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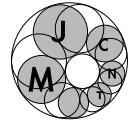
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On points with algebraically conjugate coordinates close to smooth curves

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Abstract:

Let $y = f(x)$ be a continuous differentiable function on an interval $J \subset \mathbb{R}$. In this paper we show that for any $n \in \mathbb{N}$, $n \geq 2$, sufficiently large integer Q and a real $0 < \lambda < \frac{3}{4}$ there exists a positive value $c(n, f, J)$ such that all strips $L_J(Q, \lambda) = \{(x_1, x_2) \in \mathbb{R}^2 : |x_2 - f(x_1)| \ll Q^{-\lambda}, x_1 \in J\}$ contain at least $c(n, f, J)Q^{n+1-\lambda}$ points $\alpha = (\alpha_1, \alpha_2)$ with algebraically conjugate coordinates which minimal polynomial P satisfies $\deg P \leq n$, $H(P) \leq Q$. The proof is based on a metric theorem on the measure of the set of vectors (x_1, x_2) lying in a rectangle Π of dimensions $\asymp Q^{-s_1} \times Q^{-s_2}$ with $|P(x_1)|, |P(x_2)|$ bounded from above and $|P'(x_1)|, |P'(x_2)|$ bounded from below, where P is a polynomial of degree $\deg P \leq n$ and height $H(P) \leq Q$. This theorem is a generalization of a result obtained by V. Bernik, F. Götze and O. Kukso for $s_1 = s_2 = \frac{1}{2}$ and $\lambda = \frac{1}{2}$ [10].

Keywords: algebraic numbers, Diophantine approximation, metric theory, simultaneous approximation

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1. Introduction

Let Q be a sufficiently large number. We denote by $\mathcal{P}_n(Q)$ the following class of polynomials:

$$\mathcal{P}_n(Q) = \{P \in \mathbb{Z}[t] : \deg P \leq n, H(P) \leq Q\},$$

where $H(P) = \max_{0 \leq j \leq n} |a_j|$ denotes the height of an integer polynomial $P(t) = a_n t^n + \dots + a_1 t + a_0$.

The point $\alpha = (\alpha_1, \alpha_2)$ is called an *algebraic point* if α_1 and α_2 are roots of the same polynomial $P \in \mathbb{Z}[t]$. The polynomial P of smallest degree such that $P(\alpha_1) = P(\alpha_2) = 0$ and $\gcd(|a_n|, \dots, |a_0|) = 1$ is called the minimal polynomial of the algebraic point α . Denote by $\deg(\alpha) = \deg P$ the degree of the algebraic point α , and by $H(\alpha) = H(P)$ the height of the algebraic point α . Define the following sets: $\mathbb{A}_n^2(Q)$ is the set of algebraic points α of degree at most n and of height at most Q ; $\mathbb{A}_n^2(Q, D) = \mathbb{A}_n^2(Q) \cap D$ is the set of algebraic points $\alpha \in \mathbb{A}_n^2(Q)$ lying in a domain $D \subset \mathbb{R}^2$. Denote by $\#S$ the cardinality of a finite set S , by $\mu_1 S$ the Lebesgue measure of a measurable set $S \subset \mathbb{R}$ and by $\mu_2 S$ the Lebesgue measure of a measurable set $S \subset \mathbb{R}^2$. Further, denote by $c_j > 0$, $j \in \mathbb{N}$, positive values which do not depend on $H(P)$ or Q . We are also going to use the Vinogradov symbol $A \ll B$, which means that there exists a value $c > 0$ such that $A \leq c \cdot B$ and c doesn't depend on B or A .

An important and interesting topic in the theory of Diophantine approximation is the distribution of algebraic numbers [1, 7, 8, 12]. In this paper we consider problems related to the distribution of algebraic points in domains of small measure and the distribution of algebraic points near smooth curves.

Consider rectangles $\Pi = I_1 \times I_2$ where $\mu_1 I_1 = c_{1,1} \cdot Q^{-s_1}$ and $\mu_1 I_2 = c_{1,2} \cdot Q^{-s_2}$ under the conditions $0 < s_1 + s_2 \leq 1$, $s_1, s_2 < 1$, $\Pi \cap \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - x_2| < \varepsilon\} = \emptyset$ and $c_{1,1} c_{1,2} \geq c_0$. The condition $|x_1 - x_2| > \varepsilon$ means that we exclude from consideration a strip F of small measure such that the coordinates $(x_1, x_2) \in F$ are well approximated by points of form (α, α) .

We can prove the following theorem.

THEOREM 1. *For any rectangle $\Pi = I_1 \times I_2$ satisfying the following conditions:*

1. $\mu_1 I_i = c_{1,i} Q^{-s_i}$ where $s_i < 1$ and $0 < s_1 + s_2 \leq 1$, $i = 1, 2$;
2. $\Pi \cap \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - x_2| < \varepsilon\} = \emptyset$;
3. $c_{1,1} c_{1,2} > c_0(n, \varepsilon, \mathbf{d}) > 0$ for $s_1 + s_2 = 1$, where $\mathbf{d} = (d_1, d_2)$ is the mid-point of Π ;

there exists a constant $c_2 = c_2(n, \varepsilon, \mathbf{d}) > 0$, such that

$$\#\mathbb{A}_n^2(Q, \Pi) \geq c_2 Q^{n+1} \mu_2 \Pi,$$

for $Q > Q_0(n, \varepsilon, \mathbf{d}, \mathbf{s})$.

For $s_1 + s_2 > 1$, we can find a rectangle Π such that the statement of Theorem 1 does not hold. The example of such rectangle is $\Pi = (0, 0.5Q^{-1}) \times (0, 0.5)$. It is easy to prove [9] that the interval $(0, 0.5Q^{-1})$ doesn't contain algebraic numbers of any degree and height $\leq Q$. Let us introduce some restrictions on the domains to be used in the following proofs.

Consider a square $\bar{\Pi} = I_1 \times I_2$ of size $\mu_1 I_1 = \mu_1 I_2 = c_3 Q^{-s}$ such that $\frac{1}{2} < s < \frac{3}{4}$. Given positive u_1, u_2 under the condition $u_1 + u_2 = 1$ let us say that the square $\bar{\Pi}$ is (u_1, u_2) -ordinary square if it doesn't contain points $(x'_1, x'_2) \in \mathbb{R}^2$ such that there exists a polynomial $P \in \mathcal{P}_2(Q)$ of the form $P(t) = b_2 t^2 + b_1 t + b_0$ satisfying the system of inequalities

$$\begin{cases} |P(x'_i)| \ll Q^{-u_i}, & i = 1, 2, \\ |b_2| < Q^{s-\frac{1}{2}}. \end{cases} \quad (1)$$

Otherwise, the square $\bar{\Pi}$ is going to be called (u_1, u_2) -special.

For (u_1, u_2) -ordinary squares, the following result holds.

THEOREM 2. For any $(\frac{1}{2}, \frac{1}{2})$ -ordinary square $\bar{\Pi} = I_1 \times I_2$ under the following conditions:

1. $\mu_1 I_i = c_3 Q^{-s}$, where $\frac{1}{2} < s < \frac{3}{4}$, $i = 1, 2$;
2. $\bar{\Pi} \cap \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - x_2| < \varepsilon\} = \emptyset$;
3. $c_3 > c_0(n, \varepsilon, \mathbf{d}) > 0$, where $\mathbf{d} = (d_1, d_2)$ is the midpoint of $\bar{\Pi}$;

there exists a constant $c_4 = c_4(n, \varepsilon, \mathbf{d}) > 0$, such that

$$\#\mathbb{A}_n^2(Q, \bar{\Pi}) \geq c_4 Q^{n+1} \mu_2 \bar{\Pi}$$

for $Q > Q_0(n, \varepsilon, \mathbf{d}, s)$.

Another interesting and important topic is the distribution of algebraic points near smooth curves. The result presented in this paper is a natural generalization of problems related to distribution of rational points near smooth curves [3, 4, 12–14].

In 2014 a lower bound for the number of algebraic points lying at a distance of at most $Q^{-\lambda}$, $0 < \lambda < \frac{1}{2}$, from a smooth curve was obtained by V. Bernik, F. Götze and O. Kukso [10]. We improve on this result and obtain an identical estimate for $0 < \lambda < \frac{3}{4}$.

THEOREM 3. *Let $y = f(x)$ be a continuous differentiable function on an interval $J = [a, b]$ such that $\sup_{x \in J} |f'(x)| := c_5 < \infty$ and $\#\{x \in \mathbb{R} : f(x) = x\} < \infty$. Denote by $L_J(Q, \lambda)$ the following set:*

$$L_J(Q, \lambda) = \left\{ (x_1, x_2) \in \mathbb{R}^2 : |x_2 - f(x_1)| < \left(\frac{1}{2} + c_5 \right) \cdot c_3 Q^{-\lambda}, \quad x_1 \in J \right\},$$

for $0 < \lambda < \frac{3}{4}$. Then there exists a positive value $c_6(J, f, n) > 0$ such that

$$\#\left\{ \mathbb{A}_n^2(Q) \cap L_J(Q, \lambda) \right\} \geq c_6 Q^{n+1-\lambda}$$

for $Q > Q_0(J, f, n, \lambda)$.

2. Auxiliary statements

For a polynomial P with roots $\alpha_1, \alpha_2, \dots, \alpha_n$, let

$$S(\alpha_i) = \left\{ x \in \mathbb{R} : |x - \alpha_i| = \min_{1 \leq j \leq n} |x - \alpha_j| \right\}.$$

From now on, we assume that the roots of the polynomial P are sorted by distance from $\alpha_i = \alpha_{i,1}$:

$$|\alpha_{i,1} - \alpha_{i,2}| \leq |\alpha_{i,1} - \alpha_{i,3}| \leq \dots \leq |\alpha_{i,1} - \alpha_{i,n}|.$$

LEMMA 1. *Let $x \in S(\alpha_i)$. Then*

$$|x - \alpha_i| \leq n \cdot \frac{|P(x)|}{|P'(x)|}, \quad |x - \alpha_i| \leq 2^{n-1} \cdot \frac{|P(x)|}{|P'(\alpha_i)|}, \tag{2}$$

$$|x - \alpha_i| \leq \min_{1 \leq j \leq n} \left(2^{n-j} \frac{|P(x)|}{|P'(\alpha_i)|} |\alpha_i - \alpha_{i,2}| \dots |\alpha_i - \alpha_{i,j}| \right)^{1/j}. \tag{3}$$

The first inequality follows from the identity

$$|P'(x)||P(x)|^{-1} \leq \sum_{j=1}^n |x - \alpha_j|^{-1}.$$

For a proof of the second and the third inequalities see [1, 2].

LEMMA 2. *Let I be an interval, and let $A \subset \mathbb{R}$ be a measurable set, $A \subset I$, $\mu_1 A \geq \frac{1}{2} \mu_1 I$. If for some $v > 0$ and all $x \in A$ the inequality $|P(x)| < c_7 Q^{-v}$, then*

$$|P(x)| < 6^n (n+1)^{n+1} c_7 Q^{-v}$$

for all points $x \in I$, where $n = \deg P$.

The proof of this lemma can be found in [6].

LEMMA 3. *Let δ, η_1, η_2 be real positive numbers, and let $P_1, P_2 \in \mathbb{Z}[t]$ be co-prime polynomials of degrees at most n such that*

$$\max(H(P_1), H(P_2)) < K,$$

where $K > K_0(\delta)$. Let $J_1, J_2 \subset \mathbb{R}$ be intervals of sizes $\mu J_1 = K^{-\eta_1}$, $\mu J_2 = K^{-\eta_2}$. If for some $\tau_1, \tau_2 > 0$ and for all $(x_1, x_2) \in J_1 \times J_2$, the inequalities

$$\max(|P_1(x_i)|, |P_2(x_i)|) < K^{-\tau_i}, \quad i = 1, 2,$$

hold, then

$$\tau_1 + \tau_2 + 2 + 2 \max(\tau_1 + 1 - \eta_1, 0) + 2 \max(\tau_2 + 1 - \eta_2, 0) < 2n + \delta. \quad (4)$$

The proof of this lemma can be found in [17].

LEMMA 4. *Let $P \in \mathbb{Z}[t]$ be a reducible polynomial, $P = P_1 \cdot P_2$, $\deg P = n \geq 2$. Then there exist $c_8, c_9 > 0$ such that*

$$c_8 H(P) < H(P_1)H(P_2) < c_9 H(P).$$

The proof of Lemma 4 can be found, for example, in [1].

3. Proof of Theorem 1

Before we start it should be noted that there exists a constant $h_n = h_n(\mathbf{d}) > 0$ such that for every point $(x_1, x_2) \in \Pi$ and every $\mathbf{v} = (v_1, v_2)$, $v_1, v_2 \geq -1$ with $v_1 + v_2 = n - 1$ there exists a polynomial $P \in \mathcal{P}_n(Q)$ satisfying the inequalities:

$$|P(x_i)| < h_n \cdot Q^{-v_i}, \quad i = 1, 2,$$

for $Q > Q_0$. This simple fact follows from Dirichlet's principle and estimates $\#\mathcal{P}_n(Q) > 2^n Q^{n+1}$ and $|P(x_i)| < \left((|d_i| + 1)^{n+1} - 1 \right) |d_i|^{-1} \cdot Q$, where $\mathbf{d} = (d_1, d_2)$ is the midpoint of Π .

To prove Theorem 1, we are going to rely on the following Lemma 5.

LEMMA 5. *For all rectangles $\Pi = I_1 \times I_2$ under the conditions:*

1. $\mu_1 I_i = c_{1,i} Q^{-s_i}$ where $s_i < 1$ and $0 < s_1 + s_2 \leq 1$, $i = 1, 2$;
2. $\Pi \cap \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - x_2| < \varepsilon\} = \emptyset$;
3. $c_{1,1} c_{1,2} > c_0(n, \varepsilon, \mathbf{d}) > 0$ for $s_1 + s_2 = 1$, where $\mathbf{d} = (d_1, d_2)$ is the midpoint of Π ;

let $L = L(Q, \delta_n, \mathbf{v}, \Pi)$ be the set of points $(x_1, x_2) \in \Pi$ such that there exists a polynomial $P \in \mathcal{P}_n(Q)$ satisfying the following system of inequalities:

$$\begin{cases} |P(x_i)| < h_n \cdot Q^{-v_i}, & v_i > 0, \\ \min_i \{|P'(x_i)|\} < \delta_n \cdot Q, \\ v_1 + v_2 = n - 1, & i = 1, 2. \end{cases} \quad (5)$$

Then for $\delta_n \leq \delta_0(n, \varepsilon, \mathbf{d})$ and $Q > Q_0(n, \varepsilon, \mathbf{s}, \mathbf{v}, \mathbf{d})$, the estimate

$$\mu_2 L < \frac{1}{4} \mu_2 \Pi$$

holds.

PROOF. Denote by L_1 the set of points $(x_1, x_2) \in \Pi$ such that the system of inequalities (5) has a solution in irreducible polynomials $P \in \mathcal{P}_n(Q)$ under condition $|P'(x_1)| < \delta_n \cdot Q$, by L_2 the set of points $(x_1, x_2) \in \Pi$ such that the system

of inequalities (5) has a solution in irreducible polynomials $P \in \mathcal{P}_n(Q)$ under condition $|P'(x_2)| < \delta_n \cdot Q$ and by L_3 the set of points $(x_1, x_2) \in \Pi$ such that the system of inequalities (5) has a solution in reducible polynomials $P \in \mathcal{P}_n(Q)$. Thus, $L = L_1 \cup L_2 \cup L_3$.

Let us estimate the measure of L_1 . The main idea is to split the range of the possible values of $|P'(x_i)|$, $|P'(\alpha_i)|$, where $x_i \in S(\alpha_i)$, $i = 1, 2$ into a total of $r = r(n) = (n - 1)^2$ sub-ranges and consider them separately.

