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Simultaneous distribution of primitive lattice points in convex planar domain

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Abstract: Let Ω denote a compact convex subset of $\mathbf{R}^2$. Suppose that Ω contains the origin as an inner point. Suppose that Ω is bounded by the curve $\partial\Omega$, parametrized by $x = r_\Omega(\theta)\cos\theta$, $y = r_\Omega(\theta)\sin\theta$, where the function r_Ω is continuous and piecewise C^3 on $[0, \pi/4]$. For each real $R \geqslant 1$ we consider the dilation $\Omega_R = \{(Rx, Ry)|(x, y) \in \Omega\}$ of Ω, and the set $\mathcal{F}(\Omega, R)$ of all primitive lattice points inside Ω_R. The purpose of this paper is the study of simultaneous distribution for lengths of segments connecting the origin and primitive lattice points of $\mathcal{F}(\Omega, R)$. For every $\alpha, \beta \in [0, 1]$, consider the set $P(\alpha, \beta, R)$ of fundamental parallelograms for $\mathbf{Z}^2$ of the shape $t_1 v + t_2 w$ with $t_1, t_2 \in [0, 1]$, defined by points $v = (|v|\cos\theta_v, |v|\sin\theta_v)$, $w = (|w|\cos\theta_w, |w|\sin\theta_w) \in \mathcal{F}(\Omega, R)$, such that $\frac{|v|}{R} \leqslant \alpha r_\Omega(\theta_v)$ and $\frac{|w|}{R} \leqslant \beta r_\Omega(\theta_w)$. We establish an asymptotic formula

$$\frac{\#P(\alpha, \beta, R)}{\#\mathcal{F}(\Omega, R)} = 2\int_0^\beta\int_0^\alpha [\alpha' + \beta' \geqslant 1]d\alpha' d\beta' + O\big(R^{-\frac{1}{3}}\log^{\frac{2}{3}} R\big),$$

where $[\cdot]$ denotes the value of logical expression.

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1. Introduction

Let Ω be a compact convex domain in a plane. Using polar coordinates we write

$$\Omega = \{(r, \varphi)|\ 0 \leqslant r \leqslant r(\varphi) \leqslant 1,\ 0 \leqslant \varphi \leqslant \varphi_0 \leqslant \pi/4\}, \tag{1}$$

where $r = r(\varphi)$ is continuous on $[0, \varphi_0]$. For each real $R \geqslant 1$ we consider the domain Ω_R consisting of points (Rx, Ry) with $(x, y) \in \Omega$. Let $\mathcal{F}(\Omega, R)$ denote the set of primitive integer points of Ω_R. We can write $\mathcal{F}(\Omega, R)$ as

$$\mathcal{F}(\Omega, R) = \left\{ A_j \in \Omega_R \cap \mathbf{Z}^2 \;\middle|\; \begin{array}{l} A_j = (x_j, y_j),\ \text{g.c.d.}(x_j, y_j) = 1, \\ \theta_j = \arctan\left(\dfrac{y_j}{x_j}\right), \\ \theta_{j+1} = \arctan\left(\dfrac{y_{j+1}}{x_{j+1}}\right), \\ \theta_j < \theta_{j+1},\ 1 \leqslant j < N \end{array} \right\}, \tag{2}$$

where N denotes the cardinality of $\mathcal{F}(\Omega, R)$. We say that the points A_j and A_{j+1} are *consecutive points*, and we say that the rays which have the vertex at $(0, 0)$ and pass through A_j and A_{j+1} respectively are *consecutive rays*.

Boca F. P., Cobeli C., Zaharescu A. have investigated in [1] the distribution of normalized gaps

$$\frac{N}{2\pi}(\theta_2 - \theta_1), \ldots, \frac{N}{2\pi}(\theta_N - \theta_{N-1}) \tag{3}$$

between the angles $\theta_1 < \theta_2 < \cdots < \theta_N$. They have obtained an exact formula for this distribution.

