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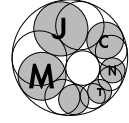
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Some identities of Ramanujan type

Yuri Nesterenko (Moscow)

*Dedicated to the memory
of Theodor Schneider.*

Abstract: Identities of quasi-modular type are proved for some functions that look like Eisenstein series. They generalize some classical identities of S. Ramanujan and E. Grosswald.

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The Eisenstein series $E_{2k}(\tau)$, $k \geq 1$, can be defined as

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \quad q = e^{2\pi i \tau}, \quad \tau \in \mathbb{C}, \operatorname{Im} \tau > 0,$$

where B_{2k} are Bernoulli numbers and $\sigma_m(n) = \sum_{d|n} d^m$. This notation will be used for any integer m . It is a classical result that the Eisenstein series for $k \geq 4$ can be expressed as polynomials in $E_4(\tau)$, $E_6(\tau)$ with rational coefficients, i. e. for any $k \geq 4$ there exists a polynomial $A_k \in \mathbb{Q}[x, y]$ such that $E_{2k}(\tau) = A_k(E_4(\tau), E_6(\tau))$. Denote $\delta = \frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq}$. Due to Ramanujan, see [10], [6], Ch. X, Theorem 5.3,

we have the following system of differential equations

$$\begin{aligned}\delta E_2 &= \frac{1}{12}(E_2^2 - E_4), \\ \delta E_4 &= \frac{1}{3}(E_2 E_4 - E_6), \\ \delta E_6 &= \frac{1}{2}(E_2 E_6 - E_4^2).\end{aligned}$$

Eisenstein series have period 1 and

$$\begin{aligned}E_{2k}(-1/\tau) &= \tau^{2k} E_{2k}(\tau), \quad k \geq 2, \\ E_2(-1/\tau) &= \tau^2 E_2(\tau) + \frac{6\tau}{\pi i},\end{aligned}$$

see [6], Ch. I, X. It is well known that the functions $e^{\pi i \tau}$, $E_2(\tau)$, $E_4(\tau)$, $E_6(\tau)$ are algebraically independent over \mathbb{C} but at any imaginary quadratic ξ the field generated over \mathbb{Q} by the three numbers $E_2(\xi)$, $E_4(\xi)$, $E_6(\xi)$ has transcendence degree 2. This degeneracy allowed with the choice $\xi = i$ to prove the algebraic independence of π , e^π and $\Gamma(1/4)$, see [8].

In this paper we study some algebraically independent functions connected to Eisenstein series and having degeneracy of transcendence degree of the field generated by their values at some points.

For every pair of integers u, v , $0 \leq u < v$, let us define functions $g_{u,v}(\tau)$ depending on complex variable τ , $\text{Im } \tau > 0$, as follows:

$$g_{u,v}(\tau) = \sum_{n=1}^{\infty} n^u \sigma_{-v}(n) q^n, \quad q = e^{2\pi i \tau}.$$

For odd integers $v > 1$ this set of functions satisfies the following differential equations:

$$\begin{aligned}\delta g_{u,v}(\tau) &= g_{u+1,v}(\tau), \quad 0 \leq u < v-1, \\ \delta g_{v-1,v}(\tau) &= B_{v+1} \frac{1 - A_{v+1}(E_4(\tau), E_6(\tau))}{2v+2}.\end{aligned}$$

The latter equation is based on the equality $n^v \sigma_{-v}(n) = \sigma_v(n)$.

In the case $v = 1$ we have

$$\delta g_{0,1}(\tau) = \frac{1}{24}(1 - E_2(\tau)).$$

In 2010 P. Kozlov ¹⁾ used this system of differential equations to prove that for any fixed m all the functions

$$E_2(\tau), E_4(\tau), E_6(\tau), g_{u,v}(\tau), \quad 0 \leq u < v \leq m,$$

are algebraically independent over $\mathbb{C}(q)$.