Without loss of generality, we will assume that $|d_1| < |d_2|$. Let us show that the inequality

$$|P'(x_i)| \geq 2c_{10} \cdot Q^{\frac{1}{2} - \frac{v_i}{2}} \quad (6)$$

yields the following bounds on $P'(\alpha_i)$:

$$\frac{1}{2}|P'(x_i)| \leq |P'(\alpha_i)| \leq 2|P'(x_i)|,$$

where $c_{10} = n(n - 1) \cdot \max\{h_n, 1\} \cdot (3 \max\{1, |d_2|\})^{n-1} \cdot (1 + |d_2|)^{-1}$. Let us write a Taylor expansion of $P'(t)$:

$$P'(x_i) = P'(\alpha_i) + P''(\alpha_i)(x_i - \alpha_i) + \dots + \frac{1}{(n - 1)!} P^{(n)}(\alpha_i)(x_i - \alpha_i)^{n-1}. \quad (7)$$

Using Lemma 1 and the estimates (5) for $Q > Q_0$, we have:

$$|x_i - \alpha_i| \leq \frac{1}{2} n h_n c_{10}^{-1} \cdot Q^{-\frac{v_i+1}{2}} < \max\{1, |d_2|\} \cdot Q^{-\frac{v_i+1}{2}}.$$

Then, for $s_i > 0$ and $Q > Q_0$ we get $|x_i - d_1| < 1/2$ and thus:

$$|\alpha_i| \leq |x_i| + \frac{1}{2} < |d_2| + 1.$$

From this estimates we obtain the following inequality for every term in (7):

$$\begin{aligned} & \left| \frac{1}{(k - 1)!} P^{(k)}(\alpha_i)(x_i - \alpha_i)^{k-1} \right| < \\ & < C_{n-1}^{k-1} \cdot \frac{n(n + 1 - k)(|d_2| + 1)^{n-k+1}}{|d_2|} \cdot \max\{1, |d_2|\}^{k-1} \cdot Q^{1 - \frac{(k-1)(1+v_i)}{2}} < \end{aligned}$$

$$\leq C_{n-1}^{k-1} \cdot \frac{n(n-1)(|d_2|+1)^{n-k+1}}{|d_2|} \cdot \max\{1, |d_2|\}^{k-1} Q^{\frac{1}{2}-\frac{v_i}{2}},$$

for $k \geq 2$. Thus, the estimate

$$\begin{aligned} & \left| 12P''(\alpha_i)(x_i - \alpha_i) + \dots + \frac{1}{(n-1)!} P^{(n)}(\alpha_i)(x_i - \alpha_i)^{n-1} \right| < \\ & < n(n-1) (3 \max\{1, |d_2|\})^{n-1} \cdot (1 + |d_2|^{-1}) \cdot Q^{\frac{1}{2}-\frac{v_i}{2}} < \frac{1}{2} \cdot |P'(x_i)| \end{aligned}$$

holds and by substituting these inequality to (7) we get

$$\frac{1}{2} \cdot |P'(x_i)| \leq |P'(\alpha_i)| \leq 2|P'(x_i)|.$$

This means that $|P'(\alpha_i)| \in T_i$, where

$$T_1 = \left[c_{10} \cdot Q^{\frac{1}{2}-\frac{v_1}{2}}; 2\delta_n \cdot Q \right), \quad T_2 = \left[c_{10} \cdot Q^{\frac{1}{2}-\frac{v_2}{2}}; n \cdot \frac{(|d_2|+1)^n - 1}{|d_2|} \cdot Q \right)$$

if the inequalities (6) hold. Let us divide the intervals T_i into sub-intervals $T_{i,j} = [d_{j,i} Q^{t_{j,i}}; d_{j-1,i} Q^{t_{j-1,i}})$, $2 \leq j \leq n$, where

$$t_{k,i} = \begin{cases} 1, & k=1, \\ \frac{1}{2} - \frac{(k-1)v_i}{2(n-1)}, & 2 \leq k \leq n, \end{cases} \quad d_{k,i} = \begin{cases} 2\delta_n, & k=1, i=1, \\ n \cdot \frac{(|d_2|+1)^n - 1}{|d_2|}, & k=1, i=2, \\ 1, & 2 \leq k \leq n-1, \\ c_{10}, & k=n, \end{cases}$$

Now we are going to consider the following cases:

- the case of polynomials of the second degree $n = 2$ (see Section 3.1);
- the case of irreducible polynomials:

$|P'(\alpha_1)| \in T_{1,j_1}$, $|P'(\alpha_2)| \in T_{2,j_2}$, where $2 \leq j_1, j_2 \leq n-1$ (see Section 3.2);

$|P'(\alpha_1)| \in T_{1,n}$, $|P'(\alpha_2)| \in T_{2,n}$ (see Section 3.3);

$$|P'(x_1)| \leq 2c_{10}Q^{\frac{1}{2}-\frac{v_1}{2}}, |P'(x_2)| \leq 2c_{10}Q^{\frac{1}{2}-\frac{v_2}{2}} \quad (\text{see Section 3.4});$$

$|P'(\alpha_1)| \in T_{1,j_1}$, $|P'(\alpha_2)| \in T_{2,n}$ or $|P'(\alpha_1)| \in T_{1,n}$, $|P'(\alpha_2)| \in T_{2,j_2}$, where $2 \leq j_1, j_2 \leq n-1$ (see Section 3.5);

$|P'(\alpha_1)| \in T_{1,j_1}$, $|P'(x_2)| \leq 2c_{10}Q^{\frac{1}{2}-\frac{v_2}{2}}$ or $|P'(x_1)| \leq 2c_{10}Q^{\frac{1}{2}-\frac{v_1}{2}}$, $|P'(\alpha_2)| \in T_{2,j_2}$, where $2 \leq j_1, j_2 \leq n$ (see Section 3.5);

- the case of reducible polynomials (see Section 3.6).

We are going to use induction on the degree n . Let us prove the following statement, which will serve as the base of induction.

3.1. The base of induction: polynomials of the second degree

STATEMENT 1. For all rectangles Π under the conditions 1–3 let $L_{2,2} = L_{2,2}(Q, \delta_2, \gamma_2, \Pi)$ be the set of points $(x_1, x_2) \in \Pi$ such that there exists a polynomial $P \in \mathcal{P}_2(Q)$ satisfying the system of inequalities

$$\begin{cases} |P(x_i)| < h_2 \cdot Q^{-\gamma_{2,i}}, & \gamma_{2,i} > 0, \\ \min_i \{|P'(x_i)|\} < \delta_2 \cdot Q, \\ \gamma_{2,1} + \gamma_{2,2} = 1, & i = 1, 2. \end{cases} \quad (8)$$

Then for any $r > 0$ and for $\delta_2 < \delta_0(r, \varepsilon, \mathbf{d})$ and $Q > Q_0(n, \varepsilon, \mathbf{s}, \gamma_2, \mathbf{d})$, the estimate

$$\mu_2 L_{2,2} < \frac{1}{4r} \cdot \mu_2 \Pi$$

holds.

PROOF. Let $P(t)$ be a polynomial of the form $b_2 t^2 + b_1 t + b_0$. Let us estimate the values $|P'(\alpha_1)|$ and $|P'(\alpha_2)|$. By the third inequality of Lemma 1, for every polynomial P satisfying the inequalities (8) at a point $(x_1, x_2) \in \Pi$, we have the following estimates:

$$|x_i - \alpha_i| < \left(|P(x_i)| |b_2|^{-1} \right)^{1/2} < h_2^{1/2} Q^{-\frac{\gamma_{2,i}}{2}} < \frac{\varepsilon}{8}, \quad (9)$$

for $Q > Q_0$ and $x_i \in S(\alpha_i)$, $i = 1, 2$.

From (9) and condition 2 we obtain that

$$|\alpha_1 - \alpha_2| > |x_1 - x_2| - |x_1 - \alpha_1| - |x_2 - \alpha_2| > \frac{3}{4} \cdot \varepsilon$$

and

$$|\alpha_1 - \alpha_2| < |x_1| + |x_2| + |x_1 - \alpha_1| + |x_2 - \alpha_2| < |d_1| + |d_2| + 1 + \frac{\varepsilon}{4}.$$

This leads to the following lower bounds for $|P'(\alpha_i)|$:

$$\left(|d_1| + |d_2| + 1 + \frac{\varepsilon}{4}\right) \cdot |b_2| > |P'(\alpha_i)| = \sqrt{D} = |b_2| \cdot |\alpha_1 - \alpha_2| > \frac{3}{4} \cdot \varepsilon \cdot |b_2|, \quad (10)$$

where D is the discriminant of the polynomial P . The inequalities (9) also yield upper bounds for $|P'(x_i)|$:

$$|P'(x_i)| \leq |b_2| \cdot (|\alpha_1 - x_i| + |\alpha_2 - x_i|) \leq \left(|d_2| + |d_1| + 1 + \frac{\varepsilon}{4}\right) \cdot |b_2|. \quad (11)$$

Now upper bounds for $|P'(\alpha_i)|$ can be obtained from the Taylor expansion of the polynomial P' :

$$|P'(\alpha_i)| \leq |P'(x_i)| + |P''(x_i)| \cdot |x_i - \alpha_i| \leq |P'(x_i)| + \frac{\varepsilon}{2} \cdot |b_2|. \quad (12)$$

Then, the estimates (10), (12) mean that

$$|b_2| < 4\varepsilon^{-1} \cdot \min_i \{|P'(x_i)|\} < 4\delta_2\varepsilon^{-1}Q. \quad (13)$$

From Lemma 1 and the estimates (10) it follows that the set $L_{2,2}$ is contained in a union $\bigcup_{P \in \mathcal{P}_2(Q)} \sigma_P$, where

$$\sigma_P = \left\{ (x_1, x_2) \in \Pi : |x_i - \alpha_i| < 2h_2\varepsilon^{-1}Q^{-\gamma_{2,i}}|b_2|^{-1}, i = 1, 2 \right\}.$$

Simple calculations show that the measure of the set σ_P is lower than the measure of the rectangle Π :

$$\mu_2\sigma_P \leq 2^4h_2^2\varepsilon^{-2}Q^{-1}|b_2|^{-2} < \frac{1}{4}c_{1,1}c_{1,2}Q^{-1} = \frac{1}{4}\mu_2\Pi$$

for $c_{1,1}c_{1,2} > 2^6h_2^2\varepsilon^{-2}$.

Let us estimate the measure of $L_{2,2}$:

$$\mu_2 L_{2,2} \leq \mu_2 \bigcup_{P \in \mathcal{P}_2(Q)} \sigma_P \leq \sum_{P \in \mathcal{P}_2(Q)} \mu_2 \sigma_P \leq 2^4 h_2^2 \varepsilon^{-2} Q^{-1} \sum_{\substack{b_2, b_1, b_0 \leq Q: \\ P(t) = b_2 t^2 + b_1 t + b_0, \\ \sigma_P \neq \emptyset}} |b_2|^{-2}.$$

To do this, we need to estimate the number of polynomials $P \in \mathcal{P}_2(Q)$ such that the system (8) holds for some point $(x_1, x_2) \in \Pi$, where b_2 is fixed.

Let the inequalities (8) hold for polynomial P and point $(x_{0,1}, x_{0,2}) \in \Pi$. Let us estimate the value of the polynomial P at d_i . From the Taylor expansion of P , we have

$$P(d_i) = P(x_{0,i}) + P'(x_{0,i})(x_{0,i} - d_i) + \frac{1}{2} P''(x_{0,i})(x_{0,i} - d_i)^2.$$

It means that $|P(d_i)| \leq |P(x_{0,i})| + |P'(x_{0,i})| \mu_1 I_i + |b_2| (\mu_1 I_i)^2$. Thus, from (11) for $Q > Q_0$ we can obtain the estimate

$$|P(d_i)| < |P(x_{0,i})| + c_{11} \cdot |b_2| \mu_1 I_i \leq 2c_{11} \cdot \max\{1, |b_2| \mu_1 I_i\}.$$

Without loss of generality, let us assume that $\mu_1 I_1 \leq \mu_1 I_2$.

Consider the system of equations

$$\begin{cases} b_2 d_1^2 + b_1 d_1 + b_0 = l_1, \\ b_2 d_2^2 + b_1 d_2 + b_0 = l_2 \end{cases} \quad (14)$$

in three variables $b_2, b_1, b_0 \in \mathbb{Z}$, where $|l_i| \leq 2c_{11} \cdot \max\{1, |b_2| \mu_1 I_i\}$, $i = 1, 2$.

Let us estimate the number of possible pairs (b_1, b_0) such that the system (14) is satisfied for a fixed b_2 . To obtain this estimate, we consider the system of linear equations (14) for two different combinations $b_2, b_{0,1}, b_{0,0}$ and $b_2, b_{j,1}, b_{j,0}$:

$$\begin{cases} b_2 d_1^2 + b_{0,1} d_1 + b_{0,0} = l_{0,1}, \\ b_2 d_1^2 + b_{j,1} d_1 + b_{j,0} = l_{j,1}, \\ b_2 d_2^2 + b_{0,1} d_2 + b_{0,0} = l_{0,2}, \\ b_2 d_2^2 + b_{j,1} d_2 + b_{j,0} = l_{j,2}. \end{cases}$$

Subtracting the second equation from the first and the fourth equation from the third leads to the following system in two variables $b_{0,1} - b_{j,1}$ and $b_{0,0} - b_{j,0}$:

$$\begin{cases} (b_{0,1} - b_{j,1})d_1 + (b_{0,0} - b_{j,0}) = l_{0,1} - l_{j,1}, \\ (b_{0,1} - b_{j,1})d_2 + (b_{0,0} - b_{j,0}) = l_{0,2} - l_{j,2}. \end{cases} \quad (15)$$

The determinant of the system (15) can be written as

$$|\Delta| = \begin{vmatrix} d_1 & 1 \\ d_2 & 1 \end{vmatrix} = |d_1 - d_2| > \varepsilon > 0.$$

Since the determinant does not vanish, we can use Cramer's rule to solve the system (15). Using the inequalities $|l_{0,i} - l_{j,i}| \leq 4c_{11} \cdot \max\{1, |b_2| \mu_1 I_i\}$, $i = 1, 2$, we estimate the determinant Δ_1 as follows:

$$|\Delta_1| \leq 8c_{11} \cdot \max\{1, |b_2| \mu_1 I_2\}.$$

Hence by Cramer's rule we have

$$|b_{0,1} - b_{j,1}| \leq \frac{|\Delta_1|}{|\Delta|} \leq 8\varepsilon^{-1} c_{11} \cdot \max\{1, |b_2| \mu_1 I_2\}.$$

This inequality means that all possible values of the coefficient b_1 lie in an interval J_1 of length $\mu_1 J_1 = 2^4 \varepsilon^{-1} c_{11} \cdot \max\{1, |b_2| \mu_1 I_2\}$ centered at $b_{0,1}$. Since the values of the coefficient b_1 are integers, the number of these values does not exceed the measure of the interval J_1 .

In addition, let us fix the value of the coefficient b_1 . Choose a value $b_1 \in J_1$ and consider two different combinations $(b_2, b_1, b_{0,0})$ and $(b_2, b_1, b_{j,0})$. In this case, the system (14) can be transformed as follows:

$$\begin{cases} |b_{0,0} - b_{j,0}| \leq 4c_{11} \cdot \max\{1, |b_2| \mu_1 I_1\}, \\ |b_{0,0} - b_{j,0}| \leq 4c_{11} \cdot \max\{1, |b_2| \mu_1 I_2\}. \end{cases}$$

Similarly, we have $b_0 \in J_0$, where J_0 is an interval of length $\mu_1 J_0 = 8c_{11} \cdot \max\{1, |b_2| \mu_1 I_1\}$ centered at $b_{0,0}$, and the number of possible values for b_0 does not exceed the measure of the interval J_0 .