A.Ustinov has noted in the paper [2] that the problem of the distribution of values (3) can be easily solved if we know the simultaneous distribution of lengths of segments d_j, d_{j+1} $(1 \leqslant j < N)$, where $d_j = \sqrt{x_j^2 + y_j^2}$. He has established an asymptotic formula for simultaneous distribution of d_j, d_{j+1} $(1 \leqslant j < N)$ when Ω is a triangle:

Theorem 1. *Let* Ω *be a triangle with vertices* $(0, 0), (1, 0), (1, \tan(\varphi_0))$ *and* $r(\varphi) = 1/\cos(\varphi)$. *Let*

$$\Phi(R) = \Phi(R; \varphi_0, \alpha, \beta) = \left\{ (A_j, A_{j+1}) \in \mathcal{F}^2(\Omega, R) \;\middle|\; \begin{array}{l} A_j = (x_j, y_j), \\ d_j \leqslant \alpha R r(\theta_j), \\ d_{j+1} \leqslant \beta R r(\theta_{j+1}), \\ \theta_{j+1} \leqslant \varphi_0, \\ 1 \leqslant j < \#\mathcal{F}(\Omega, R) \end{array} \right\}, \tag{4}$$

$$N_{\varphi_0}(R) = \sum_{j=0}^{\#\mathcal{F}(\Omega,R)-1} [\theta_{j+1} \leqslant \varphi_0]. \tag{5}$$

Then for any $\alpha, \beta \in [0, 1]$, $\varphi_0 \in [0, \pi/4]$, $R \geqslant 2$ *one has*

$$\frac{\#\Phi(R)}{N_{\varphi_0}(R)} = \mathcal{I}(\alpha, \beta) + O\big(R^{-\frac{1}{2}} \log^3 R\big) \text{ as } R \to \infty,$$

where

$$\mathcal{I}(\alpha, \beta) = 2\int_0^\beta\int_0^\alpha [\alpha' + \beta' \geqslant 1] d\alpha' d\beta' = \begin{cases} 0, & \text{if } \alpha + \beta \leqslant 1, \\ (\alpha + \beta - 1)^2, & \text{otherwise.} \end{cases} \tag{6}$$

In the present work we consider a more general situation:

THEOREM 2. *Let the domain* Ω *be given by* (1). *Let* $r(\varphi)$ *be a real function with three continuous derivatives for* $\varphi \in [0, \varphi_0]$. *Suppose that for* $\varphi \in [0, \varphi_0]$ *functions*

$$x(\varphi) = r(\varphi) \cos(\varphi),\ y(\varphi) = r(\varphi) \sin(\varphi), \Psi(\varphi) = x''(\varphi) - 2x'(\varphi) \tan(\varphi)$$

satisfy the following conditions:

1. $x'(\varphi) \leqslant 0, y'(\varphi) \geqslant 0,\ |x'(\varphi)|, y'(\varphi) < \infty$.
2. *The equation* $\Psi(\varphi) = 0$ *has a finite number of solutions in* $[0, \varphi_0]$.
3. *There is no* $\varphi \in [0, \varphi_0]$ *such that* $\Psi(\varphi) = \Psi'(\varphi) = 0$.

Then for any $\alpha, \beta \in [0, 1]$, $\varphi_0 \in [0, \pi/4]$,

$$\frac{\#\Phi(R)}{N_{\varphi_0}(R)} = \mathcal{I}(\alpha, \beta) + O\big(R^{-\frac{1}{3}} \log^{\frac{2}{3}} R\big) \text{ as } R \to \infty,$$

where $\Phi(R), N_{\varphi_0}(R), \mathcal{I}(\alpha, \beta)$ *are given by* (4)–(6).

Remark 1. In particular case when the equation $\Psi(\varphi) = 0$ has no solutions in $[0, \varphi_0]$, the error term is $O(R^{-\frac{1}{2}+\varepsilon})$.

In this paper we always assume that the boundary $\partial\Omega$ of Ω satisfies the conditions of Theorem 2.

2. Formula for $\#\Phi(R)$

STATEMENT 1. *For any consecutive points* $A_j = (x_j, y_j)$, $A_{j+1} = (x_{j+1}, y_{j+1})$ *of* $\mathcal{F}(\Omega, R)$ *the point* $(x_j + x_{j+1}, y_j + y_{j+1})$ *does not lie in* Ω_R.

PROOF. Let $A = (x_j + x_{j+1}, y_j + y_{j+1})$ and $A' = (\frac{x_j+x_{j+1}}{d}, \frac{y_j+y_{j+1}}{d})$, where $d = \text{g.c.d.}(x_j + x_{j+1}, y_j + y_{j+1})$. Suppose that $A \in \Omega_R$. Then $A' \in \Omega_R$ and this means that $A' \in \mathcal{F}(\Omega, R)$. We observe that the point A' lies inside the angle generated by consecutive rays, which pass through points A_j, A_{j+1}. This contradicts (2). □

STATEMENT 2. *If* α *and* β *are non-negative real numbers and* $\alpha + \beta < 1$, *then* $\#\Phi(R) = 0$.