The main result of the current paper is the following

THEOREM 1. *Let τ be a complex number, $\text{Im } \tau > 0$. Then for any pair of integers u, v with $0 \leq u < v$ and odd $v \geq 3$, the following identity holds:*

$$\begin{aligned} g_{u,v}(-1/\tau) &= \\ &= (-1)^{\frac{v-1}{2}-u} \sum_{\ell=0}^u \frac{u!}{\ell!} \binom{v-\ell-1}{v-u-1} (-i\tau)^{1-v+u+\ell} (2\pi)^{-u+\ell} g_{\ell,v}(\tau) + R(\tau, u, v), \quad (1) \end{aligned}$$

where

$$\begin{aligned} 2R(\tau, u, v) &= (-1)^{\frac{v-1}{2}} \frac{(2\pi)^{v-u} u!}{(v+1)!} B_{v+1} (-i\tau)^{1+u} + \\ &+ \frac{(2\pi)^{v-u}}{(v-u)!} \sum_{\frac{u+1}{2} \leq \ell \leq \frac{v+1}{2}} (-1)^{\frac{v-1}{2}+u-\ell} \binom{v-u}{v+1-2\ell} \frac{B_{2\ell} B_{v+1-2\ell}}{2\ell} (-i\tau)^{u+1-2\ell} + \\ &+ (-1)^{\frac{v-1}{2}+u} \zeta(v) \frac{(2\pi)^{-u} (v-1)!}{(v-u-1)!} (-i\tau)^{1+u-v} - \begin{cases} 0, & \text{if } u > 0; \\ \zeta(v), & \text{if } u = 0; \end{cases} \end{aligned}$$

B_{2m} with $m \geq 1$ are Bernoulli numbers and $B_0 = 1$.

In particular, for $u = 0$ and $u = 1$ we derive the following corollary.

COROLLARY 1. *Let τ be a complex number, $\text{Im } \tau > 0$. Then, for any odd integer $v \geq 3$, the following identities hold:*

¹⁾ The corresponding paper is in preparation.

$$g_{0,v}(-1/\tau) = \tau^{1-v} g_{0,v}(\tau) + \frac{\zeta(v)}{2} (\tau^{1-v} - 1) + \\ + i(-1)^{\frac{v+1}{2}} \frac{(2\pi)^v}{2 \cdot (v+1)!} \sum_{\ell=0}^{\frac{v+1}{2}} \binom{v+1}{2\ell} B_{2\ell} B_{v+1-2\ell} \tau^{1-2\ell}, \quad (2)$$

and

$$g_{1,v}(-1/\tau) = \tau^{3-v} \left(g_{1,v}(\tau) + \frac{1-v}{2\pi i \tau} \cdot g_{0,v}(\tau) \right) + \zeta(v) \frac{1-v}{4\pi i} \tau^{2-v} + \\ + (-1)^{\frac{v-1}{2}} \frac{(2\pi)^{v-1}}{2 \cdot (v+1)!} \sum_{\ell=0}^{\frac{v+1}{2}} \binom{v+1}{2\ell} (2\ell-1) B_{2\ell} B_{v+1-2\ell} \tau^{2-2\ell}. \quad (3)$$

The identities (2) and (3) were proved first by E. Grosswald, see [3]. Let γ and β be positive real numbers such that $\gamma\beta = \pi^2$. Denote $\tau = i\gamma/\pi$. Then $-1/\tau = i\beta/\pi$ and with this choice the identity (2) coincides with the famous Ramanujan identity, see [2], Ch. 14, Entry 21(i). The identity (3) can be proved independently of Theorem 1 by differentiating (2), see [1], where the corresponding special case is derived in this way from the identity of Ramanujan.

Identities (2) and (3) for $\tau = i$ give

$$g_{0,3}(i) = \frac{7\pi^3}{360} - \frac{\zeta(3)}{2}, \quad g_{0,7}(i) = \frac{19\pi^7}{113400} - \frac{\zeta(7)}{2},$$

and

$$g_{1,5}(i) + \frac{1}{\pi} g_{0,5}(i) = \frac{13\pi^4}{7560} - \frac{\zeta(5)}{2\pi}.$$

It is interesting that the experimentally discovered list of linear relations for zeta-values and q -series from [9] does not contain the last one. Note that formulas for $\zeta(4n+3)$, $n \geq 0$, were proved first in 1901 by M. Lerch, see [7]. Taking $v = 3$, $u = 2$ and $\tau = i$ in Theorem 1 we derive

$$\zeta(3) = \frac{2}{45} \pi^3 - 8\pi^2 g_{2,3}(i) - 4\pi g_{1,3}(i) - 2g_{0,3}(i).$$

The equality (2) implies

$$g_{0,4n+3}(i) \in \mathbb{Q}(\pi, \zeta(4n+3)).$$

COROLLARY 2. Let $v > 1$ be an odd integer and ρ be a cubic root of unity satisfying $\text{Im } \rho > 0$. Denote by L the field generated over \mathbb{Q} by the numbers $\pi, \zeta(v)$ and the values $g_{u,v}(\rho)$ for any $w \equiv \frac{v-1}{2} \pmod{3}$, $0 \leq w < v$. Then all the numbers

$$g_{u,v}(\rho), \quad 0 \leq u < v,$$

are algebraic over the field L .