The following estimate

$$\#(b_1, b_0) \leq \mu_1 J_1 \cdot \mu_1 J_0 = \begin{cases} 2^7 \varepsilon^{-1} c_{11}^2 \cdot |b_2|^2 \mu_2 \Pi, & |b_2| \geq (\mu_1 I_1)^{-1}, \\ 2^7 \varepsilon^{-1} c_{11}^2 \cdot |b_2| \mu_1 I_2, & (\mu_1 I_2)^{-1} \leq |b_2| \leq (\mu_1 I_1)^{-1}, \\ 2^7 \varepsilon^{-1} c_{11}^2, & |b_2| \leq (\mu_1 I_2)^{-1}, \end{cases} \quad (16)$$

holds for a fixed value of the coefficient b_2 .

Let us use the estimates (13) and (16) to consider the following three cases.

Case 1: $(\mu_1 I_1)^{-1} \leq |b_2| \leq 4\delta_2 \varepsilon^{-1} Q$.

In this case, the first estimate of (16) holds, and we have

$$\mu_2 L_{2,2} \leq 2^{11} \varepsilon^{-3} c_{11}^2 h_2^2 \cdot Q^{-1} \mu_2 \Pi \cdot 4\delta_2 \varepsilon^{-1} Q < \frac{1}{12r} \mu_2 \Pi,$$

for $\delta_2 < 2^{-17} r^{-1} \varepsilon^4 c_{11}^{-2} h_2^{-2}$.

Case 2: $(\mu_1 I_2)^{-1} \leq |b_2| \leq (\mu_1 I_1)^{-1}$.

Then the second estimate of (16) holds, and we have

$$\mu_2 L_{2,2} \ll Q^{-1} \mu_1 I_2 \sum_{(\mu_1 I_2)^{-1} \leq |b_2| \leq (\mu_1 I_1)^{-1}} |b_2|^{-1} \ll Q^{-1} \ln Q \cdot \mu_1 I_2.$$

Consequently, for $\varepsilon_1 = \frac{1-s_1}{2}$ and $Q > Q_0$ we obtain

$$\mu_2 L_{2,2} \ll Q^{-1+\varepsilon_1} \mu_1 I_2 \ll Q^{-\varepsilon_1} \mu_2 \Pi \leq \frac{1}{12r} \mu_2 \Pi.$$

Case 3: $1 \leq |b_2| \leq (\mu_1 I_2)^{-1}$.

In this case, the third estimate of (16) holds, leading to

$$\mu_2 L_{2,2} \leq 2^{11} \varepsilon^{-3} c_{11}^2 h_2^2 \cdot Q^{-1} \sum_{1 \leq |b_2| \leq (\mu_1 I_2)^{-1}} |b_2|^{-2} \leq \frac{1}{12r} \mu_2 \Pi,$$

for $c_{1,1} c_{1,2} > 2^{12} r \pi^2 c_{11}^2 \varepsilon^{-3} h_2^2$. □

3.2. The induction step: reducing the degree of the polynomial

Let us return to the proof of Lemma 5. For $|P'(\alpha_1)| \in T_{1,j_1}$ and $|P'(\alpha_2)| \in T_{2,j_2}$, we have the following system of inequalities:

$$\begin{cases} |P(x_i)| < h_n \cdot Q^{-v_i}, & v_i > 0, \\ d_{j_i,i} Q^{t_{j_i,i}} \leq |P'(\alpha_i)| < d_{j_i-1,i} Q^{t_{j_i-1,i}}, \\ v_1 + v_2 = n - 1, & i = 1, 2. \end{cases} \quad (17)$$

Without loss of generality, assume that $j_1 \leq j_2$. Denote by L_{j_1,j_2} the set of points $(x_1, x_2) \in \Pi$ such that the system of inequalities (17) has a solution in polynomials $P \in \mathcal{P}_n(Q)$. By Lemma 1, it follows that L_{j_1,j_2} is contained in a union $\bigcup_{P \in \mathcal{P}_n(Q)} \sigma_P$, where

$$\sigma_P = \left\{ (x_1, x_2) \in \Pi : |x_i - \alpha_i| < 2^{n-1} h_n \cdot Q^{-v_i} |P'(\alpha_i)|^{-1}, i = 1, 2 \right\}. \quad (18)$$

It means that the following estimate for $\mu_2 L_{j_1,j_2}$ holds:

$$\mu_2 L_{j_1,j_2} \leq \mu_2 \bigcup_{P \in \mathcal{P}_n(Q)} \sigma_P \leq \sum_{P \in \mathcal{P}_n(Q)} \mu_2 \sigma_P.$$

Together with the sets σ_P consider the following expanded sets

$$\sigma'_P = \sigma'_{P,1} \times \sigma'_{P,2} = \left\{ (x_1, x_2) \in \Pi : |x_i - \alpha_i| < c_{12} Q^{-\gamma_{j_2,i}} |P'(\alpha_i)|^{-1}, i = 1, 2 \right\}. \quad (19)$$

where $\gamma_{j_2,i} = \frac{(j_2 - 1)v_i}{n - 1}$. Simple calculations show that the measure of the set σ'_P is smaller than the measure of the rectangle Π for $Q > Q_0$:

$$\mu_2 \sigma'_P \leq 4c_{12}^2 \cdot Q^{1-j_2} Q^{-t_{1,j_1} - t_{2,j_2}} < 4c_{12}^2 \cdot Q^{-\frac{j_2+1}{2}} < \mu_2 \Pi.$$

Using (18) and (19), we find that the measures $\mu_2 \sigma_P$ and $\mu_2 \sigma'_P$ are connected as follows:

$$\mu_2 \sigma_P \leq 2^{2n-2} h_n^2 c_{12}^{-2} \cdot Q^{-n+j_2} \mu_2 \sigma'_P. \quad (20)$$

Fix the vector $\mathbf{b}_{j_2} = (a_n, \dots, a_{j_2+1})$, where a_n, \dots, a_{j_2+1} are the coefficients of the polynomial $P \in \mathcal{P}_n(Q)$. Denote by $\mathcal{P}_n(\mathbf{b}_{j_2}) \subset \mathcal{P}_n(Q)$ a subclass of polynomials with the same vector of coefficients \mathbf{b}_{j_2} . The number of subclasses $\mathcal{P}_n(\mathbf{b}_{j_2})$ is equal to the number of vectors \mathbf{b}_{j_2} which can be estimated as follows:

$$\# \{ \mathbf{b}_{j_2} \} = (2Q + 1)^{n-j_2} < 2^{2n} Q^{n-j_2}. \quad (21)$$

We are going to apply Sprindžuk's method of essential and non-essential sets [1]. A set σ'_{P_1} , $P_1 \in \mathcal{P}_n(\mathbf{b}_{j_2})$ is called *essential* if for every σ'_{P_2} , $P_2 \in \mathcal{P}_n(\mathbf{b}_{j_2})$, $P_2 \neq P_1$, the inequality

$$\mu_2 (\sigma'_{P_1} \cap \sigma'_{P_2}) < \frac{1}{2} \mu_2 \sigma'_{P_1}, \quad (22)$$

is satisfied. Otherwise, σ'_{P_1} is called *non-essential*.

The case of essential sets. For essential sets, we have the following estimate:

$$\sum_{\substack{P \in \mathcal{P}_n(\mathbf{b}_{j_2}) \\ \sigma'_P \text{-essential}}} \mu_2 \sigma'_P \leq 4\mu_2 \Pi. \quad (23)$$

Then from (20), (21) and (23) we can write

$$\sum_{\mathbf{b}_{j_2}} \sum_{\substack{P \in \mathcal{P}_n(\mathbf{b}_{j_2}) \\ \sigma'_P \text{-essential}}} \mu_2 \sigma'_P \leq 2^{4n-2} h_n^2 c_{12}^{-2} \sum_{\substack{P \in \mathcal{P}_n(\mathbf{b}_{j_2}) \\ \sigma'_P \text{-essential}}} \mu_2 \sigma'_P < \frac{1}{24r} \mu_2 \Pi, \quad (24)$$

for $c_{12} = 2^{2n+3} r^{1/2} h_n$.

The case of non-essential sets. If a set σ'_{P_1} is non-essential, then there exists a set σ'_{P_2} such that $\mu_2 (\sigma'_{P_1} \cap \sigma'_{P_2}) > \frac{1}{2} \mu_2 \sigma'_{P_1}$. Consider the polynomial $R = P_2 - P_1$, $\deg R \leq j_2$, $H(R) \leq 2Q$, on the set $(\sigma'_{P_1} \cap \sigma'_{P_2})$. Let us estimate the values $|R(x_i)|$ and $|R'(x_i)|$, $i = 1, 2$.

Let us write Taylor expansions of the polynomials P_1 and P_2 in the interval $\sigma'_{P_1, i} \cap \sigma'_{P_2, i}$, $i, j = 1, 2$:

$$P_j(x_i) = P'_j(\alpha_{j,i})(x_i - \alpha_{j,i}) + \dots + \frac{1}{n!} \cdot P_j^{(n)}(\alpha_{j,i})(x_i - \alpha_{j,i})^n,$$

where $\alpha_{j,i} \in \sigma'_{P_{j,i}}$. From the estimate (19), we have:

$$|P'_j(\alpha_{j,i})(x_i - \alpha_{j,i})| \leq c_{12}Q^{-\gamma_{j_2,i}},$$

$$\left| \frac{1}{k!} P_j^{(k)}(\alpha_{j,i})(x_i - \alpha_{j,i})^k \right| \leq c_{13,k}Q^{1-k\gamma_{j_2,i}-kt_{j_2,i}} \leq c_{13,k}Q^{1-\frac{k}{2}+\frac{k}{2}\gamma_{j_2,i}-k\gamma_{j_2,i}} \leq c_{13,k}Q^{-\gamma_{j_2,i}},$$

for $k \geq 2$ and $Q > Q_0$.

Thus, the estimate $|R(x_i)| < |P_1(x_i)| + |P_2(x_i)| < c_{13} \cdot Q^{-\gamma_{j_2,i}}$ holds. From Lemma 2 it follows that for every point $(x_1, x_2) \in \sigma'_{P_1}$, the inequalities

$$|R(x_i)| < c_{14} \cdot Q^{-\gamma_{j_2,i}}, \quad i = 1, 2,$$

are satisfied.

Now let us write Taylor expansions of the polynomials P'_1 and P'_2 in the interval $\sigma'_{P_{1,i}} \cap \sigma'_{P_{2,i}}$, $j, i = 1, 2$:

$$P'_j(x_i) = P'_j(\alpha_{j,i}) + \dots + \frac{1}{(n-1)!} P_j^{(n)}(\alpha_{j,i})(x_i - \alpha_{j,i})^{n-1},$$

where $\alpha_{j,i} \in \sigma'_{P_{j,i}}$. From the estimate (19), we have:

$$\left| \frac{1}{(k-1)!} P_j^{(k)}(\alpha_i)(x_i - \alpha_i)^{k-1} \right| \leq c_{15,k}Q^{1+(k-1)\left(\frac{\gamma_{j_2,i}}{2}-\gamma_{j_2,i}-\frac{1}{2}\right)} \leq c_{15,k}|P'(\alpha_i)|$$

for $k \geq 2$ and $Q > Q_0$. Thus, we obtain $|R'(x_i)| \leq |P'_1(x_i)| + |P'_2(x_i)| \leq c_{15}|P'(\alpha_i)|$. From Lemma 2 it follows that for a sufficiently large $Q > Q_0$ the following inequalities hold:

$$\min_i \{|R'(x_i)|\} \leq c_{16} \min_i \{|P'(\alpha_i)|\} \leq \begin{cases} 2c_{16}\delta_n Q, & j_1 = j_2 = 2, \\ c_{16}Q^{\frac{1}{2}}, & j_1 \neq 2 \text{ or } j_2 \neq 2, \end{cases}$$

for every point $(x_1, x_2) \in \sigma'_{P_1}$. Thus, the measure of L_{j_1, j_2} for non-essential sets does not exceed the respective measure for the system

$$\begin{cases} |R(x_i)| < h_{j_2} Q_1^{-\gamma_{j_2, i}}, & \gamma_{j_2, i} > 0, \\ \min_i \{|R'(x_i)|\} < \delta_{j_2} Q_1, \\ \gamma_{1, j_2} + \gamma_{2, j_2} = j_2 - 1, & i = 1, 2, \end{cases} \quad (25)$$

where $Q_1 = \min_i \{(h_{j_2}/c_{14})^{1/\gamma_{j_2, i}}\} \cdot Q$ and $\delta_{j_2} = 2c_{16} \cdot \left(\min_i \{(h_{j_2}/c_{14})^{1/\gamma_{j_2, i}}\}\right)^{-1} \cdot \delta_n$.

It should be mentioned that if polynomial $R(t) = a_1 t - a_0$ is linear, then by Lemma 1 we obtain:

$$\left| x_i - \frac{a_0}{a_1} \right| \ll Q_1^{-\gamma_{j_2, i}} < \frac{\varepsilon}{4}, \quad i = 1, 2$$

for $Q_1 > Q_0$. Hence, we immediately have $|x_1 - x_2| < \varepsilon$ which contradicts to condition 2 for polynomial Π . Thus, $\deg R \geq 2$ and we can use induction. Since $j_2 < n$, by the induction hypothesis the measure of solutions of the system (25) is bounded from above by $\frac{1}{24r} \mu_2 \Pi$ for $\delta_{j_2} \leq \delta_0$ and $Q_1 > Q_0$. Thus,

$$\sum_{\mathbf{b}_{j_2}} \sum_{\substack{P \in \mathcal{P}_n(\mathbf{b}_{j_2}) \\ \sigma'_P \text{-non-essential}}} \mu_2 \sigma_P \leq \sum_{\mathbf{b}_{j_2}} \sum_{\substack{P \in \mathcal{P}_n(\mathbf{b}_{j_2}) \\ \sigma'_P \text{-non-essential}}} \mu_2 \sigma'_P \leq \frac{1}{24r} \mu_2 \Pi$$

and together with the estimate (24), this implies that

$$\mu_2 L_{j_1, j_2} \leq \sum_{\mathbf{b}_{j_2}} \sum_{\substack{P \in \mathcal{P}_n(\mathbf{b}_{j_2}) \\ \sigma'_P \text{-essential}}} \mu_2 \sigma_P + \sum_{\mathbf{b}_{j_2}} \sum_{\substack{P \in \mathcal{P}_n(\mathbf{b}_{j_2}) \\ \sigma'_P \text{-non-essential}}} \mu_2 \sigma_P \leq \frac{1}{12r} \mu_2 \Pi.$$

3.3. The case of sub-intervals $T_{1,n}$ and $T_{2,n}$

For $|P'(\alpha_1)| \in T_{1,n}$ and $|P'(\alpha_2)| \in T_{2,n}$ we have the following system of inequalities:

$$\begin{cases} |P(x_i)| < h_n \cdot Q^{-v_i}, & v_i > 0, \\ c_{10}Q^{\frac{1}{2}-\frac{v_i}{2}} \leq |P'(\alpha_i)| < Q^{\frac{1}{2}-\frac{v_i}{2}+\frac{v_i}{2(n-1)}}, \\ v_1 + v_2 = n - 1, & i = 1, 2. \end{cases} \quad (26)$$

By Lemma 1, the set $L_{n,n}$ of solutions of the system (26) is contained in a union $\bigcup_{P \in \mathcal{P}_n(Q)} \sigma_P$, where

$$\sigma_P = \left\{ (x_1, x_2) \in \Pi : |x_i - \alpha_i| \leq 2^{n-1} h_n c_{10}^{-1} Q^{-\frac{v_i+1}{2}}, i = 1, 2 \right\}. \quad (27)$$

This leads to the following estimate for $\mu_2 L_{n,n}$:

$$\mu_2 L_{n,n} \leq \mu_2 \bigcup_{P \in \mathcal{P}_n(Q)} \sigma_P \leq \sum_{P \in \mathcal{P}_n(Q)} \mu_2 \sigma_P.$$

In this case we can not apply induction since the degree of the polynomial can not be reduced. Let us use a different method to estimate the measure $\mu_2 L_{n,n}$.