PROOF. Suppose that $\Phi(R)$ is a nonempty set. Then there is a pair $A_j = (x_j, y_j)$, $A_{j+1} = (x_{j+1}, y_{j+1})$ of consecutive elements of $\mathcal{F}(\Omega, R)$ satisfying the relations

$$x_j = \alpha' R r(\theta_j) \cos(\theta_j)\ ,\ x_{j+1} = \beta' R r(\theta_{j+1}) \cos(\theta_{j+1}),$$
$$y_j = \alpha' R r(\theta_j) \sin(\theta_j)\ ,\ y_{j+1} = \beta' R r(\theta_{j+1}) \sin(\theta_{j+1})$$

for some $\alpha' \in [0, \alpha]$ and $\beta' \in [0, \beta]$. The condition $\alpha + \beta < 1$ leads to the conclusion that the point $A = (x_j + x_{j+1}, y_j + y_{j+1})$ lies below the straight line passing through A_j and A_{j+1}. Therefore $A \in \Omega_R$. This contradicts Statement 1. □

STATEMENT 3. *For any consecutive points* $A_j = (x_j, y_j)$ *and* $A_{j+1} = (x_{j+1}, y_{j+1})$ *of* $\mathcal{F}(\Omega, R)$ *we have*

$$x_j y_{j+1} - x_{j+1} y_j = \pm 1.$$

PROOF. We consider the triangle with vertices $(0, 0)$, A_j, A_{j+1}. According to Statement 1 the triangle does not contain elements of the lattice $\mathbf{Z}^2$. So the parallelogram with vertices $(0, 0)$, A_j, A_{j+1}, $(x_j + x_{j+1}, y_j + y_{j+1})$ is a fundamental parallelogram of the lattice $\mathbf{Z}^2$. It is known that the area of this parallelogram is equal to $\mid x_j y_{j+1} - x_{j+1} y_j \mid$ and the determinant of the lattice $\mathbf{Z}^2$ is equal to 1. Hence Statement 3 follows. □

LEMMA 1. Let

$$\mathcal{T}_+(R) = \left\{ (P,P',Q,Q') \,\middle|\, \begin{array}{l} P'Q - PQ' = 1, \\ Q \leqslant Q',\, P \leqslant Q,\, P' \leqslant Q', P' \leqslant Q' \tan(\varphi_0), \\ (Q,P) \in \Omega_{\alpha R},\, (Q',P') \in \Omega_{\beta R},\, (Q+Q',P+P') \notin \Omega_R \end{array} \right\},$$

$$\mathcal{T}_-(R) = \left\{ (P,P',Q,Q') \,\middle|\, \begin{array}{l} P'Q - PQ' = -1, \\ Q \leqslant Q',\, P \leqslant Q,\, P' \leqslant Q', P \leqslant Q \tan(\varphi_0), \\ (Q,P) \in \Omega_{\beta R},\, (Q',P') \in \Omega_{\alpha R},\, (Q+Q',P+P') \notin \Omega_R \end{array} \right\}$$

be sets of 4-tuples $(P, P', Q, Q') \in \mathbf{Z}^4$. Then

$$\#\Phi(R) = \#\mathcal{T}(R) = \#\mathcal{T}_-(R) + \#\mathcal{T}_+(R),$$

where $\mathcal{T}(R) = \mathcal{T}_-(R) \bigcup \mathcal{T}_+(R)$.

PROOF. It follows from definitions of $\mathcal{T}_-(R)$ and $\mathcal{T}_+(R)$ that $\mathcal{T}_-(R) \bigcap \mathcal{T}_+(R) = \varnothing$.

Let $A_j = (x_j, y_j)$, $A_{j+1} = (x_{j+1}, y_{j+1})$ be consecutive points of $\mathcal{F}(\Omega, R)$ and $(A_j, A_{j+1}) \in \Phi(R)$. By (1), (2), (4) and Statement 1, Statement 3, setting

$$(P, P', Q, Q') = \begin{cases} (y_j, y_{j+1}, x_j, x_{j+1}), \text{ if } x_j \leqslant x_{j+1}, \\ (y_{j+1}, y_j, x_{j+1}, x_j), \text{ if } x_j > x_{j+1}, \end{cases}$$

we have $(P, P', Q, Q') \in \mathcal{T}(R)$. Hence $\#\Phi(R) \leqslant \#\mathcal{T}(R)$.