PROOF. To prove this Corollary let us note that $-\frac{1}{\rho} = \rho + 1$ and $g_{u,v}(-1/\rho) = g_{u,v}(\rho)$. The coefficient of $g_{u,v}(\tau)$ on the right-hand side of (1) is equal to τ^{1-v+2u} . The Corollary can be proved by induction on u . If $u \equiv \frac{v-1}{2} \pmod{3}$, $u \geq 0$, it is valid by the definition of the field L . If $u \not\equiv \frac{v-1}{2} \pmod{3}$, $u \geq 0$, the coefficient of $g_{u,v}(\rho)$ on the right-hand side of (1) with $\tau = \rho$ is distinct from 1. That is why $g_{u,v}(\rho)$ is a linear combination of $g_{w,v}(\rho)$, $w < u$, $\zeta(v)$ and 1 with coefficients algebraic over the field $\mathbb{Q}(\pi)$. \square

To prove Theorem 1 we use the method proposed by Grosswald in [3].

LEMMA 1. For any integers u, v , $0 \leq u < v$, and $\tau \in \mathbb{C}$, $\text{Im } \tau > 0$, the following representation holds:

$$g_{u,v}(\tau) = \frac{1}{2\pi i} \int_C \Gamma(s+u)\zeta(s)\zeta(s+v)(-2\pi i\tau)^{-s-u} ds, \quad (4)$$

where the path of integration C is a straight line $\sigma = \gamma > 1$ from $\gamma - i\infty$ to $\gamma + i\infty$ and we use the branch of the logarithmic function $\ln(-2\pi i\tau)$ under the integral satisfying $|\arg(-i\tau)| < \pi/2$. With this condition, the integral converges absolutely.

PROOF. To prove the absolute convergence, we denote $\sigma = \text{Re } s$ and $t = \text{Im } s$. The functions $\zeta(s)$ and $\zeta(s+u)$ are bounded on the line $\sigma = \gamma$. For $\tau = x + iy$, $y > 0$, we have $\arg \tau = \arctan \frac{x}{y} = \gamma$, $0 < \gamma < \pi$, whence

$$\left| (-2\pi i\tau)^{-s-u} \right| = e^{\text{Re}(\ln(-2\pi i\tau)(-s-u))} = e^{(-\gamma-u)\ln|2\pi\tau| + t(\gamma-\pi/2)} = c_1 e^{t(\gamma-\pi/2)},$$

where c_1 does not depend on t .

We shall use the following formula valid for all sufficiently large $|t|$:

$$\Gamma(\sigma + it) = O(|t|^{\sigma-1/2})e^{-\frac{\pi}{2}|t|}, \quad a \leq \sigma \leq b, \quad (5)$$

see for example [5], § 218. The constant in $O(\cdot)$ depends only on a and b .

Finally, the integrand in (4) is bounded on the path C by

$$c_2 |t|^{\gamma+u-1/2} e^{t(\gamma-\pi/2)-|t|\frac{\pi}{2}} < e^{-|t|\lambda},$$

where $2\lambda = \min(\gamma, \pi - \gamma) > 0$, c_2 depends only on γ and $|t|$ is sufficiently large. This bound implies the convergence of the integral (4).

We shall use the two classical identities

$$I = \frac{1}{2\pi i} \int_C z^{-s} \Gamma(s) ds = e^{-z}, \quad \operatorname{Re} z > 0,$$

see for example [4], (1.11), and

$$\zeta(s)\zeta(s+v) = \sum_{n=1}^{\infty} \frac{\sigma_{-v}(n)}{n^s}, \quad s \in \mathbb{C}, \quad \operatorname{Re} s > 1,$$

see [14], Ch. 1, § 3, (1). The series converges uniformly on the path C . These identities imply

$$\begin{aligned} & \frac{1}{2\pi i} \int_C \Gamma(s+u)\zeta(s)\zeta(s+v)(-2\pi i\tau)^{-s-u} ds = \\ &= \sum_{n=1}^{\infty} n^u \sigma_{-v}(n) \frac{1}{2\pi i} \int_C \Gamma(s+u)(-2\pi i\tau n)^{-s-u} ds = \\ &= \sum_{n=1}^{\infty} n^u \sigma_{-v}(n) \frac{1}{2\pi i} \int_{C_1} \Gamma(\xi)(-2\pi i\tau n)^{-\xi} d\xi = \sum_{n=1}^{\infty} n^u \sigma_{-v}(n) e^{2\pi i n \tau} = g_{u,v}(\tau). \end{aligned}$$