Cover the rectangle Π by a system of disjoint rectangles $\Pi_k = J_{1,k} \times J_{2,k}$, where $\mu_1 J_{i,k} = Q^{-\frac{v_i+1}{2} + \varepsilon_{2,i}}$, $i = 1, 2$, such that $\Pi \subset \bigcup_k \Pi_k$ and $\Pi_k \cap \Pi \neq \emptyset$. Thus, the number of rectangles Π_k can be estimated as follows:

$$\begin{aligned} & 2 \max \left\{ \frac{\mu_1 I_1}{\mu_1 J_{1,k}}, 1 \right\} \cdot 2 \max \left\{ \frac{\mu_1 I_2}{\mu_1 J_{2,k}}, 1 \right\} = \\ & = \begin{cases} 4Q^{\frac{n+1}{2} - \varepsilon_{2,1} - \varepsilon_{2,2}} \mu_2 \Pi, & s_i < \frac{v_i + 1}{2}, \\ 4Q^{\frac{v_1+1}{2} - \varepsilon_{2,1}} \mu_1 I_1, & s_1 < \frac{v_1 + 1}{2}, s_2 \geq \frac{v_2 + 1}{2}, \\ 4Q^{\frac{v_2+1}{2} - \varepsilon_{2,2}} \mu_1 I_2, & s_1 \geq \frac{v_1 + 1}{2}, s_2 < \frac{v_2 + 1}{2}. \end{cases} \quad (28) \end{aligned}$$

We are going to say that a polynomial P belongs to Π_k if there is a point $(x_1, x_2) \in \Pi_k$ such that the inequalities (26) are satisfied.

Now let us prove that there is no rectangle Π_k containing two or more irreducible polynomials $P \in \mathcal{P}_n(Q)$. Assume the converse: let $P_1, P_2 \in \Pi_k$ be irreducible polynomials and let the inequalities (26) hold for each polynomial P_j at a point $(x_{j,1}, x_{j,2}) \in \Pi_k$, $j = 1, 2$. Thus, for $Q > Q_0$ and for every point $(x_1, x_2) \in \Pi_k$, the estimates

$$|x_i - \alpha_{j,i}| \leq |x_i - x_{j,i}| + |x_{j,i} - \alpha_{j,i}| \leq 2Q^{-\frac{v_i+1}{2} + \varepsilon_{2,i}}, \quad (29)$$

are satisfied, where $x_{j,i} \in S(\alpha_{j,i})$.

Let us estimate the values $|P_j(x_i)|$, $i, j = 1, 2$ where $(x_1, x_2) \in \Pi_k$. Let us write Taylor expansions of P_j in the interval $J_{i,k}$:

$$P_j(x_i) = P'_j(\alpha_{j,i})(x_i - \alpha_{j,i}) + \dots + \frac{1}{n!} \cdot P_j^{(n)}(\alpha_{j,i})(x_i - \alpha_{j,i})^n.$$

From estimates (26) and (29) we obtain that

$$\begin{aligned} |P'_j(\alpha_{j,i})(x_i - \alpha_{j,i})| &\ll Q^{-v_i + \frac{v_i}{2(n-1)} + \varepsilon_{2,i}}, \\ \left| \frac{1}{k!} \cdot P_j^{(k)}(\alpha_{j,i})(x_i - \alpha_{j,i})^k \right| &\ll Q^{1 - \frac{k}{2} - \frac{kv_i}{2} + k\varepsilon_{2,i}} \ll Q^{-v_i + \frac{v_i}{2(n-1)} + \varepsilon_{2,i}} \end{aligned}$$

for $\varepsilon_{2,i} < \frac{v_i}{2(n-1)(k-1)}$ and $Q > Q_0$.

Then we can write the following estimate:

$$|P_j(x_i)| \ll Q^{-v_i + \frac{v_i}{2(n-1)} + \varepsilon_{2,i}} < Q^{-v_i + \frac{v_i}{2(n-1)} + \varepsilon_{2,i} + \varepsilon_3}, \quad (30)$$

where $\varepsilon_{2,i} < \frac{v_i}{2(n-1)^2}$.

From Lemma 3 for $\eta_i = \frac{v_i + 1}{2} - \varepsilon_{2,i}$ and $\tau_i = v_i - \frac{v_i}{2(n-1)} - \varepsilon_{2,i} - \varepsilon_3$, $i = 1, 2$, we have

$$\begin{aligned} \tau_1 + \tau_2 + 2 &= (n-1) - \frac{1}{2} - \varepsilon_{2,1} - \varepsilon_{2,2} + 2 - 2\varepsilon_3 = n + \frac{1}{2} - \varepsilon_{2,1} - \varepsilon_{2,2} - 2\varepsilon_3, \\ 2(\tau_i + 1 - \eta_i) &= 2 \left(v_i - \frac{v_i}{2(n-1)} - \varepsilon_{2,i} - \varepsilon_3 + 1 - \frac{v_i + 1}{2} + \varepsilon_{2,i} \right) = v_i + 1 - \frac{v_i}{n-1} - 2\varepsilon_3. \end{aligned}$$

Substitution of this expressions into (4) leads to the inequality

$$\tau_1 + \tau_2 + 2 + 2(\tau_1 + 1 - \eta_1) + 2(\tau_2 + 1 - \eta_2) = 2n + \frac{1}{2} - \varepsilon_{2,1} - \varepsilon_{2,2} - 6\varepsilon_3 \geq 2n + \frac{1}{8}$$

for $\varepsilon_{2,i} = \frac{v_i}{4(n-1)^2}$, $\varepsilon_3 = \frac{1}{48}$. This contradicts to Lemma 3 with $\delta = \frac{1}{8}$.

Hence, every rectangle Π_k contains at most one polynomial $P \in \mathcal{P}_n(Q)$. In this case, we have the following estimate for the measure of the set $L_{n,n}$:

$$\mu_2 L_{n,n} \leq \sum_{\Pi_k} \mu_2 \sigma_P,$$

and together with the estimates (27) and (28) this leads to

$$\mu_2 L_{n,n} \ll Q^{-\varepsilon_{2,1} - \varepsilon_{2,2}} \mu_2 \Pi < \frac{1}{12r} \mu_2 \Pi$$

for $Q > Q_0$ and $s_i < \frac{v_i + 1}{2}$, $i = 1, 2$. If $s_i \geq \frac{v_i + 1}{2}$, then we obtain the estimate

$$\mu_2 L_{n,n} \leq \sum_{P \in \mathcal{P}_n(Q)} \mu_2 \sigma_P \ll Q^{-\varepsilon_{2,i}} \mu_1 I_1 \mu_1 I_2 < \frac{1}{12r} \mu_2 \Pi$$

for $Q > Q_0$.

3.4. The case of a small derivative

Let us discuss a situation where $|P'(x_i)| \leq 2c_{10}Q^{\frac{1}{2} - \frac{v_i}{2}}$, $i = 1, 2$. In this case, we can show that $|P'(\alpha_i)| \leq 2^{n-1}c_{10}Q^{\frac{1}{2} - \frac{v_i}{2}}$, where $x_i \in S(\alpha_i)$.

Indeed, let $|P'(\alpha_i)| > 2^{n-1}c_{10}Q^{\frac{1}{2} - \frac{v_i}{2}}$. Let us write a Taylor expansion of the polynomial P' :

$$P'(x_i) = P'(\alpha_i) + P''(\alpha_i)(x_i - \alpha_i) + \dots + \frac{1}{(n-1)!} P^{(n)}(\alpha_i)(x_i - \alpha_i)^{n-1}.$$

Using our assumption and repeating analogous computations to those from the beginning of the proof of Lemma 5 (see page 61) we have:

$$\left| P'(\alpha_i)(x_i - \alpha_i) + \dots + \frac{1}{(n-1)!} P^{(n)}(\alpha_i)(x_i - \alpha_i)^{n-1} \right| \leq c_{10}Q^{\frac{1}{2} - \frac{v_i}{2}}.$$

This leads to the following upper bound for $|P'(\alpha_i)|$:

$$|P'(\alpha_i)| \leq 3c_{10}Q^{\frac{1}{2} - \frac{v_i}{2}},$$

which contradicts our assumption for $n \geq 3$.

Now let $L_{n+1,n+1} \subset \Pi$ be the set of points satisfying the system

$$\begin{cases} |P(x_i)| < h_n Q^{-v_i}, & v_i > 0, \\ |P'(\alpha_i)| < 2^{n-1} c_{10} Q^{\frac{1}{2} - \frac{v_i}{2}}, \\ v_1 + v_2 = n - 1, & i = 1, 2. \end{cases} \quad (31)$$

The polynomials $P \in \mathcal{P}_n(Q)$ satisfying (31) are going to be classified according to the distribution of their roots and the size of the leading coefficient $|a_m|$. This classification was introduced by Sprindžuk [1].

For every polynomial $P \in \mathcal{P}_n(Q)$ of degree $3 \leq m \leq n$ we define numbers $\rho_{1,j}$ and $\rho_{2,j}$, $2 \leq j \leq m$, as solutions of equations

$$|\alpha_{1,1} - \alpha_{1,j}| = Q^{-\rho_{1,j}}, \quad |\alpha_{2,1} - \alpha_{2,j}| = Q^{-\rho_{2,j}}.$$

Let us also define the vectors $\mathbf{k}_1 = (k_{1,2}, \dots, k_{1,m})$ and $\mathbf{k}_2 = (k_{2,2}, \dots, k_{2,m})$ with integer coefficients as solutions of the inequalities

$$k_{i,j} \varepsilon_4 - \varepsilon_4 \leq \rho_{i,j} < k_{i,j} \varepsilon_4, \quad i = 1, 2, j = \overline{2, m},$$

where $\varepsilon_4 > 0$ is some small constant.

Denote by $\mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u) \subset \mathcal{P}_n(Q)$ a subclass of polynomials with the same pair of vectors $(\mathbf{k}_1, \mathbf{k}_2)$ and the following bounds on leading coefficients: $Q^u \leq |a_m| < Q^{u+\varepsilon_4}$, where $u \in \mathbb{Z} \cdot \varepsilon_4$. Since $1 \leq |a_m| \leq Q$, the following estimate holds for u : $0 \leq u \leq 1 - \varepsilon_4$. The roots of the polynomial P are bounded, and we can write $Q \gg |\alpha_{j_1} - \alpha_{j_2}| \gg H^{-m+1} \gg Q^{-m+1}$, which leads to the estimates $-\frac{1}{\varepsilon_4} \leq k_{i,j} \leq \frac{m-1}{\varepsilon_4} + 1$. Thus, an integer vector $\mathbf{k}_i = (k_{i,2}, \dots, k_{i,m})$ can take at most $\left(\frac{m}{\varepsilon_4} + 1\right)^{m-1}$ values, the number of subclasses $\mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$ can be estimated as follows:

$$\#\{m, \mathbf{k}_1, \mathbf{k}_2, l\} \leq c_{16}^2 c_{17}, \quad (32)$$

where $c_{16} = \sum_{i=2}^n \left(\frac{i}{\varepsilon_4} + 1\right)^{i-1}$, $c_{17} = \varepsilon_4^{-1} + 1$.

Let $p_{i,j}$, $i = 1, 2$, $j = \overline{1, m}$ be defined as follows:

$$\begin{cases} p_{i,j} = (k_{i,j+1} + \dots + k_{i,m}) \cdot \varepsilon_4, & 1 \leq j \leq m - 1, \\ p_{i,j} = 0, & j = m. \end{cases} \quad (33)$$

For a polynomial $P \in \mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$, we can write the following estimates for its derivatives at the root α_i :

$$\begin{aligned} Q^{u-p_{i,1}} \leq |P'(\alpha_i)| &= |a_m| |\alpha_{i,1} - \alpha_{i,2}| \dots |\alpha_{i,1} - \alpha_{i,m}| \leq Q^{u-p_{i,1}+(m+1)\varepsilon_4}, \\ |P^{(j)}(\alpha_i)| &\ll |a_m| |\alpha_{i,1} - \alpha_{i,j+1}| \dots |\alpha_{i,1} - \alpha_{i,m}| \ll Q^{u-p_{i,j}+(m+1)\varepsilon_4}, \end{aligned} \quad (34)$$

$j = \overline{2, m}$.

Consider polynomials which solve the system (31). We can assume that the following inequalities hold:

$$\begin{aligned} Q^{u-p_{1,1}} \leq |P'(\alpha_1)| &\ll Q^{\frac{1}{2}-\frac{v_1}{2}}, \\ Q^{u-p_{2,1}} \leq |P'(\alpha_2)| &\ll Q^{\frac{1}{2}-\frac{v_2}{2}}, \end{aligned}$$

which leads to the inequalities

$$p_{1,1} > u + \frac{v_1 - 1}{2}, \quad p_{2,1} > u + \frac{v_2 - 1}{2}. \quad (35)$$

Now let us obtain an estimate for the measure of the set $L_{n+1, n+1}$. From Lemma 1 it follows that this set is contained in a union $\bigcup_{m, \mathbf{k}_1, \mathbf{k}_2, u} \bigcup_{P \in \mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)} \sigma_P$,

where

$$\sigma_P = \left\{ (x_1, x_2) \in \Pi : |x_i - \alpha_i| \leq \min_{1 \leq j \leq m} \left(2^{m-j} \frac{h_n \cdot Q^{-v_i}}{|P'(\alpha_{i,1})|} \cdot |\alpha_{i,1} - \alpha_{i,2}| \dots |\alpha_{i,1} - \alpha_{i,j}| \right)^{1/j} \right\}.$$

This, together with the previous notation (33) and the estimates (34), yields the formula

$$\sigma_P = \left\{ (x_1, x_2) \in \Pi : |x_i - \alpha_i| \leq \frac{1}{2} \cdot \min_{1 \leq j \leq m} \left((2^m h_n)^{1/j} \cdot Q^{\frac{-u-v_i+p_{i,j}}{j}} \right), i = 1, 2 \right\} \quad (36)$$

for $P \in \mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$. If the inequalities

$$(2^m h_n)^{1/m_i} \cdot Q^{\frac{-u-v_i+p_{i,m_i}}{m_i}} \leq (2^m h_n)^{1/k} \cdot Q^{\frac{-u-v_i+p_{i,k}}{k}}, \quad 1 \leq k \leq m, i = 1, 2, \quad (37)$$

are satisfied, then the numbers $j = m_1$ and $j = m_2$ provide the best estimates for the roots α_1 and α_2 respectively, and the inequalities

$$\sigma_P = \left\{ (x_1, x_2) \in \Pi : |x_i - \alpha_i| \leq \frac{1}{2} \cdot (2^m h_n)^{1/m_i} \cdot Q^{-\frac{-u-v_i+p_{i,m_i}}{m_i}}, i = 1, 2 \right\} \quad (38)$$

hold.