Conversely, putting

$$(y_j, y_{j+1}, x_j, x_{j+1}) = \begin{cases} (P, P', Q, Q'), \text{ if } (P, P', Q, Q') \in \mathcal{T}_+(R), \\ (P', P, Q', Q), \text{ if } (P, P', Q, Q') \in \mathcal{T}_-(R), \end{cases}$$

we observe that $A_j = (x_j, y_j)$, $A_{j+1} = (x_{j+1}, y_{j+1})$ are consecutive points of $\mathcal{F}(\Omega, R)$ and $(A_j, A_{j+1}) \in \Phi(R)$. So $\#\Phi(R) \geqslant \#\mathcal{T}(R)$. The desired conclusion follows. □

Now we are ready to calculate $\#\mathcal{T}_+(R)$. In our context we put $q = Q'$, $u = P'$, $v = Q$. Then Lemma 1 yields the representation

$$\#\mathcal{T}_+(R) = \sum_{q<R} \sum_{u,v=1}^{q} \delta_q(uv-1), \tag{7}$$

where

$$u \leqslant q\tan(\varphi_0),\ (q,u) \in \Omega_{\beta R},\ (vq, uv-1) \in \Omega_{\alpha q R},\ (q(q+v), u(q+v)-1) \notin \Omega_{qR}.$$

Here

$$\delta_q(uv-1) = \begin{cases} 1, \text{ if } q|(uv-1), \\ 0, \text{ otherwise} \end{cases}$$

is the indicator function of divisibility by q.

The domain $\{(u,v)|(vq, uv-1) \in \Omega_{\alpha q R},\ (q(q+v), u(q+v)-1) \notin \Omega_{qR}\}$ is bounded by curves

$$\begin{aligned} \{(u, f_1(u))\} =\ & \{(u,v)|v = \alpha R x(t),\ u = q\tan(t) + \frac{1}{\alpha R x(t)},\ t \in [0,\varphi_0]\}, \\ \{(u, f_2(u))\} =\ & \{(u,v)|v = R x(t) - q,\ u = q\tan(t) + \frac{1}{R x(t)},\ t \in [0,\varphi_0]\}, \end{aligned}$$

so (7) may be expressed as

$$\#\mathcal{T}_+(R) = \sum_{q<R} \sum_{\substack{u\in(0,q\tan(\varphi_0)] \\ (q,u)\in\Omega_{\beta R}}} \sum_{f_2(u)<v\leqslant\min\{q,f_1(u)\}} \delta_q(uv-1).$$

We replace the functions $f_1(u), f_2(u)$ by functions $g_1(u,\alpha), g_2(u)$, which we define by

$$\{(u, g_1(u,\alpha))\} = \{(u,v)|v = \alpha R x(t),\ u = q\tan(t),\ t \in [0,\varphi_0]\}, \tag{8}$$

$$\{(u, g_2(u))\} = \{(u,v)|v = R x(t) - q,\ u = q\tan(t),\ t \in [0,\varphi_0]\}. \tag{9}$$

This replacing gives the error term $O(1)$. Define

$$S(R,\alpha,\beta) = \sum_{q<R} \sum_{u\in I(q,\beta)} \sum_{g_2(u)<v\leqslant\min\{q,g_1(u,\alpha)\}} \delta_q(uv-1), \tag{10}$$

$$I(q,\beta) = \{u \in (0,q] | (q,u) \in \Omega_{\beta R},\ u \leqslant q\tan(\varphi_0)\}.$$

Then it is clear that

$$\#\mathcal{T}_+(R) = S(R,\alpha,\beta) + O(1). \tag{11}$$

We need the following estimates concerning the number of solutions of congruence $uv \equiv 1(\bmod q)$ in the domain $\{(u,v)|u \in (X_1,X_2], v \in (0,f(u)]\}$, obtained by A. Ustinov [3]:

LEMMA 2. Let X_1, X_2, Y be a real non-negative numbers, which do not exceed q. Then

$$\sum_{u\in(X_1,X_2]}\sum_{v\in(0,Y]}\delta_q(uv\pm 1) = \frac{Y}{q}\sum_{\substack{u\in(X_1,X_2]\\ g.c.d.(q,u)=1}} 1 + O(R_1[q]),$$

where

$$R_1[q] \ll \sigma(q)\log^2(q+1)\sqrt{q}.$$

Here $\sigma(q)$ is the number of divisors of q.

