=\frac{1}{e_j}\det\mathcal{L}_j$, then $e_i>e_j$.

THEOREM 5.5. *If the number e_1 is of the form p^r for p a prime, then the tiling has the translation property.*

PROOF. We set $\alpha=\det\mathcal{L}_1/e_1$. Let $\mathbf{z}\in\widehat{G}_{\mathcal{L}_1}$ be an element of order α. We define a sequence $1=n_1<n_2<\ldots<n_s$ of indices such that $\mathbf{z}$ belongs to $\widehat{G}_{\mathcal{L}_{n_1}},\dots,\widehat{G}_{\mathcal{L}_{n_s}}$ and to no other lattice. This sequence is nonempty by Corollary 4.4. To shorten the notation we will write $\mathcal{L}_k^{\mathbf{z}}$, $\mathbf{v}_k^{\mathbf{z}}$ instead of $\mathcal{L}_{n_k}$, $\mathbf{v}_{n_k}$.

The maximum order over all elements in $\widehat{G}_{\mathcal{L}_k^{\mathbf{z}}}$ is $\det\mathcal{L}_k^{\mathbf{z}}/e_k^{\mathbf{z}}$. By the ordering condition we have $\det\mathcal{L}_k^{\mathbf{z}}/e_k^{\mathbf{z}}\leqslant\alpha$. But $\widehat{G}_{\mathcal{L}_k^{\mathbf{z}}}$ contains the element $\mathbf{z}$ of order α. Hence $\det\mathcal{L}_k^{\mathbf{z}}/e_k^{\mathbf{z}}=\alpha$. Let us now apply now Proposition 4.3 to get the following equation

$$\sum_{k=1}^{s} \frac{\chi_{\mathbf{z}}(\mathbf{v}_k^{\mathbf{z}})}{\alpha e_k^{\mathbf{z}}} = 0, \tag{16}$$

where we wrote $\det \mathcal{L}_k^{\mathbf{z}} = \alpha e_k^{\mathbf{z}}$. Each term $\chi_{\mathbf{z}}(\mathbf{v}_k^{\mathbf{z}})$ is a root of unity. Define

$$a_k = \chi_{\mathbf{z}}(\mathbf{v}_1^{\mathbf{z}})^{-1} \chi_{\mathbf{z}}(\mathbf{v}_k^{\mathbf{z}}).$$

Equation (16) now takes the following form:

$$\sum_{k=1}^{s} \frac{a_k}{e_k^{\mathbf{z}}} = 0. \tag{17}$$

Denote by g the minimal positive integer such that $a_k^g = 1$ for all $k = 1, \dots, s$. Next we need two lemmas.

LEMMA 5.6. *If there are $k \neq l$ such that $e_k^{\mathbf{z}} = e_l^{\mathbf{z}}$, then the translational property holds.*

PROOF OF LEMMA 5.6. The sublattices $\mathcal{L}_k^{\mathbf{z}}$ and $\mathcal{L}_l^{\mathbf{z}}$ have the same determinant and multiplicity, and share an element $\mathbf{z}$ of order equal to the order of each sublattice. By Corollary 5.4 we obtain that $\mathcal{L}_k^{\mathbf{z}} = \mathcal{L}_l^{\mathbf{z}}$. So the translational property holds. □

LEMMA 5.7. *Suppose that there exists a prime q an integer $l > 0$, and an index $k \in \{1, \dots, s\}$ such that $q^l | e_k^{\mathbf{z}}$. Then there exists $k' \in \{1, \dots, s\}$, $k' \neq k$ such that $q^l | e_{k'}^{\mathbf{z}}$.*

PROOF OF LEMMA 5.7. Assume the contrary, so that k is the unique index for which $q^l | e_k^{\mathbf{z}}$. Let B be the least common multiple of $e_1^{\mathbf{z}}, \dots, e_s^{\mathbf{z}}$ and $B_k = B/e_k^{\mathbf{z}}$. By the above assumption, $q \nmid B_k$ and for any $n \neq k$ we have $q | B_n$.

Equation (17) can be now be rewritten as

$$B_k + \sum_{n \neq k} B_n \frac{a_n}{a_k} = B_k + \sum_{n \neq k} B_m \varepsilon^{\gamma_n} = 0, \tag{18}$$

where ε is a root of unity of order g, and $\gamma_n \in \{0, \dots, g-1\}$. The above expression is a polynomial in ε. Let us denote this polynomial by $P(\varepsilon)$. By the assumption on $B_1, \dots, B_s$, we have.

$$P(\varepsilon) = B_k + qQ(\varepsilon),$$

where Q is a polynomial with integer coefficients.

Now let H be the minimal integer polynomial for a g-th root of unity. This is a monic, symmetric polynomial. Since $P(\varepsilon) = 0$, H divides P. Since H is monic, the quotient $R = P/H \in \mathbb{Z}[x]$ has integer coefficients. We end up with the following relation in the ring $\mathbb{Z}[x]$:

$$B_k + qQ(x) = R(x)H(x). \tag{19}$$

Let us now reduce this equation modulo q. We get $B_k \bmod q = (R \bmod q)(H \bmod q)$, where $(H \bmod q)$ has positive degree (because H is monic) and $B_k \not\equiv 0 \bmod q$. This cannot hold, for either $\deg(R \bmod q) \geq 0$ and the right-hand side has positive degree, or $R \equiv 0 \bmod q$ so the left-hand side must be zero. □

Remark 5.8. We point out that Lemma 5.7 works without any assumption on the translational property. It is a direct consequence of Proposition 4.3, that is of the tiling condition. The result is valid also in higher dimensions, if we define e as the quotient of the determinant over the order.

Conclusion of the proof of Theorem 5.5

We apply now Lemma 5.7 to $q^l = p^r = e_1^{\mathbf{z}} = e_1$. We find another index $k > 1$ such that $p^r | e_k^{\mathbf{z}}$. But $e_k^{\mathbf{z}} \leq e_1 = p^r$ by the ordering, so we must have $e_k^{\mathbf{z}} = e_1$. Now Lemma 5.6 ensures the translational property. □

6. Open questions

We end with some open questions which arise naturally from the results above.

Problem 6.1. For dimensions $d \geq 2$, give a necessary and sufficient condition, in terms of the arithmetic of the sublattice translates, for a lattice tiling to possess the translation property.

We call a lattice tiling *primitive* if it is not a split tiling. In other words, a primitive lattice tiling is a tiling that cannot be formed by splitting another tiling which has a smaller number of sublattice translates.

Problem 6.2. What are the divisibility relations between the determinants of the sublattices in a primitive tiling?

And, of course, perhaps the most surprising state of affairs is that Question 1.2 from the introduction, concerning two dimensional lattice tilings, remains open.

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