Let us cover the rectangle Π by a system of disjoint rectangles $\Pi_{m_1, m_2} = J_{m_1} \times J_{m_2}$, where $\mu_1 J_{m_i} = Q^{-\frac{u+v_i-p_{i,m_i}}{m_i} + \varepsilon_5}$, such that $\Pi \subset \bigcup \Pi_{m_1, m_2}$ and $\Pi_{m_1, m_2} \cap \Pi \neq \emptyset$. The number of rectangles Π_{m_1, m_2} can be estimated as follows:

$$\#\Pi_{m_1, m_2} \leq 4 \cdot Q^{\frac{u+v_1-p_{1,m_1}}{m_1} + \frac{u+v_2-p_{2,m_2}}{m_2} - 2\varepsilon_5} \mu_2 \Pi. \quad (39)$$

Now let us show that there is no rectangle Π_{m_1, m_2} containing two or more irreducible polynomials. Let $P_1, P_2 \in \Pi_{m_1, m_2}$ be irreducible polynomials, and let the inequalities (31) hold for polynomials P_j at points $(x_{j,1}, x_{j,2}) \in \Pi_{m_1, m_2}$, $j = 1, 2$. Thus, estimates

$$|x_i - \alpha_{j,i}| \leq |x_i - x_{j,i}| + |x_{j,i} - \alpha_{j,i}| \leq 2 \cdot Q^{-\frac{u+v_i-p_{i,m_i}}{m_i} + \varepsilon_5} \quad (40)$$

are satisfied for every point $(x_1, x_2) \in \Pi_{m_1, m_2}$ and for $Q > Q_0$, where $x_{j,i} \in S(\alpha_{j,i})$.

Let us estimate $|P_j(x_i)|$, where $(x_1, x_2) \in \Pi_{m_1, m_2}$. Let us write Taylor expansions of the polynomials P_j in the interval J_{m_i} :

$$P_j(x_i) = P'_j(\alpha_{j,i})(x_i - \alpha_{j,i}) + \dots + \frac{1}{m!} \cdot P_j^{(m)}(\alpha_{j,i})(x_i - \alpha_{j,i})^m.$$

By estimates (34), (37) and (40) we have

$$\left| \frac{1}{k!} \cdot P_j^{(k)}(\alpha_{j,i})(x_i - \alpha_{j,i})^k \right| \ll Q^{-v_i + (m+1)\varepsilon_4 + k\varepsilon_5}.$$

This leads to the following estimates for $|P_j(x_i)|$:

$$|P_j(x_i)| \ll Q^{-v_i + (m+1)\varepsilon_4 + m\varepsilon_5} < Q^{-v_i + (m+1)(\varepsilon_4 + \varepsilon_5)}. \quad (41)$$

From Lemma 3 with $\eta_i = \frac{u + v_i - p_{i,m_i}}{m_i} - \varepsilon_5$ and $\tau_i = v_i - (m+1)(\varepsilon_4 + \varepsilon_5)$,

where $i = 1, 2$ and $\varepsilon_4 = \frac{1}{12(m+1)}$, $\varepsilon_5 = \frac{1}{4(3m+1)}$, we obtain

$$\begin{aligned} \tau_1 + \tau_2 + 2 &= n + 1 - \frac{1}{6} - 2(m+1)\varepsilon_5, \\ 2(\tau_i + 1 - \eta_i) &= 2v_i + 2 - 2 \cdot \frac{u + v_i - p_{i,m_i}}{m_i} - \frac{1}{6} - 2m\varepsilon_5. \end{aligned}$$

Let us estimate the expression $2(\tau_i + 1 - \eta_i)$ by applying the inequalities (35):

$$2(\tau_i + 1 - \eta_i) \geq \begin{cases} v_i + 2 - u + \frac{2p_{i,m_i}}{m} - \frac{1}{6} - 2m\varepsilon_5, & m_i \geq 2, \\ v_i + 1 - \frac{1}{6} - 2m\varepsilon_5, & m_i = 1, \end{cases} \geq v_i + 1 - \frac{1}{6} - 2m\varepsilon_5.$$

Substituting these expressions into (4) yields

$$\tau_1 + \tau_2 + 2 + 2(\tau_1 + 1 - \eta_1) + 2(\tau_2 + 1 - \eta_2) = 2n + \frac{3}{2} - (6m + 2)\varepsilon_5 > 2n + \frac{1}{2},$$

which contradicts to Lemma 3 with $\delta = \frac{1}{2}$.

This means that every rectangle Π_{m_1, m_2} contains at most one polynomial $P \in \mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$. Thus, the measure of solutions of the system (31) can be estimated as follows:

$$\mu_2 L_{n+1, n+1} \leq \sum_{m, \mathbf{k}_1, \mathbf{k}_2, u} \sum_{P \in \mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)} \mu_2 \sigma_P \leq \sum_{m, \mathbf{k}_1, \mathbf{k}_2, u} \sum_{\Pi_{m_1, m_2}} \mu_2 \sigma_P.$$

Thus, by estimates (32), (38) and (39), we can obtain the inequality

$$\mu_2 L_{n+1, n+1} \ll Q^{-2\varepsilon_5} \cdot \mu_2 \Pi < \frac{1}{12r} \mu_2 \Pi$$

for $Q > Q_0$.

3.5. Mixed cases

The case of sub-intervals $T_{1,n}, T_{2,j} (T_{1,j}, T_{2,n}), j = \overline{2, n - 1}$

Consider the system of inequalities

$$\begin{cases} |P(x_i)| < h_n \cdot Q^{-v_i}, & v_i > 0, \\ c_{10}Q^{\frac{1}{2}-\frac{v_1}{2}} \leq |P'(\alpha_1)| < Q^{\frac{1}{2}-\frac{v_1}{2}+\frac{v_1}{2(n-1)}}, \\ Q^{\frac{1}{2}-\frac{(j-1)v_2}{2(n-1)}} \leq |P'(\alpha_2)| < Q^{\frac{1}{2}-\frac{v_2(j-2)}{2(n-1)}}, \\ v_1 + v_2 = n - 1, & i = 1, 2. \end{cases} \quad (42)$$

Let $L_{n,j}$ be the set of solutions of the system (42). In this case we need to consider two different sets. Let $L_{n,j}^1$ and $L_{n,j}^2$ be the sets of points $(x_1, x_2) \in \Pi$ such that there exists a polynomial $P \in \mathcal{P}_n(Q)$ satisfying the system (42) under condition $c_{10}Q^{\frac{1}{2}-\frac{v_1}{2}} \leq |P'(\alpha_1)| < Q^{\frac{1}{2}-\frac{v_1}{2}+\frac{v_1}{4(n-1)}}$ and $Q^{\frac{1}{2}-\frac{v_1}{2}+\frac{v_1}{4(n-1)}} \leq |P'(\alpha_1)| < Q^{\frac{1}{2}-\frac{v_1}{2}+\frac{v_1}{2(n-1)}}$ respectively.

As in the case of small derivatives, we classify polynomials $P \in \mathcal{P}_n(Q)$ according to the distribution of their roots and the size of their leading coefficients. We will consider the subclasses of polynomials $\mathcal{P}_m(Q, \mathbf{k}_2, u)$ with the same vector \mathbf{k}_2 and the following bounds on leading coefficient: $Q^u < |a_m| < Q^{u+\varepsilon_4}$, where $0 \leq u \leq 1 - \varepsilon_4$, $0 < \varepsilon_4 < 1$ and $u \in \mathbb{Z} \cdot \varepsilon_4$. Then

$$\#\{m, \mathbf{k}_2, u\} \leq c_{17} \cdot c_{16}. \quad (43)$$

From Lemma 1, the set $L_{n,j}^g, g = 1, 2$ is contained in a union $\bigcup_{m, \mathbf{k}_2, u} \bigcup_{P \in \mathcal{P}_m(Q, \mathbf{k}_2, u)} \sigma_P$, where

$$\sigma_P = \left\{ (x_1, x_2) \in \Pi : \begin{cases} |x_1 - \alpha_1| \leq 2^{m-1} h_n \max\{c_{10}^{-1}, 1\} \cdot Q^{-\frac{v_1}{2}-\frac{1}{2}-\frac{v_1(g-1)}{4(n-1)}}, \\ |x_2 - \alpha_2| \leq 2^{m-1} h_n Q^{-v_2+p_{2,1}-u} \end{cases} \right\}. \quad (44)$$

Define the value $l = v_2 - p_{2,1} + u - k_{2,2}\varepsilon_4$ and let us write l as $l = [l] + \{l\}$, where $[l]$ is the integer part of l and $\{l\}$ is the fractional part. Now let us cover the rectangle Π by a system of disjoint rectangles $\Pi_k = J_{1,k} \times J_{2,k}$, where

$\mu_1 J_{1,k} = Q^{-\frac{v_1}{2} - \frac{1}{2} - \frac{v_1(g-1)}{4(n-1)} + \varepsilon_6}$ and $\mu_1 J_{2,k} = Q^{-k_{2,2}\varepsilon_4 - \{l\}}$, such that $\Pi \subset \bigcup_k \Pi_k$ and $\Pi_k \cap \Pi \neq \emptyset$. The number of rectangles $\Pi_k \in \Pi$ can be estimated as

$$\#\{\Pi_k\} \leq 4Q^{\frac{v_1}{2} + \frac{1}{2} + \frac{v_1(g-1)}{4(n-1)} + k_{2,2}\varepsilon_4 - \varepsilon_6 + \{l\}} \mu_2 \Pi. \quad (45)$$

Assume that every rectangle Π_k contains no more than $2^m Q^{l+\frac{\varepsilon_6}{2}}$ points (α_1, α_2) , where α_1, α_2 are the roots of polynomial $P \in \mathcal{P}_m(Q, \mathbf{k}_2, u)$. Then by inequalities (43), (44) and (45) it follows that the measure of the set $L_{n,j}^g$ can be estimated as:

$$\begin{aligned} \mu_2 L_{n,j}^g &\leq 2^{3m+4} h_n^2 \max\{c_{10}^{-1}, 1\} c_{16} c_{17} \cdot Q^{-v_2 + p_{2,1} - u + k_{2,2}\varepsilon_4 - \frac{\varepsilon_6}{2} + l + \{l\}} \mu_2 \Pi \leq \\ &Q^{-\frac{\varepsilon_6}{2}} \mu_2 \Pi \leq \frac{1}{24r} \mu_2 \Pi, \end{aligned} \quad (46)$$

where $Q > Q_0$.

Now assume that there exists a rectangle Π_k containing more than $2^m Q^{l+\frac{\varepsilon_6}{2}}$ polynomials $P_j \in \mathcal{P}_m(Q, \mathbf{k}_2, u)$. From the Taylor expansions of polynomials P_j in the interval $J_{2,k}$, the estimates (34) and condition $(\alpha_{j,1}, \alpha_{j,2}) \in \Pi_k$ it follows that

$$\left| \frac{1}{k!} \cdot P_j^{(k)}(\alpha_{j,2})(x_2 - \alpha_{j,2})^k \right| \ll Q^{u - p_{2,k} + (m+1)\varepsilon_4 - k \cdot k_{2,2}\varepsilon_4 - k\{l\}} < Q^{u - p_{2,1} - k_{2,2}\varepsilon_4 - \{l\} + (m+1)\varepsilon_4},$$

which allows us to write

$$|P_j(x_2)| < Q^{u - p_{2,1} - k_{2,2}\varepsilon_4 - \{l\} + (m+2)\varepsilon_4}. \quad (47)$$

Similarly, repeating the calculations by analogy with Section 3.3 (see inequality (30)), we have

$$|P_j(x_1)| < Q^{-v_1 + \frac{v_1}{4(n-1)} + 2\varepsilon_6} \quad (48)$$

for $\varepsilon_6 < \frac{v_1}{(n-1)^2}$.

By Dirichlet's principle we can find at least $\left[Q^{\frac{\varepsilon_6}{2}} \right] + 1$ polynomials from $\mathcal{P}_m(Q, \mathbf{k}_2, u)$ contained in Π_k such that their coefficients a_m, \dots, a_{m+1-l} coincide. Let us call them $P_1, \dots, P_{\left[Q^{\frac{\varepsilon_6}{2}} \right] + 1}$. If $l = 0$, then we can simply ignore this step.

Let us consider the differences $R_{i,j} = P_i - P_j$, $1 \leq i < j \leq \left[Q^{\frac{\varepsilon_6}{2}} \right] + 1$.

From the inequalities (47) and (48), we obtain that at every point of the rectangle Π_k the polynomials $R_{i,j}$ satisfy

$$\begin{cases} |R_{i,j}(x_1)| < 2Q^{-v_1 + \frac{v_1}{4(n-1)} + 2\varepsilon_6}, & |R_{i,j}(x_2)| < 2Q^{u - p_{2,1} - k_{2,2}\varepsilon_4 - \{l\} + (m+2)\varepsilon_4}, \\ \deg R_{i,j} \leq m - [l] = m - v_2 + p_{2,1} - u + k_{2,2}\varepsilon_4 + \{l\}. \end{cases} \quad (49)$$

Assume that among polynomials $R_{i,j}$ we can find at least two polynomials without common roots. Then we can apply Lemma 3 with $\tau_1 = v_1 - \frac{v_1}{4(n-1)} - 2\varepsilon_6$, $\tau_2 = -u + p_{2,1} + k_{2,2}\varepsilon_4 + \{l\} - (m+2)\varepsilon_4$, $\eta_1 = \frac{v_1}{2} + \frac{1}{2} + \frac{v_1(g-1)}{4(n-1)} - \varepsilon_6$, $\eta_2 = k_{2,2}\varepsilon_4 + \{l\}$, so that we have

$$\tau_1 + 1 = v_1 + 1 - \frac{v_1}{4(n-1)} - 2\varepsilon_6, \quad \tau_2 + 1 = 1 - u + p_{2,1} + k_{2,2}\varepsilon_4 + \{l\} - (m+2)\varepsilon_4,$$

$$2(\tau_1 + 1 - \eta_1) = v_1 + 1 - \frac{gv_1}{2(n-1)} - 2\varepsilon_6, \quad 2(\tau_2 + 1 - \eta_2) = 2 - 2u + 2p_{2,1} - 2(m+2)\varepsilon_4.$$

Substituting these expressions into (4) for $\varepsilon_4 = \frac{1 - \{l\}}{9(m+2)}$ and $\varepsilon_6 = \frac{1 - \{l\}}{12}$ yields

$$\begin{aligned} \tau_1 + \tau_2 + 2 + 2(\tau_1 + 1 - \eta_1) + 2(\tau_2 + 1 - \eta_2) &= 2v_1 + 5 - \frac{(2+g)v_1}{4(n-1)} + 3p_{2,1} + k_{2,2}\varepsilon_4 - 3u + \\ &+ \{l\} - 3(m+2)\varepsilon_4 - 4\varepsilon_6 \geq 2n - 2v_2 + 2p_{2,1} + 2k_{2,2}\varepsilon_4 - 2u + (p_{2,1} - k_{2,2}\varepsilon_4) + \\ &+ (1-u) + \{l\} + 2 - \frac{2(1 - \{l\})}{3} - \frac{v_1}{n-1} \geq 2(m - [l]) - \frac{\{l\}}{3} + \frac{1}{3} \end{aligned}$$

This inequality contradict to Lemma 3 for $\delta = \frac{1 - \{l\}}{3}$.

The case when among polynomials $R_{i,j}$, $1 \leq i < j \leq \left[Q^{\frac{\varepsilon_6}{2}} \right] + 1$ we can not find two polynomials without common roots is considered in [10].

Hence, we obtain

$$\mu_2 L_{n,j} \leq \mu_2 L_{n,j}^1 + \mu_2 L_{n,j}^2 \leq \frac{1}{12r} \mu_2 \Pi.$$

The case where one derivative is small and the other derivative lies in the sub-interval $T_{2,j}$, $j = \overline{2, n}$ ($T_{2,j}$)

Given the estimate for $|P'(\alpha_1)|$ obtained in Section 3.4 for $|P'(x_1)| \leq 2c_{10}Q^{\frac{1}{2}-\frac{v_1}{2}}$ consider the system of inequalities

$$\left\{ \begin{array}{l} |P(x_i)| < h_n \cdot Q^{-v_i}, \quad v_i > 0, \\ |P'(\alpha_1)| < 2^{n-1}c_{10}Q^{\frac{1}{2}-\frac{v_1}{2}}, \\ Q^{\frac{1}{2}-\frac{(j-1)v_2}{2(n-1)}} \leq |P'(\alpha_2)| < Q^{\frac{1}{2}-\frac{(j-2)v_2}{2(n-1)}}, \\ v_1 + v_2 = n - 1, \quad i = 1, 2. \end{array} \right. \quad (50)$$

Denote by $L_{n+1,j}$ the set of points $(x_1, x_2) \in \Pi$ such that there exists a polynomial $P \in \mathcal{P}_n(Q)$ satisfying the system (50). Once again let us classify polynomials $P \in \mathcal{P}_n(Q)$ according to the distribution of their roots and the size of leading coefficients. We will consider the subclasses of polynomials $\mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$ defined above.