LEMMA 3. Let $f(x)$ be a non-negative real function two times differentiable for $[X_1,X_2]$ $(0 \leqslant X_1, X_2 \leqslant q)$, whose derivatives satisfy the condition

$$\frac{1}{A} \ll |f''(x)| \ll \frac{w}{A}$$

for some constants $A > 0, w \geqslant 1$. Then the asymptotic formula

$$\sum_{u\in(X_1,X_2]}\sum_{0<v\leqslant f(u)}\delta_q(uv\pm 1) = \frac{1}{q}\sum_{\substack{u\in(X_1,X_2]\\ g.c.d.(q,u)=1}} f(u) + O(R_2[q,A,X_2-X_1]),$$

is valid. Here

$$R_2[q,A,X] \ll_w \sigma^{\frac{2}{3}}(q)XA^{-\frac{1}{3}} + X^{\varepsilon}(\sqrt{A}+\sqrt{q}).$$

Now we turn to (10). We write $S(R,\alpha,\beta)$ as

$$S(R,\alpha,\beta) = S_1'(R,\alpha,\beta) + S_1''(R,\alpha,\beta) - S_2(R,\alpha,\beta), \tag{12}$$

where

$$S_1'(R,\alpha,\beta)=\sum_{q\leqslant R}\ \sum_{u\in I'(q,\alpha,\beta)}\ \sum_{v\leqslant q}\delta_q(uv-1),$$
$$S_1''(R,\alpha,\beta)=\sum_{q\leqslant R}\ \sum_{u\in I''(q,\alpha,\beta)}\ \sum_{v\leqslant g_1(u,\alpha)}\delta_q(uv-1),$$
$$S_2(R,\alpha,\beta)=\sum_{q\leqslant R}\ \sum_{u\in I(q,\beta)}\ \sum_{v\leqslant g_2(u)}\delta_q(uv-1).$$

Here intervals $I'(q,\alpha,\beta), I''(q,\alpha,\beta)$ are defined by

$$I'(q,\alpha,\beta)=\{u\in I(q,\beta)|g_2(u)<q\leqslant g_1(u,\alpha)\},$$
$$I''(q,\alpha,\beta)=\{u\in I(q,\beta)|g_2(u)<g_1(u,\alpha)\leqslant q\}.$$

According to Lemma 2 and the bound $\sum\limits_{q<R}\sigma(q)\ll R\log R,$ we have

$$S_1'(R,\alpha,\beta)=\sum_{q<R}\frac{1}{q}\sum_{\substack{u\in I'(q,\alpha,\beta)\\ \text{g.c.d.}(q,u)=1}} q+O\big(R^{\frac{3}{2}}\log^3 R\big). \tag{13}$$

To estimate two other sums $S_1''(R,\alpha,\beta)$ and $S_2(R,\alpha,\beta)$ we must consider the fact that for fixed natural q the second derivatives of $g_1(u,\alpha)$ and $g_2(u)$ lie within closed intervals containing zero.

Lemma 4. For $S_1''(R,\alpha,\beta)$ and $S_2(R,\alpha,\beta)$ it follows that

$$S_1''(R,\alpha,\beta)=\sum_{q<R}\frac{1}{q}\sum_{\substack{u\in I''(q,\alpha,\beta)\\ g.c.d.(q,u)=1}} g_1(u,\alpha)+O\big(R^{2-\frac{1}{3}}\log^{\frac{2}{3}}R\big),\ R\to\infty,$$
$$S_2(R,\alpha,\beta)=\sum_{q<R}\frac{1}{q}\sum_{\substack{u\in I(q,\beta)\\ g.c.d.(q,u)=1}} g_2(u,\alpha)+O\big(R^{2-\frac{1}{3}}\log^{\frac{2}{3}}R\big),\ R\to\infty.$$

Proof. We will prove the lemma for $S_1''(R,\alpha,\beta)$ only as we can easily adapt the proof below for the sum $S_2(R,\alpha,\beta)$. By (8) we conclude that

$$g_1''(u,\alpha)=\frac{\alpha R}{q^2}\cos^4(t)\Psi(t),\ t=\arctan\left(\frac{u}{q}\right),$$

where the function $\Psi(t)$ is defined in Theorem 2. This function vanishes at a finite number of points. Without loss of generality we suppose that the equation $\Psi(t)=0$ has only one solution which we denote by t_0. We denote the corresponding value of the variable u by u_0.

If $t_0 \notin (0,\varphi_0]$, application of Lemma 2 (with $A=\frac{q^2}{R}$) to inner sums over u, v of the sum $S_1''(R,\alpha,\beta)$ gives

$$S_1''(R,\alpha,\beta)=\sum_{q<R}\frac{1}{q}\sum_{\substack{u\in I''(q,\alpha,\beta)\\ \text{g.c.d.}(q,u)=1}} g_1(u,\alpha)+O\big(R^{\frac{3}{2}+\varepsilon}\big), \tag{14}$$

since

$$\sum_{q<R} R_2\left[q,\frac{q^2}{R},q\right]\ll R^{\frac{3}{2}+\varepsilon}.$$

For this case Lemma 4 is proved.