We used the change of variables $\xi = s + u$ in the calculations and the path C_1 is derived by the corresponding translation of the path C . \square

PROOF OF THEOREM 1. Denote $\rho = -1/\tau$, choose a large positive number N , and define

$$I = \frac{1}{2\pi i} \int_{ABCD} \Gamma(s+u)\zeta(s)\zeta(s+v)(-2\pi i\rho)^{-s-u} ds, \quad (6)$$

where $ABCD$ is the rectangular path of integration with vertices $A = 3/2 - iN$, $B = 3/2 + iN$, $C = -v - 1/2 + iN$, $D = -v - 1/2 - iN$.

Due to Lemma 1 we have

$$\frac{1}{2\pi i} \int_{AB} \Gamma(s+u)\zeta(s)\zeta(s+v)(-2\pi i\rho)^{-s-u} ds = g_{u,v}(\rho) + o(1),$$

as N tends to infinity.

On the path BC we have $s = \sigma + iN$, $-v - 1/2 \leq \sigma \leq 3/2$, and

$$|\zeta(s)| \leq c_3 N^{v+1}, \quad |\zeta(s+u)| \leq c_3 N^{v-u+1},$$

see [5], § 228. Due to (5) this implies

$$\frac{1}{2\pi i} \int_{BC} \Gamma(s+u)\zeta(s)\zeta(s+v)(-2\pi i\rho)^{-s-u} ds = o(1),$$

as N tends to infinity. The same bound is valid on the path DA .

Making the change of variable in integration along the path DC from s to $1 - v - s$ we get

$$\begin{aligned} J &= \frac{1}{2\pi i} \int_{CD} \Gamma(s+u)\zeta(s)\zeta(s+v)(-2\pi i\rho)^{-s-u} ds = \\ &= -\frac{1}{2\pi i} \int_{AB} \Phi(1-v-s)(-i\rho)^{-1+v-u+s} ds, \end{aligned}$$

where

$$\Phi(s) = \Gamma(s+u)\zeta(s)\zeta(s+v)(2\pi)^{-s-u}.$$

Due to the functional equation for the Riemann zeta-function

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s)\zeta(1-s),$$

see [14], Ch. 2, and the formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s},$$

see [15], § 12.14, the following identity holds:

$$\Phi(1-v-s) = (-1)^{\frac{v-1}{2}-u} \zeta(s) \zeta(s+v) (2\pi)^{-s-u} \Gamma(s) (s+v-u) \dots (s+v-1).$$

With some complex coefficients a_j , $0 \leq j \leq u$, we have

$$(s+v-u) \dots (s+v-1) = \sum_{\ell=0}^u a_{\ell} s(s+1) \dots (s+\ell-1).$$

To find these coefficients we note that

$$\begin{aligned} R(s) &= \frac{(s+v-u) \dots (s+v-1)}{s(s+1) \dots (s+j)} = \\ &= \sum_{\ell=0}^{j-1} \frac{a_{\ell}}{(s+\ell) \dots (s+j)} + \frac{a_j}{s+j} + \sum_{\ell=j+1}^u a_{\ell} (s+j+1) \dots (s+\ell-1) \end{aligned}$$

and

$$\begin{aligned} a_j &= \frac{1}{2\pi i} \int_{|s|=j+1} R(s) ds = \sum_{\ell=0}^j \operatorname{Res}_{s=-\ell} R(s) = \sum_{\ell=0}^{\min(j, v-u-1)} (-1)^{\ell} \frac{(v-u-\ell) \dots (v-\ell-1)}{(j-\ell)! \ell!} = \\ &= \frac{u!}{j!} \sum_{\ell \geq 0} (-1)^{\ell} \binom{v-\ell-1}{u} \binom{j}{\ell} = \frac{u!}{j!} \binom{v-j-1}{v-u-1}. \end{aligned}$$

The binomial coefficients in the latter sum have to be interpreted as 0, if the lower index is larger than the upper one. For the latter equality see (5) in [12], § 1.3.