From Lemma 1 by analogy with Section 3.4 (see inequality (36)) we conclude that the set $L_{n+1,j}$ is contained in a union $\bigcup_{m, \mathbf{k}_1, \mathbf{k}_2, u} \bigcup_{P \in \mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)} \sigma_P$, where

$$\sigma_P = \left\{ (x_1, x_2) \in \Pi : \begin{array}{l} |x_1 - \alpha_1| \leq \frac{1}{2} \min_{1 \leq j \leq m} \left((2^m h_n)^{1/j} \cdot Q^{\frac{-u-v_1+p_{1,j}}{j}} \right), \\ |x_2 - \alpha_2| \leq 2^{m-1} h_n Q^{-u-v_2+p_{2,1}} \end{array} \right\}$$

for $P \in \mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$.

If the inequalities (37) hold for $i = 1$, then the estimate numbered as $j = m_1$ is optimal for the root α_1 , and we have

$$\sigma_P = \left\{ (x_1, x_2) \in \Pi : \begin{array}{l} |x_1 - \alpha_1| \leq \frac{1}{2} \cdot (2^m h_n)^{1/m_1} \cdot Q^{\frac{-u-v_1+p_{1,m_1}}{m_1}}, \\ |x_2 - \alpha_2| \leq 2^{m-1} h_n Q^{-u-v_2+p_{2,1}} \end{array} \right\}. \quad (51)$$

Define the value $l = v_2 - p_{2,1} + u - k_{2,2}\varepsilon_4$ as in the previous case and let us cover the rectangle Π by a system of disjoint rectangles $\Pi_k = J_{1,k} \times J_{2,k}$,

where $\mu_1 J_{1,k} = Q^{-\frac{u+v_1-p_{1,m_1}}{m_1}+\varepsilon_7}$ and $\mu_1 J_{2,k} = Q^{-k_{2,2}\varepsilon_4-\{l\}}$, such that $\Pi \subset \bigcup_k \Pi_k$ and $\Pi_k \cap \Pi \neq \emptyset$. The number of rectangles $\Pi_k \in \Pi$ can be estimated as

$$\#\{\Pi_k\} \leq 4Q^{\frac{u+v_1-p_{1,m_1}}{m_1}+k_{2,2}\varepsilon_4+\{l\}-\varepsilon_7} \mu_2 \Pi. \quad (52)$$

Let every rectangle Π_k contain no more than $2^m Q^{l+\frac{\varepsilon_7}{2}}$ polynomials $P \in \mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$. Then by inequalities (50), (32) and (52) it follows that the measure of the set $L_{n+1,j}$ can be estimated as:

$$\mu_2 L_{n+1,j} \ll Q^{-u-v_2+p_{2,1}+k_{2,2}\varepsilon_4-\frac{\varepsilon_7}{2}+l+\{l\}} \mu_2 \Pi \ll Q^{-\frac{\varepsilon_7}{2}} \mu_2 \Pi \leq \frac{1}{12r} \mu_2 \Pi,$$

where $Q > Q_0$.

Now assume that there exists a rectangle Π_k containing more than $2^m Q^{l+\frac{\varepsilon_7}{2}}$ points (α_1, α_2) , where α_1, α_2 are the roots of polynomial $P_j \in \mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$. Using the calculations described in the previous case (see estimate (47)) and in Section 3.4 (see estimate (41)) for every point $(x_1, x_2) \in \Pi_k$ we have:

$$|P_j(x_1)| < Q^{-v_1+(m+1)(\varepsilon_4+\varepsilon_7)}, \quad |P_j(x_2)| < Q^{u-p_{2,1}-k_{2,2}\varepsilon_4-\{l\}+(m+2)\varepsilon_4}. \quad (53)$$

By Dirichlet's principle we can find at least $\left[Q^{\frac{\varepsilon_7}{2}} \right] + 1$ polynomials from $\mathcal{P}_m(Q, \mathbf{k}_1, \mathbf{k}_2, u)$ contained in Π_k such that their coefficients a_m, \dots, a_{m+1-l} coincide. Let us call them $P_1, \dots, P_{\left[Q^{\frac{\varepsilon_7}{2}} \right] + 1}$. Thus, let us consider the differences $R_{i,j} = P_i - P_j$, where $1 \leq i < j \leq \left[Q^{\frac{\varepsilon_7}{2}} \right] + 1$.

From the inequalities (53), we obtain that at every point of the rectangle Π_k the polynomials $R_{i,j}$ satisfy

$$\begin{cases} |R_{i,j}(x_1)| < 2Q^{-v_1+(m+1)(\varepsilon_4+\varepsilon_7)}, & |R_{i,j}(x_2)| < 2Q^{u-p_{2,1}-k_{2,2}\varepsilon_4-\{l\}+(m+2)\varepsilon_4}, \\ \deg R_{i,j} \leq m - [l] = m - v_2 + p_{2,1} - u + k_{2,2}\varepsilon_4 + \{l\}. \end{cases}$$

Assume that among polynomials $R_{i,j}$ we can find at least two polynomials without common roots and apply Lemma 3 with $\tau_1 = v_1 - (m+1)(\varepsilon_4 + \varepsilon_7)$, $\tau_2 = -u + p_{2,1} + k_{2,2}\varepsilon_4 + \{l\} - (m+2)\varepsilon_4$, $\eta_1 = \frac{u + v_1 - p_{1,m_1}}{m_1} - \varepsilon_7$, $\eta_2 = k_{2,2}\varepsilon_4 + \{l\}$,

so that we have

$$\tau_1 + 1 = v_1 + 1 - (m + 1)(\varepsilon_4 + \varepsilon_7), \quad \tau_2 + 1 = 1 - u + p_{2,1} + k_{2,2}\varepsilon_4 + \{l\} - (m + 2)\varepsilon_4,$$

and repeating the arguments from the end of Section 3.4 we obtain

$$2(\tau_1 + 1 - \eta_1) \geq v_1 + 1 - 2(m + 1)\varepsilon_4 - 2m\varepsilon_7, \quad 2(\tau_2 + 1 - \eta_2) = 2 - 2u + 2p_{2,1} - 2(m + 2)\varepsilon_4.$$

Substituting these expressions into (4) for $\varepsilon_4 = \frac{1}{48(m + 2)}$ and $\varepsilon_7 = \frac{1}{8(3m + 1)}$ yields

$$\begin{aligned} \tau_1 + \tau_2 + 2 + 2(\tau_1 + 1 - \eta_1) + 2(\tau_2 + 1 - \eta_2) &\geq 2v_1 + 5 + 3p_{2,1} + k_{2,2}\varepsilon_4 - 3u + \{l\} - \frac{1}{4} \geq \\ &\geq 2n - 2v_2 + 2p_{2,1} + 2k_{2,2}\varepsilon_4 - 2u + \{l\} + \frac{7}{4} \geq 2(m - [l]) - \{l\} + 1 + \frac{3}{4} \geq 2(m - [l]) + \frac{3}{4}. \end{aligned}$$

This inequality contradicts to Lemma 3 with $\delta = \frac{3}{4}$.

If among polynomials $R_{i,j}$, $1 \leq i < j \leq \left[Q^{\frac{\varepsilon_7}{2}} \right] + 1$ we can not find two polynomials without common roots then we use the arguments described in [10].

This section concludes the proof of Lemma in case of irreducible polynomials.

We have

$$\mu_2 L_1 \leq \sum_{2 \leq i, j \leq n+1} \mu_2 L_{i,j} \leq (n - 1)^2 \cdot \frac{1}{12r} \cdot \mu_2 \Pi \leq \frac{1}{12} \cdot \mu_2 \Pi.$$

Similarly we obtain $\mu_2 L_2 \leq \frac{1}{12} \cdot \mu_2 \Pi$.

3.6. The case of reducible polynomials

Let us estimate the measure of the set L_3 . Let a polynomial P of degree n be a product of several (not necessarily different) irreducible polynomials P_1, P_2, \dots, P_s , $s \geq 2$, where $\deg P_i = n_i \geq 2$ and $n_1 + \dots + n_s = n$. Then by Lemma 4 we have:

$$H(P_1) \cdot H(P_2) \cdot \dots \cdot H(P_s) \leq c_9 H(P) \leq c_9 Q.$$

On the other hand, by the definition of height, we have $H(P_i) \geq 1$, and thus $H(P_i) \leq c_9 Q = Q_1$, $i = 1, \dots, s$.

Denote by $L_3(k)$ a set of points $(x_1, x_2) \in \Pi$ such that there exists a polynomial $R \in \mathcal{P}_k(Q_1)$ satisfying the inequality:

$$|R(x_1)R(x_2)| < h_n^2 Q_1^{-k+\frac{1}{2}}. \quad (54)$$

If a polynomial $P \in \mathcal{P}_n(Q)$ satisfies the inequalities (5) at a point $(x_1, x_2) \in \Pi$, we can write

$$|P(x_1)P(x_2)| = |P_1(x_1)P_1(x_2)| \cdot \dots \cdot |P_s(x_1)P_s(x_2)| \leq h_n^2 Q^{-n+1}.$$

Since $n = n_1 + \dots + n_s$ and $s \geq 2$, it is easy to see that at least one of the inequalities

$$|P_i(x_1)P_i(x_2)| \leq h_n^2 Q^{-n_i+\frac{1}{2}}, \quad i = 1, \dots, s,$$

is satisfied at the point (x_1, x_2) . Hence, $(x_1, x_2) \in L_3(n_j)$ and we have

$$L_3 \subset \bigcup_{k=2}^{n-2} L_3(k).$$

Let us estimate the measure of the set $L_3(k)$, $2 \leq k \leq n-2$. Denote by $L_3^1(k, t)$ a set of points $(x_1, x_2) \in \Pi$ such that there exists a polynomial $P \in \mathcal{P}_k(Q_1)$ satisfying the inequalities:

$$\begin{cases} |P(x_1)| < h_n^2 Q_1^t, & |P(x_2)| < h_n^2 Q_1^{-k+1-t}, \\ \min_i \{|P'(\alpha_i)|\} < \delta_k Q_1, & x_i \in S(\alpha_i), i = 1, 2. \end{cases} \quad (55)$$

and by $L_3^2(k, t)$ a set of points $(x_1, x_2) \in \Pi$ such that there exists a polynomial $P \in \mathcal{P}_k(Q_1)$ satisfying the inequality:

$$\begin{cases} |P(x_1)| < h_n^2 Q_1^t, & |P(x_2)| < h_n^2 Q_1^{-k+\frac{3}{4}-t}, \\ |P'(\alpha_i)| > \delta_k Q_1, & x_i \in S(\alpha_i), \quad i = 1, 2. \end{cases} \quad (56)$$

By the definition of the set $L_3(k)$ it is easy to see that:

$$L_3(k) \subset \left(\bigcup_{i=0}^{2k} L_3^1(k, 1-i/2) \right) \cup \left(\bigcup_{i=0}^{4k+1} L_3^2(k, 1-i/4) \right).$$

The system (55) is a system of the form (5). Hence, as the polynomials $P \in \mathcal{P}_k(Q_1)$ are irreducible and $k < n$, we can apply the induction hypothesis to obtain the following estimate:

$$\mu_2 L_3^1(k, t) < \frac{1}{2^6 n^2} \cdot \mu_2 \Pi \tag{57}$$

for $Q_1 > Q_0$ and sufficiently small δ_k .

Now let us estimate the measure of the set $L_3^2(k, t)$. From Lemma 1 it follows that $L_3^2(k, t)$ is contained in a union $\bigcup_{P \in \mathcal{P}_k(Q)} \sigma_P(t)$, where

$$\sigma_P(t) := \left\{ (x_1, x_2) \in \Pi : \begin{array}{l} |x_1 - \alpha_1| \leq 2^{k-1} h_n^2 \cdot Q^t \cdot |P'(\alpha_1)|^{-1}, \\ |x_2 - \alpha_2| \leq 2^{k-1} h_n^2 \cdot Q^{-k+\frac{3}{4}-t} \cdot |P'(\alpha_2)|^{-1}. \end{array} \right\}$$

Let us estimate the value of the polynomial P at a central point \mathbf{d} of the square Π . A Taylor expansion of the polynomial P can be written as follows:

$$P(d_i) = P'(\alpha_i)(d_i - \alpha_i) + \frac{1}{2} P''(\alpha_i)(d_i - \alpha_i)^2 + \dots + \frac{1}{k!} \cdot P^{(k)}(\alpha_i)(d_i - \alpha_i)^k. \tag{58}$$

If polynomial P satisfy (56) at point $(x_{0,1}, x_{0,2}) \in \Pi$ then:

$$\begin{aligned} |d_1 - \alpha_1| &\leq |d_1 - x_{0,1}| + |x_{0,1} - \alpha_1| \leq \mu_1 I_1 + 2^{k-1} h_n^2 \delta_k^{-1} \cdot Q_1^{t-1}, \\ |d_2 - \alpha_2| &\leq |d_2 - x_{0,2}| + |x_{0,2} - \alpha_2| \leq \mu_1 I_2 + 2^{k-1} h_n^2 \delta_k^{-1} \cdot Q_1^{-k+\frac{3}{4}-t-1}. \end{aligned} \tag{59}$$

Without loss of generality, let us assume that $t \geq -k + \frac{3}{4} - t$. Then we can rewrite the estimates (59) as follows:

$$|d_1 - \alpha_1| \leq \begin{cases} c_{18} \cdot \mu_1 I_1, & t < 1 - s_1, \\ c_{18} \cdot Q_1^{t-1}, & 1 - s_1 \leq t \leq 1, \end{cases} \quad |d_2 - \alpha_2| \leq \mu_1 I_2.$$

where $c_{18} = 2^{k-1} h_n^2 \delta_k^{-1} + c_{1,1}$. We mention that $\Pi = I_1 \times I_2$, $\mu_1 I_i = c_{1,i} Q^{-s_i}$, $i = 1, 2$ and $s_1 \geq s_2$.