Let $t_0\in(0,\varphi_0]$. Put

$$u_{max}=\max_{u\in I''(q,\alpha,\beta)}\{u\},\ \ k=[\log_2(u_{max})],$$
$$S(q,J)=\sum_{u\in J\cap I''(q,\alpha,\beta)}\ \sum_{v\leqslant g_1(u,\alpha)}\delta_q(uv-1),$$

where J is the interval.

Let $\Delta\in(0,1)$ be a real number to be specified later. We divide the interval $(0,u_{max}]$ into subintervals $J^{(0)},\ J_i\ (1\leqslant i\leqslant k+1):$

$$J^{(0)}=(u_0-\Delta q,u_0+\Delta q]\bigcap I''(q,\alpha,\beta),$$

$$J_i=\begin{cases}(2^{i-1},2^i]\bigcap I''(q,\alpha,\beta), & \text{if } J^{(0)}=\varnothing,\\ (2^{i-1},2^i]\bigcap I''(q,\alpha,\beta)\backslash(J^{(0)}\bigcap(2^{i-1},2^i]), & \text{otherwise}\end{cases}$$

(some of these intervals may be empty). For the above reasons we write the sum $S_1''(R,\alpha,\beta)$ as

$$S_1''(R,\alpha,\beta)=\sum_{q<R}\ \sum_{1\leqslant i\leqslant k+1}S(q,J_i)+\sum_{q<R}S(q,J^{(0)}).$$

The set $\{J_i\}_{i=1}^{k+1}$ has subintervals for which intersections with $J^{(0)}$ are non-empty. We denote these ones as $J^{(1)}, J^{(2)}$. We apply Lemma 2 to $S(q, J^{(0)})$, replacing $g_1(u, \alpha)$ with the constant $g_1(u_0 - \Delta q, \alpha)$. As $\mid g_1'(u, \alpha) \mid \ll \frac{R}{q}$. Then this replacing gives the error term $O(R\Delta^2)$. To other sums we apply Lemma 3 with

$$A = \frac{q^2}{R} \cdot \begin{cases} \Delta^{-1} & \text{— for } J^{(1)}, J^{(2)}, \\ q \cdot 2^{-i} & \text{— for } J_i, \text{ not coinciding with } J^{(1)}, J^{(2)}. \end{cases}$$

We obtain

$$S_1''(R, \alpha, \beta) = \sum_{q<R} \frac{1}{q} \sum_{\substack{u \in I''(q,\alpha,\beta) \\ \text{g.c.d.}(q,u)=1}} g_1(u, \alpha) + O(R''),$$

where

$$R'' = R_1'' + R_2'', \tag{15}$$

$$R_1'' \ll \sum_{q<R} R_1[q] + \sum_{q<R} R\Delta^2, \tag{16}$$

$$R_2'' \ll \sum_{q<R} \sum_{i<\log q} R_2\left[q, \frac{q^3}{R \cdot 2^i}, 2^i\right] + \sum_{q<R} R_2\left[q, \frac{q^2}{R\Delta}, \Delta\right]. \tag{17}$$

The sums on the right of (16) may be estimated by $R^{\frac{3}{2}} \log^3 R$ and $R^2\Delta^2$ respectively. Using Lemma 3 we represent the sum in the right hand side of (17) as a sum of three terms $\Sigma_1, \Sigma_2, \Sigma_3$:

$$\Sigma_1 = \sum_{q<R} \sum_{j<\log q} \sigma^{\frac{2}{3}}(q) 2^j \left(\frac{R \cdot 2^j}{q^3}\right)^{\frac{1}{3}} + \sum_{q<R} \sigma^{\frac{2}{3}}(q) \Delta \left(\frac{R\Delta}{q^2}\right)^{\frac{1}{3}},$$

$$\Sigma_2 = \sum_{q<R} \sum_{j<\log q} 2^{\varepsilon j} \sqrt{\frac{q^3}{R \cdot 2^j}} + \sum_{q<R} \Delta^{\varepsilon} \sqrt{\frac{q^2}{R\Delta}},$$

$$\Sigma_3 = \sum_{q<R} \sum_{j<\log q} 2^{\varepsilon j} \sqrt{q} + \sum_{q<R} \Delta^{\varepsilon} \sqrt{q}.$$