Since $(-i\rho)^{-1} = -i\tau$, we get

$$\begin{aligned} J &= (-1)^{\frac{v+1}{2}-u} \sum_{\ell=0}^u a_{\ell} (-i\tau)^{1-v+u+\ell} (2\pi)^{-u+\ell} \times \\ &\times \frac{1}{2\pi i} \int_{AB} \Gamma(s+\ell) \zeta(s) \zeta(s+v) (-2\pi i\tau)^{-s-\ell} ds = \end{aligned}$$

$$= (-1)^{\frac{v+1}{2}-u} \sum_{\ell=0}^u a_{\ell}(-i\tau)^{1-v+u+\ell} (2\pi)^{-u+\ell} g_{\ell,v}(\tau) + o(1)$$

as $N \rightarrow \infty$.

The integrand in (6) can have poles inside the path $ABCD$ only at integer points $s = -k$, $-1 \leq k \leq v$. Hence, letting N tend to infinity, we derive

$$\begin{aligned} g_{u,v}(\rho) + (-1)^{\frac{v+1}{2}-u} \sum_{\ell=0}^u a_{\ell}(-i\tau)^{1-v+u+\ell} (2\pi)^{-u+\ell} g_{\ell,v}(\tau) &= \\ = \sum_{k=-1}^v \operatorname{Res}_{s=-k} (\Gamma(s+u)\zeta(s)\zeta(s+v)(-2\pi i\rho)^{-s-u}). \end{aligned}$$

To finish the proof of the Theorem it remains to calculate the sum of the residues

$$r_k = \operatorname{Res}_{s=-k} (\Gamma(s+u)\zeta(s)\zeta(s+v)(-2\pi i\rho)^{-s-u}), \quad -1 \leq k \leq v.$$

For the residue at the point $s = 1$, that is for $k = -1$, we have

$$r_{-1} = u!\zeta(v+1)(-2\pi i\rho)^{-1-u}.$$

For the point $s = 0$ we have

$$r_0 = \begin{cases} 0, & \text{if } u > 0; \\ \zeta(0)\zeta(v), & \text{if } u = 0. \end{cases}$$

Due to the fact that at the point $s = 1 - v$

$$\Gamma(s+u) = \frac{(-1)^u}{(v-u-1)!} \cdot \frac{1}{s-1+v} + O(1), \quad \zeta(s+v) = \frac{1}{s-1+v} + O(1),$$

and due to the functional equation

$$\zeta(s) = (-1)^{\frac{v-1}{2}} 2^{-v} \pi^{-v+1} (v-1)!\zeta(v)(s-1+v) + O((s-1+v)^2),$$

we have

$$r_{v-1} = (-1)^{\frac{v-1}{2}-u} 2^{-1-u} \pi^{-u} \zeta(v) \frac{(v-1)!}{(v-u-1)!} (-i\rho)^{-1+v-u}.$$

The function $\zeta(s)$ has zeros at the points $-2, -4, \dots$; hence the other poles of the integrand in (6) can be only at the points $s = -k = -2\ell + 1, 1 \leq \ell \leq \frac{v+1}{2}$. Moreover, $\Gamma(s+u)$ has poles only at $s = -k \leq -u$. We have

$$r_{2\ell-1} = \frac{(-1)^{u+1}}{(2\ell-u-1)!} \zeta(1-2\ell)\zeta(v+1-2\ell)(-2\pi i \rho)^{2\ell-1-u}, \quad u \leq 2\ell-1 \leq v.$$

Collecting all these results we derive

$$\begin{aligned} R(\tau, u, v) &= u! \zeta(v+1) (-2\pi i \rho)^{-1-u} + \\ &+ (-1)^{\frac{v-1}{2}-u} 2^{-1-u} \pi^{-u} \zeta(v) \frac{(v-1)!}{(v-u-1)!} (-i\rho)^{-1+v-u} + \sum_{\frac{u+1}{2} \leq \ell \leq \frac{v+1}{2}} \frac{(-1)^{u+1}}{(2\ell-u-1)!} \times \\ &\times \zeta(1-2\ell)\zeta(v+1-2\ell)(-2\pi i \rho)^{2\ell-1-u} + \begin{cases} 0, & \text{if } u > 0; \\ \zeta(0)\zeta(v), & \text{if } u = 0. \end{cases} \end{aligned}$$

Taking into account that

$$\zeta(0) = -\frac{1}{2}, \quad \zeta(1-2m) = -\frac{B_{2m}}{2m}, \quad \zeta(2m) = (-1)^{m-1} \frac{(2\pi)^{2m} B_{2m}}{2 \cdot (2m)!},$$

we get the statement of the Theorem. □

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YURI NESTERENKO

Lomonosov Moscow State University
Faculty Mathematics and Mechanics,
Department of Number Theory,
Leninskie Gory 1,
119992 Moscow, Russia
nester@mi.ras.ru