Using these inequalities and expression (58) allows us to write

$$|P(d_1)| < \begin{cases} c_{19} \cdot Q_1 \cdot \mu_1 I_1, & t < 1 - s_1, \\ c_{19} \cdot Q_1^t, & 1 - s_1 \leq t < 1, \end{cases} \quad |P(d_2)| < c_{19} \cdot Q_1 \cdot \mu_1 I_2. \quad (60)$$

Fix a vector $\mathbf{A}_1 = (a_k, \dots, a_2)$, where a_k, \dots, a_2 will denote the coefficients of the polynomial $P \in \mathcal{P}_k(Q_1)$. Consider a subclass $\mathcal{P}_k(\mathbf{A}_1)$ of polynomials P which satisfy (56) and have the same vector of coefficients \mathbf{A}_1 . For $Q_1 > Q_0$, the number of such vectors can be estimated as follows

$$\#\{\mathbf{A}_1\} = (2Q_1 + 1)^{k-1} < 2^k Q_1^{k-1}. \quad (61)$$

Let us estimate the value $\#\mathcal{P}_k(\mathbf{A}_1)$. Take a polynomial $P_0 \in \mathcal{P}_k(\mathbf{A}_1)$ and consider the difference between the polynomials P_0 and $P_j \in \mathcal{P}_k(\mathbf{A}_1)$ at points d_i , $i = 1, 2$. By (60), we have that:

$$|P_0(d_1) - P_j(d_1)| = |(a_{0,1} - a_{j,1})d_1 + (a_{0,0} - a_{j,0})| \leq \begin{cases} 2c_{19} \cdot Q_1 \mu_1 I_1, & t < 1 - s_1, \\ 2c_{19} \cdot Q_1^t, & 1 - s_1 \leq t \leq 1, \end{cases}$$

$$|P_0(d_2) - P_j(d_2)| = |(a_{0,1} - a_{j,1})d_2 + (a_{0,0} - a_{j,0})| \leq 2c_{19} \cdot Q_1 \mu_1 I_2.$$

This implies that the number of different polynomials $P_j \in \mathcal{P}_k(\mathbf{A}_1)$ does not exceed the number of integer solutions of the system

$$|b_1 d_i + b_0| \leq K_i, \quad i = 1, 2, \quad (62)$$

where $K_2 = 2c_{19} \cdot Q_1 \mu_1 I_2$ and $K_1 = 2c_{19} \cdot Q_1 \mu_1 I_1$ if $t < 1 - s_1$ and $K_1 = 2c_{19} \cdot Q_1^t$ if $1 - s_1 \leq t \leq 1$. It is easy to see that $K_i \geq 2c_{19} \cdot Q_1^{1-s_i} > Q_1^\varepsilon > 1$ for $Q_1 > Q_0$. Thus, using the scheme described in Section 3.1 to solve the system (62) we have

$$\#\mathcal{P}_k(\mathbf{A}_1) \leq \begin{cases} 32\varepsilon^{-1} \cdot Q_1^2 \cdot \mu_2 \Pi, & t < 1 - s_1, \\ 32\varepsilon^{-1} \cdot Q_1^{t+1} \cdot \mu_1 I_2, & 1 - s_1 \leq t \leq 1. \end{cases}$$

This estimate and the inequality (61) mean that the number N of polynomials $P \in \mathcal{P}_k(Q_1)$ satisfying the conditions (56) can be estimated as follows:

$$N \leq \begin{cases} 2^{k+5} \varepsilon^{-1} \cdot Q_1^{k+1} \cdot \mu_2 \Pi, & t < 1 - s_1, \\ 2^{k+5} \varepsilon^{-1} \cdot Q_1^{k+t} \cdot \mu_1 I_2, & 1 - s_1 \leq t \leq 1. \end{cases} \quad (63)$$

On the other hand, the measure of the set $\sigma_P(t)$ satisfies the inequality

$$\mu_2 \sigma_P(t) \leq \begin{cases} 2^{2k} h_n^4 \delta_k^{-2} \cdot Q_1^{-k-\frac{5}{4}}, & t < 1 - s_1, \\ 2^{2k} h_n^4 \delta_k^{-2} \cdot Q_1^{-k-\frac{1}{4}-t} \cdot \mu_1 I_1, & 1 - s_1 \leq t \leq 1. \end{cases} \quad (64)$$

Then, by estimates (63) and (64), for $Q_1 > Q_0$ we can write

$$\mu_2 L_3^2(k, t) \leq 2^{3k+5} \delta_k^{-2} h_n^4 \varepsilon^{-1} Q_1^{-\frac{1}{4}} \mu_2 \Pi < \frac{1}{2^7 n^2} \cdot \mu_2 \Pi. \quad (65)$$

The inequalities (57) and (65) lead to the following estimate of the measure of the set $L_3(k)$:

$$\mu_2 L_3(k) \leq \sum_{i=0}^{2k} \mu_2 L_3^1(k, 1 - i/2) + \sum_{i=0}^{4k+1} \mu_2 L_3^2(k, 1 - i/4) \leq \frac{5+6k}{2^7 n^2} \cdot \mu_2 \Pi \leq \frac{1}{12n} \cdot \mu_2 \Pi.$$

Therefore

$$\mu_2 L_3 \leq \sum_{k=2}^{n-2} \mu_2 L_3(k) \leq \frac{n-3}{12n} \cdot \mu_2 \Pi \leq \frac{1}{12} \cdot \mu_2 \Pi.$$

This proves Lemma 5 in the case of reducible polynomials.

Thus, we have

$$\mu_2 L \leq \mu_2 L_1 + \mu_2 L_2 + \mu_2 L_3 \leq \frac{1}{4} \mu_2 \Pi.$$

□

3.7. The final part of the proof

The proof of Theorem 1 is going to be based on Lemma 5. Consider a set $B = \Pi \setminus L$ for $v_1 = v_2 = \frac{n-1}{2}$. From Lemma 5 it follows that

$$\mu_2 B \geq \frac{3}{4} \mu_2 \Pi \quad (66)$$

for $Q > Q_0$.

It should be recalled that the value h_n is defined in the beginning of the section 3 such that for every point $\mathbf{x} \in \Pi$ there exists a polynomial $P \in \mathcal{P}_n(Q)$ satisfying

$$|P(x_i)| \leq h_n Q^{-\frac{n-1}{2}}, \quad i = 1, 2.$$

Then, for every point $(x_{1,1}, x_{1,2}) \in B$ there exists an irreducible polynomial $P_1 \in \mathcal{P}_n(Q)$ satisfying the system of inequalities

$$\begin{cases} |P_1(x_{1,i})| < h_n Q^{-\frac{n-1}{2}}, \\ |P'_1(x_{1,i})| > \delta_n Q, \quad i = 1, 2. \end{cases}$$

Let α_i , such that $x_{1,i} \in S(\alpha_i)$, $i = 1, 2$ be roots of the polynomial P_1 . By Lemma 1, we have

$$|x_{1,i} - \alpha_i| \leq n h_n \delta_n^{-1} Q^{-\frac{n+1}{2}}, \quad i = 1, 2. \quad (67)$$

We are going to choose a maximal system of algebraic points $\Gamma = (\gamma_1, \dots, \gamma_t)$ satisfying the following conditions

1. $H(\gamma_j) \leq Q$, $\deg(\gamma_j) \leq n$;
2. Rectangles

$$\sigma(\gamma_j) = \left\{ |x_i - \gamma_{j,i}| < n h_n \delta_n^{-1} Q^{-\frac{n+1}{2}}, i = 1, 2 \right\}, \quad j = \overline{1, t},$$

do not intersect.

Let us introduce an expanded rectangles

$$\sigma_1(\gamma_j) = \left\{ |x_i - \gamma_{j,i}| < 2n h_n \delta_n^{-1} Q^{-\frac{n+1}{2}}, i = 1, 2 \right\}, \quad j = \overline{1, t}, \quad (68)$$

and show that

$$B \subset \bigcup_{j=1}^t \sigma_1(\gamma_j). \tag{69}$$

We obtain this by proving that for every point $(x_{1,1}, x_{1,2}) \in B$ there exists a point $\gamma_j \in \Gamma$ such that $(x_{1,1}, x_{1,2}) \in \sigma_1(\gamma_j)$. Since $(x_{1,1}, x_{1,2}) \in B$, there is a point $\alpha = (\alpha_1, \alpha_2)$ such that the inequalities (67) are true. Thus, either $\alpha \in \Gamma$ and $(x_{1,1}, x_{1,2}) \in \sigma_1(\alpha)$, or there exists a point $\gamma_j \in \Gamma$ satisfying

$$|\alpha_i - \gamma_{j,i}| \leq nh_n \delta_n^{-1} Q^{-\frac{n+1}{2}}, \quad i = 1, 2.$$

Hence, $(x_{1,1}, x_{1,2}) \in \sigma_1(\gamma_j)$.

In this case, by (66),(68) and (69) we have:

$$\frac{3}{4} \mu_2 \Pi \leq \mu_2 B \leq \sum_{j=1}^t \mu_2 \sigma_1(\gamma_j) \leq t \cdot 2^4 n^2 h_n^2 \delta_n^{-2} Q^{-n-1},$$

which yields the estimate

$$t \geq c_2 Q^{n+1} \mu_2 \Pi.$$

4. Proof of Theorem 2

The proof of Theorem 2 is based on the following Lemma.

LEMMA 6. For all $\left(\frac{v_1}{n-1}, \frac{v_2}{n-1}\right)$ -ordinary rectangles $\bar{\Pi} = I_1 \times I_2$ such that:

1. $\mu_1 I_1 = \mu_1 I_2 = c_3 Q^{-s}$, where $\frac{1}{2} < s < \frac{3}{4}$;
2. $\bar{\Pi} \cap \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - x_2| < \varepsilon\} = \emptyset$;
3. $c_3 > c_0(n, \varepsilon, \mathbf{d})$, where $\mathbf{d} = (d_1, d_2)$ is the midpoint of $\bar{\Pi}$;

let $L = L(Q, \delta_n, \mathbf{v}, \bar{\Pi})$ be the set of points $(x_1, x_2) \in \bar{\Pi}$ such that there exists a polynomial $P \in \mathcal{P}_n(Q)$ satisfying the following system of inequalities

$$\begin{cases} |P(x_i)| < h_n Q^{-v_i}, & v_i > 0, \\ \min_i \{|P'(x_i)|\} < \delta_n Q, \\ v_1 + v_2 = n - 1, & i = 1, 2. \end{cases} \tag{70}$$

Then for a sufficiently small constant $\delta_n < \delta_0(n, \varepsilon, \mathbf{d})$ and a sufficiently large $Q > Q_0(n, \varepsilon, \mathbf{v}, s, \mathbf{d})$, the estimate

$$\mu_2 L < \frac{1}{4} \mu_2 \bar{\Pi}$$

holds.

PROOF. Lemma 6 can be proved in the same way as Lemma 5, we only need to replace the base of induction.

STATEMENT 2. For all $(\gamma_{2,1}, \gamma_{2,2})$ -ordinary squares $\bar{\Pi} = I_1 \times I_2$ under conditions 1–3 let $L_{2,2} = L_{2,2}(Q, \delta_2, \gamma_2, \bar{\Pi})$ be the set of points $(x_1, x_2) \in \bar{\Pi}$ such that there exists a polynomial $P \in \mathcal{P}_2(Q)$ satisfying the system of inequalities

$$\begin{cases} |P(x_i)| < h_2 Q^{-\gamma_{2,i}}, & \gamma_{2,i} > 0, \\ \min_i \{|P'(x_i)|\} < \delta_2 Q, & i = 1, 2 \\ \gamma_{2,i} + \gamma_{2,i} = 1, & |b_2| > Q^{s-\frac{1}{2}}. \end{cases} \quad (71)$$

Then for any $r > 0$, $\delta_2 \leq \delta_0(\varepsilon, r, \mathbf{d})$ and $Q > Q_0(n, \varepsilon, s, \gamma_2, \mathbf{d})$, the estimate

$$\mu_2 L_{2,2} < \frac{1}{4r} \mu_2 \bar{\Pi}$$

holds.

PROOF. Let P be a polynomial of the form $P(t) = b_2 t^2 + b_1 t + b_0$. Applying the same argument that we used to prove the Statement 1, we obtain upper and lower bounds for the absolute value of the derivative P' at roots α_1, α_2 and at points x_1, x_2 , where $x_i \in S(\alpha_i)$, $i = 1, 2$:

$$|P'(\alpha_i)| > \frac{3}{4} \cdot \varepsilon \cdot |b_2|, \quad |P'(x_i)| \leq \left(|d_1| + |d_2| + 1 + \frac{\varepsilon}{4} \right) \cdot |b_2|. \quad (72)$$

These estimates lead to the following inequality:

$$|b_2| < 4\delta_2 \varepsilon^{-1} Q.$$

From Lemma 1 and the estimates (71), (72) it follows that the set $L_{2,2}$ is contained in a union $\bigcup_{P \in \mathcal{P}_2(Q)} \sigma_P$, where

$$\sigma_P = \left\{ (x_1, x_2) \in \bar{\Pi} : |x_i - \alpha_i| < 2h_2\varepsilon^{-1}Q^{-\gamma_{2,i}}|b_2|^{-1}, i = 1, 2 \right\}. \quad (73)$$

Since the square $\bar{\Pi}$ is $(\gamma_{2,1}, \gamma_{2,2})$ -ordinary we have $|b_2| \geq Q^{s-\frac{1}{2}}$ and

$$\mu_2\sigma_P \leq 2^4 h_2^2 \varepsilon^{-2} Q^{-1} |b_2|^{-2} \leq \frac{1}{4} c_3^2 Q^{-2s} = \frac{1}{4} \mu_2 \bar{\Pi}$$

with $c_3 > 8h_2\varepsilon^{-1}$ and $s > \frac{1}{2}$.

Then we can write the following estimate for the measure of the set $L_{2,2}$:

$$\mu_2 L_{2,2} \leq \mu_2 \bigcup_{P \in \mathcal{P}_2(Q)} \sigma_P \leq \sum_{P \in \mathcal{P}_2(Q)} \mu_2 \sigma_P \leq 2^4 h_2^2 \varepsilon^{-2} Q^{-1} \sum_{\substack{b_2, b_1, b_0 \leq Q: \\ P(t) = b_2 t^2 + b_1 t + b_0 \\ \sigma_P \neq \emptyset}} |b_2|^{-2}.$$

As in the proof of Statement 1 of Lemma 5, we estimate the number of polynomials $P \in \mathcal{P}_2(Q)$ satisfying the system of inequalities (71) at some point $(x_1, x_2) \in \bar{\Pi}$ for a fixed value of b_2 .

Let us estimate the polynomial P at the points d_1, d_2 . From Taylor expansions and estimates (72) we have

$$|P(d_i)| \leq |P(x_i)| + c_{20} \cdot |b_2| \mu_1 I_i, \quad (74)$$

for a sufficiently large $Q > Q_0$. Consider a system of equations

$$\begin{cases} b_2 d_1^2 + b_1 d_1 + b_0 = l_1, \\ b_2 d_2^2 + b_1 d_2 + b_0 = l_2, \end{cases} \quad (75)$$

in three variables $b_2, b_1, b_0 \in \mathbb{Z}$, where $|l_i| \leq 2c_{20} \cdot \max\{1, |b_2| \mu_1 I_i\}$, $i = 1, 2$.

Let us estimate the number of possible pairs (b_1, b_0) such that the system (75) is satisfied for a fixed b_2 . To obtain this estimate, we consider the system (75) for two

different combinations $b_2, b_{0,1}, b_{0,0}$ and $b_2, b_{j,1}, b_{j,0}$. Simple transformations lead to the following system of linear equations in two variables $b_{0,1} - b_{j,1}$ and $b_{0,0} - b_{j,0}$:

$$\begin{cases} (b_{0,1} - b_{j,1})d_1 + (b_{0,0} - b_{j,0}) = l_{0,1} - l_{j,1}, \\ (b_{0,1} - b_{j,1})d_2 + (b_{0,0} - b_{j,0}) = l_{0,2} - l_{j,2}. \end{cases} \tag{76}$$

Since the determinant of this system does not vanish, we can use Cramer’s rule to solve it. Using inequalities $|l_{0,i} - l_{j,i}| \leq 4c_{20} \cdot \max\{1, |b_2|\mu_1 I_i\}$ we estimate the determinants $\Delta_i, i = 1, 2$ as follows:

$$|\Delta_i| \leq 8c_{20} \cdot \max\{1, |b_2|\mu_1 I_i\}.$$

Thus

$$|b_{0,i} - b_{j,i}| \leq \frac{|\Delta_i|}{|\Delta|} \leq 8c_{20}\varepsilon^{-1} \cdot \max\{1, |b_2|\mu_1 I_i\},$$

and for a fixed b_2 the following estimate holds:

$$\#(b_1, b_0) \leq \begin{cases} 2^6 c_{20}^2 \varepsilon^{-2} |b_2|^2 \mu_2 \bar{\Pi}, & |b_2| > c_3^{-1} Q^s, \\ 2^6 c_{20}^2 \varepsilon^{-2}, & Q^{s-\frac{1}{2}} < |b_2| < c_3^{-1} Q^s. \end{cases} \tag{77}$$

Depending on the absolute value $|b_2|$, let us consider the following two sets:

$$L_{2,2}^1 = \bigcup_{\substack{P \in \mathcal{P}_2(Q), \\ c_3^{-1} Q^s < |b_2| < 4\delta_2 \varepsilon^{-1} Q}} \sigma_P, \quad L_{2,2}^2 = \bigcup_{\substack{P \in \mathcal{P}_2(Q), \\ Q^{s-\frac{1}{2}} < |b_2| < c_3^{-1} Q^s}} \sigma_P.$$

The set $L_{2,2}^1$: In this case for $\delta_2 < 2^{-15} r^{-1} c_{20}^{-2} h_2^{-2} \varepsilon^5$ the measure can be estimated as:

$$\mu_2 L_{2,2} \leq 2^{10} c_{20}^2 h_2^2 \varepsilon^{-4} Q^{-1} \cdot 4\delta_2 \varepsilon^{-1} Q \mu_2 \bar{\Pi} < \frac{1}{8r} \cdot \mu_2 \bar{\Pi}.$$

The set $L_{2,2}^2$: Consider the polynomials P under condition $Q^{s-\frac{1}{2}} < |b_2| < c_3^{-1} Q^s$. For every set σ_P we define the expanded set:

$$\sigma'_P = \left\{ (x_1, x_2) \in \bar{\Pi} : |x_i - \alpha_i| < 2^5 h_2 \varepsilon^{-1} \sqrt{r} \cdot Q^{-\gamma_{2,i}} |b_2|^{-1}, i = 1, 2 \right\}. \tag{78}$$

Let us prove that for $|b_2| < c_{21} \cdot Q^{\frac{1}{2}}$, where $c_{21} = \varepsilon (2^7 h_2 \sqrt{r})^{-1} \cdot (|d_1| + |d_2| + 2)^{-1}$ this sets do not intersect.