In Σ_1 we see that the first term dominates, so we may omit the second term. Therefore

$$\Sigma_1 \ll R^{\frac{1}{3}} \sum_{q<R} \sigma^{\frac{2}{3}}(q) q^{-1} \sum_{j<\log q} 2^{\frac{4}{3}j} \ll R^{\frac{1}{3}} \sum_{q<R} \sigma^{\frac{2}{3}}(q) q^{\frac{1}{3}} \ll$$
$$\ll R^{\frac{1}{3}} \Big(\sum_{q<R} \sigma(q) \Big)^{\frac{2}{3}} \Big(\sum_{q<R} q \Big)^{\frac{1}{3}} \ll R^{1+\frac{2}{3}} \log^{\frac{2}{3}} R.$$

Also in Σ_2 the second term may be omitted and in the first term the sum over q, j is restricted to pairs with $\frac{q^3}{R \cdot 2^j} \geqslant 1$. In all intervals J_j $(1 \leqslant j \leqslant k+1)$ we have $g''(u) \gg \frac{R \cdot \Delta}{q^2}$, then $\frac{R \cdot 2^j}{q^3} \gg \frac{R \cdot \Delta}{q^2}$. So

$$\Sigma_2 \ll \sum_{q<R} \sum_{j<\log q} 2^{\varepsilon j} \sqrt{\frac{q^3}{R \cdot 2^j}} \ll \sum_{q<R} \sum_{j<\log q} 2^{\varepsilon j} \sqrt{\frac{q^2}{R \cdot \Delta}} \ll R^{-\frac{1}{2}} \Delta^{-\frac{1}{2}} \sum_{q<R} q^{1+\varepsilon} \ll R^{\frac{3}{2}+\varepsilon} \Delta^{-\frac{1}{2}}.$$

As $\Sigma_3 \ll R^{\frac{3}{2}+\varepsilon}$ we have $R_2'' \ll R^{1+\frac{2}{3}} \log^{\frac{2}{3}} R + R^{\frac{3}{2}+\varepsilon} \Delta^{-\frac{1}{2}}$. From (15) we get an estimate of the error term for $S_1''(R, \alpha, \beta)$:

$$R'' \ll R^{\frac{3}{2}} \log^3 R + R^2 \Delta^2 + R^{1+\frac{2}{3}} \log^{\frac{2}{3}} R + R^{\frac{3}{2}+\varepsilon} \Delta^{-\frac{1}{2}}.$$

Now we have to choose the parameter Δ in such a way that $R^2 \Delta^2 \asymp R^{\frac{3}{2}+\varepsilon} \Delta^{-\frac{1}{2}}$. Then we get $\Delta = R^{-\frac{1-2\varepsilon}{5}}$. This gives the result of Lemma 4 for $S_1''(R, \alpha, \beta)$. $\square$

Let $F(u, q, \alpha)$ denote the function $F(u, q, \alpha) = \min\{q, g_1(u, \alpha)\} - g_2(u)$. The relations (12), (13) and Lemma (4) give

$$S(R, \alpha, \beta) = \sum_{q<R} \frac{1}{q} \sum_{\substack{u \in I(q,\beta) \\ \text{g.c.d.}(q,u)=1}} F(u, q, \alpha) + O\big(R^{2-\frac{1}{3}} \log^{\frac{2}{3}} R\big). \tag{18}$$

By (8) and (9) we have

$$\frac{1}{q} \sum_{\substack{u \in I(q,\beta) \\ \text{g.c.d.}(q,u)=1}} F(u, q, \alpha) = \frac{1}{q} \sum_{\delta | q} \mu(\delta) \sum_{\substack{u \in I(q,\beta) \\ \delta | u}} F(u, q, \alpha).$$

From the identity

$$\sum_{\substack{u\in I(q,\beta)\\ \delta|u}} F(u,q,\alpha)=\frac{1}{\delta}\int\limits_0^q [u\in I(q,\beta)]F(u,q,\alpha)du+O(q)$$

and relations (8), (9) we have

$$\int\limits_0^q [u\in I(q,\beta)]F(u,q,\alpha)du=$$
$$=q^2\int\limits_0^1\int\limits_0^1\left[t\leqslant t_q, t\leqslant\varphi_0, \frac{R}{q}x(t)-1<v\leqslant\alpha\frac{R}{q}x(t)\right]dvd\tan(t),$$

where the value t_q is given by $q=\beta Rx(t_q)$. Now the main term in (18), which we denote as $S^*(R,\alpha,\beta)$, can be written in the form

$$S^*(R,\alpha,\beta)=\sum_{\delta<R}\mu(\delta)S'\left(\frac{R}{\delta}\right), \tag{19}$$

where

$$S'(R)=\sum_{q<R} q\int\limits_0^1\int\limits_0^1\left[t\leqslant t_q, t\leqslant\varphi_0, \frac{R}{q}x(t)-1<v\leqslant\alpha\frac{R}{q}x(t)\right]dvd\tan(t).$$

Here we take into account that the remainder $\sum_{q<R}\frac{1}{q}\sum_{\delta|q}q\ll R\log R$ is less than the error term in (18).

To evaluate $S'(R)$ we change the order of the summation and the integration, then we replace the sum with the integral, taking into account that the error term is of order R. Thus we have

$$S'(R)=R^2\frac{1}{2}\int\limits_0^{\varphi_0}x^2(t)d\tan(t)\int\limits_0^1\left[\frac{1}{\beta}-1<v<\frac{\alpha}{1-\alpha}\right]\times$$
$$\times\left(\min\left\{\frac{\alpha^2}{v^2},\beta^2\right\}-\frac{1}{(v+1)^2}\right)dv+O(R).$$

Applying Statement 2, we obtain $S'(R) = R^2 S_\Omega \cdot I(\alpha,\beta) + O(R)$, where $I(\alpha,\beta)$ is defined by the formula

$$I(\alpha,\beta) = [\alpha+\beta \geqslant 1]\cdot[\beta \geqslant 1/2]\cdot \begin{cases} (\alpha+\beta-1)^2, & \text{if } \alpha \leqslant 1/2,\\ 2(\beta-1/2)^2-(\alpha-\beta)^2, & \text{if } 1/2<\alpha\leqslant\beta,\\ 2(\beta-1/2)^2, & \text{if } \alpha>\beta \end{cases}$$

and S_Ω denotes the area of the domain Ω. Combining the above result with (10), (11), (18), (19) we get an asymptotic formula for $\#\mathcal{T}_+(R)$:

$$\#\mathcal{T}_+(R) = \frac{R^2}{\zeta(2)} S_\Omega \cdot I(\alpha,\beta) + O\big(R^{2-\frac{1}{3}}\log^{\frac{2}{3}} R\big).$$

To prove the asymptotic formula for $\#\mathcal{T}_-(R)$, we proceed similarly to (7). We deduce

$$\#\mathcal{T}_-(R) = \sum_{q<R}\sum_{u,v=1}^{q}\delta_q(uv+1),$$

where

$$u \leqslant q\tan(\varphi_0) - 1/v,\ \ (q,u)\in\Omega_{\alpha R},\ \ (vq, uv-1)\in\Omega_{\beta q R},$$
$$(q(q+v), u(q+v)-1) \notin \Omega_{qR}.$$

According to (8)–(11) we have $\mathcal{T}_-(R) = S(R,\beta,\alpha) + O(R)$. Then

$$\#\mathcal{T}_-(R) = \frac{R^2}{\zeta(2)} S_\Omega \cdot I(\beta,\alpha) + O\big(R^{2-\frac{1}{3}}\log^{\frac{2}{3}} R\big).$$

At last we note, that $I(\alpha,\beta) + I(\beta,\alpha) = \mathcal{I}(\alpha,\beta)$; and by Lemma 1 for $\#\Phi(R)$ we obtain the asymptotics

$$\#\Phi(R) = \frac{R^2}{\zeta(2)} S_\Omega \cdot \mathcal{I}(\alpha,\beta) + O\big(R^{2-\frac{1}{3}}\log^{\frac{2}{3}} R\big). \tag{20}$$

3. Proof of Theorem 2

Theorem 2 follows from (20) and the asymptotic formula for $\#\mathcal{F}(\Omega, R)$:

$$\#\mathcal{F}(\Omega, R) = \sum_{\substack{(x,y)\in F(\Omega,R)\\ \text{g.c.d.}(x,y)=1}} 1 = \sum_{(x,y)\in F(\Omega,R)} \sum_{\delta|\text{g.c.d.}(x,y)} \mu(\delta) = \sum_{\delta<R} \mu(\delta) \sum_{(x,y)\in F(\Omega,R/\delta)} 1 =$$

$$= R^2 \cdot S_\Omega \sum_{\delta<R} \frac{\mu(\delta)}{\delta^2} + O(R\log(R)) = \frac{R^2}{\zeta(2)} \cdot S_\Omega + O(R\log(R)).$$

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