Consider polynomials $P_j, j = 1, 2$ with roots $\alpha_{j,1}, \alpha_{j,2}$ and leading coefficients $|b_{j,2}| < c_{21} \cdot Q^{\frac{1}{2}}$. Without loss of generality we will assume $|b_{1,2}| < |b_{2,2}|$. Let there exists a point $(x_{0,1}, x_{0,2}) \in \sigma'_{P_1} \cap \sigma'_{P_2}$. Since P_1 and P_2 have no common roots, the resultant $R(P_1, P_2)$ doesn't vanish, and the following estimate holds:

$$1 \leq |R(P_1, P_2)| = |b_{1,2}|^2 |b_{2,2}|^2 |\alpha_{1,1} - \alpha_{2,1}| |\alpha_{1,1} - \alpha_{2,2}| |\alpha_{1,2} - \alpha_{2,1}| |\alpha_{1,2} - \alpha_{2,2}|. \quad (79)$$

By the estimates (78) we have

$$|\alpha_{1,i} - \alpha_{2,i}| \leq |\alpha_{1,i} - x_{0,i}| + |\alpha_{2,i} - x_{0,i}| < 2^6 h_2 \varepsilon^{-1} \sqrt{r} \cdot Q^{-\gamma_{2,i}} |b_{1,2}|^{-1}.$$

On the other hand for $Q > Q_0$ we get

$$\begin{aligned} |\alpha_{1,1} - \alpha_{2,2}| &\leq |\alpha_{1,1}| + |\alpha_{2,2}| \leq |d_1| + |d_2| + 2, \\ |\alpha_{1,2} - \alpha_{2,1}| &\leq |\alpha_{1,2}| + |\alpha_{2,1}| \leq |d_1| + |d_2| + 2. \end{aligned}$$

By substituting these inequalities to (79) we obtain

$$1 \leq |R(P_1, P_2)| < 2^{12} h_2^2 \varepsilon^{-2} r \cdot (|d_1| + |d_2| + 2)^2 \cdot |b_{2,2}|^2 \cdot Q^{-1} < \frac{1}{4}.$$

This contradiction yields the following estimate

$$\sum_{\substack{P \in \mathcal{P}_2(Q), \\ Q^{s-\frac{1}{2}} < |b_2| < c_{21} Q^{\frac{1}{2}}}} \mu_2 \sigma_P \leq \frac{1}{16r} \cdot \sum_{\substack{P \in \mathcal{P}_2(Q), \\ Q^{s-\frac{1}{2}} < |b_2| < c_{21} Q^{\frac{1}{2}}}} \mu_2 \sigma'_P \leq \frac{1}{16r} \cdot \mu_2 \bar{\Pi}.$$

Consider the case $|b_2| > c_{21} Q^{\frac{1}{2}}$. Let $\mathcal{P}_2(Q, k) \subset \mathcal{P}_2(Q), 1 \leq k \leq K = \left\lceil \ln_2 \left(\frac{2 - 2s}{3 - 4s} \right) \right\rceil + 2$ be a subclass of polynomials defined as follows:

$$\mathcal{P}_2(Q, k) := \left\{ P \in \mathcal{P}_2(Q) : l_{k+1} \cdot Q^{\lambda_{k+1}} \leq |b_2| \leq l_k \cdot Q^{\lambda_k} \right\},$$

where

$$\begin{aligned} \lambda_1 &= s, & l_1 &= c_3^{-1}, \\ \lambda_k &= \lambda_{k-1} - (1-s) \cdot 2^{1-k}, & l_k &= \frac{2^6 c_{20} h_2 \cdot \sqrt{rK \cdot l_{k-1}}}{\varepsilon^2 c_3} \quad \text{for } 2 \leq k \leq K, \\ \lambda_{K+1} &= \frac{1}{2}, & l_{K+1} &= c_{21}. \end{aligned}$$

This equations give $\lambda_k = s - (1-s) \cdot \left(1 - \frac{1}{2^{k-1}}\right)$ for $2 \leq k \leq K$.

Let us consider the following sets $L(k) = \bigcup_{P \in \mathcal{P}_2(Q,k)} \sigma_P$ and estimate the measure of every one of them in the following way:

$$\begin{aligned} \mu_2 L(k) &\leq \sum_{P \in \mathcal{P}_2(Q,k)} \mu_2 \sigma_P \leq \frac{2^{10} h_2^2 c_{20}^2}{\varepsilon^4} \cdot Q^{-1} \sum_{l_{k+1} Q^{\lambda_{k+1}} \leq |b_2| \leq l_k Q^{\lambda_k}} |b_2|^{-2} \leq \\ &\leq \frac{2^{10} h_2^2 c_{20}^2 l_k}{\varepsilon^4 l_{k+1}^2} \cdot Q^{-1-2\lambda_{k+1}+\lambda_k}. \end{aligned}$$

Then for $k = 1$ we obtain

$$\mu_2 L(1) \leq \frac{c_3^2}{16rK} \cdot Q^{-1-2s+1-s+s} \leq \frac{1}{16rK} \cdot c_3^2 Q^{-2s} < \frac{1}{16rK} \cdot \mu_2 \bar{\Pi};$$

for $1 \leq k \leq K - 1$ we have

$$\mu_2 L(k) \leq \frac{c_3^2}{16rK} \cdot Q^{-1+s-(1-s) \cdot \left(1 - \frac{1}{2^{k-1}}\right) - 2s + (1-s) \cdot \left(2 - \frac{1}{2^{k-1}}\right)} \leq \frac{1}{16rK} \cdot c_3^2 Q^{-2s} = \frac{1}{16rK} \cdot \mu_2 \bar{\Pi};$$

and for $k = K$, $s < \frac{3}{4}$ and $Q > Q_0$ we get

$$\begin{aligned} \mu_2 L(K) &\leq \frac{2^6 h_2^2 \cdot l_K}{\varepsilon^2 c_{20}^2} \cdot Q^{-2+s-(1-s) \cdot \left(1 - \frac{1}{2^{K-1}}\right)} \leq \frac{2^4 h_2^2 \cdot l_K}{\varepsilon^2 c_{21}^2} \cdot Q^{-3+2s+(1-s) \cdot \frac{3-4s}{2-2s}} \leq \\ &\leq \frac{2^4 h_2^2 \cdot l_K}{\varepsilon^2 c_{21}^2} \cdot Q^{-\frac{3}{2}} < \frac{1}{16rK} \cdot \mu_2 \bar{\Pi}. \end{aligned}$$

Then, we obtain following estimate for the measure of the set $L_{2,2}^2$

$$\mu_2 L_{2,2}^2 \leq \sum_{\substack{P \in \mathcal{P}_2(Q), \\ Q^{s-\frac{1}{2}} < |b_2| < c_{21} Q^{\frac{1}{2}}}} \mu_2 \sigma_P + \sum_{1 \leq k \leq K} \mu_2 L(k) \leq \frac{1}{8r} \cdot \mu_2 \bar{\Pi},$$

and thus

$$\mu_2 L_{2,2} \leq \mu_2 L_{2,2}^1 + \mu_2 L_{2,2}^2 \leq \frac{1}{4r} \cdot \mu_2 \bar{\Pi}. \quad \square$$

Now Lemma 6 can be proved by repeating the proof of Lemma 5. □

Theorem 2 is proved by applying the results of Lemma 6 to the proof of Theorem 1.

5. Proof of Theorem 3

To prove Theorem 3 we are going to use the results of Theorem 1 and Theorem 2. For this purpose we need to consider the set $J \setminus D = \bigcup_k J_k$, where $D := \{x \in J : |f(x) - x| < \frac{1}{2}\varepsilon\}$. It is easy to see, that for sufficiently small ε the following estimate for the measure of the set $J \setminus D$ holds:

$$\mu_1(J \setminus D) \geq \frac{3}{4} \mu_1 J.$$

Now for every strip $L_{J_k}(Q, \lambda)$ we have $L_{J_k}(Q, \lambda) \cap \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - x_2| < \varepsilon\} = \emptyset$.

Let us consider an interval $J_k = [a_k, b_k]$ and the strip $L_{J_k}(Q, \lambda)$ for a fixed $0 < \lambda < \frac{3}{4}$. Divide the strip $L_{J_k}(Q, \lambda)$ into sets

$$E_j := \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \in J_{k,j}, |x_2 - f(x_1)| < \left(\frac{1}{2} + c_5\right) \cdot c_3 Q^{-\lambda} \right\},$$

where $J_{k,j} = [x_j, x_{j+1}]$, $x_j = x_{j-1} + c_3 Q^{-\lambda}$, $x_0 = a_k$ and $1 \leq j \leq t_k$. The number of sets E_j can be estimated as follows:

$$t_k > \frac{\mu_1 J_k}{\mu_1 J_{k,j}} - 1 > \frac{1}{2} c_3^{-1} \mu_1 J_k \cdot Q^\lambda$$

for $Q > Q_0$.

Let $\bar{f}_j = \frac{1}{2} \cdot \left(\max_{x \in J_{k,j}} f(x) + \min_{x \in J_{k,j}} f(x) \right)$. Consider the rectangles

$$\Pi_j = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \in J_{k,j}, \left| x_2 - \bar{f}_j \right| \leq \frac{1}{2} c_5 c_3 Q^{-\lambda} \right\}.$$

By mean value theorem, since f is continuous and differentiable function on every interval $J_{k,j}$ and $\sup_{x \in J_{k,j}} |f'(x)| \leq \sup_{x \in J} |f'(x)| := c_5$, we obtain:

$$\left| \max_{x \in J_{k,j}} f(x) - \min_{x \in J_{k,j}} f(x) \right| \leq |f'(\xi)| \cdot \mu_1 J_{k,j} < c_5 c_3 \cdot Q^{-\lambda}.$$

It means that $\Pi_j \subset E_j$ for every $1 \leq j \leq t_k$.

Case 1: $0 < \lambda \leq \frac{1}{2}$.

In this case, we apply the result of Theorem 1. From Theorem 1 it follows that every rectangle Π_j , $j = \overline{1, t_k}$, contains at least $\frac{1}{2} c_2 c_3^2 c_5 Q^{n+1-2\lambda}$ algebraic points of degree at most n and height at most Q . Since we have $t_k > \frac{1}{2} c_3^{-1} \mu_1 J_k Q^\lambda$ and $\sum_k \mu_1 J_k \geq \mu_1 J \setminus D > \frac{3}{4} \mu_1 J$, there must be at least $c_6 Q^{n+1-\lambda}$ algebraic points $\alpha \in L_J(Q, \lambda) \cap \mathbb{A}_n^2(Q)$.

Case 2: $\frac{1}{2} < \lambda < \frac{3}{4}$.

Theorem 2 will be used in that case. Let us count the number of $\left(\frac{1}{2}, \frac{1}{2}\right)$ -special squares Π_i . By definition, a $\left(\frac{1}{2}, \frac{1}{2}\right)$ -special square contains the points $(x_{0,1}, x_{0,2})$ such that there exists a polynomial $P \in \mathcal{P}_2(Q)$ satisfying the system of inequalities

$$\begin{cases} |P(x_{0,i})| < h_2 Q^{-\frac{1}{2}}, & i = 1, 2, \\ |b_2| \leq Q^{\lambda-\frac{1}{2}}. \end{cases} \tag{80}$$

Repeating the steps of the proof of Statement 1 from the beginning till inequality (13), we obtain the following estimates:

$$|P'(\alpha_1)| = |P'(\alpha_2)| > \frac{3}{4} \cdot \varepsilon \cdot |b_2|.$$

Thus, by Lemma 1 the set of points (x_1, x_2) satisfying the system (80) for a fixed polynomial P is a subset of the following square:

$$\sigma_P = \left\{ (x_1, x_2) \in \mathbb{R}^2 : |x_i - \alpha_i| \leq 2h_2 \varepsilon^{-1} Q^{-\frac{1}{2}} |b_2|^{-1}, i = 1, 2 \right\}.$$

Let us estimate the number of squares Π_j , such that $\Pi_j \cap \sigma_P \neq \emptyset$. It is easy to see that the width of the strip $L_{J_k}(Q, \lambda)$ is smaller than the heights of the squares σ_P for sufficiently large c_3 . Hence, every square σ_P intersects with at most $4h_2\varepsilon^{-1}c_3^{-1}Q^{\lambda-\frac{1}{2}}|b_2|^{-1}$ squares Π_j . Therefore, the number m_1 of $(\frac{1}{2}, \frac{1}{2})$ -special squares Π_i can be estimated as

$$m_1 \leq \sum_{P \in \mathcal{P}_2(Q)} 4h_2\varepsilon^{-1}c_3^{-1}Q^{\lambda-\frac{1}{2}}|b_2|^{-1} \leq 4h_2\varepsilon^{-1}c_3^{-1}Q^{\lambda-\frac{1}{2}} \sum_{b_2, b_1, b_0} |b_2|^{-1}$$

Now we need to estimate the number of polynomials $P \in \mathcal{P}_2(Q)$ satisfying the system of inequalities (80) at some point $(x_1, x_2) \in L_{J_k}(Q, \lambda)$ for a fixed value of b_2 . Since the function f is continuously differentiable on the interval J , and $\sup_{x \in J_k} |f'(x)| < c_5$, we get by the mean value theorem that

$$\left| \max_{x \in J_k} f(x) - \min_{x \in J_k} f(x) \right| < c_5 \cdot \mu_1 J_k,$$

which implies that the set $L_{J_k}(Q, \lambda)$ is contained in a rectangle $\Pi = I_1 \times I_2$, where $\mu_1 I_2 = c_5 \mu_1 I_1 = c_5 \mu_1 J_k$.

Let us estimate the polynomial P at the midpoint (d_1, d_2) of the rectangle Π . Using the steps of the proof of Statement 1 we obtain

$$|P(d_1)| \leq c_{22} \cdot |b_2| \mu_1 J_k, \quad |P(d_2)| \leq c_{22} c_5 \cdot |b_2| \mu_1 J_k.$$

and, hence, for a fixed value of b_2 the number of polynomials $P \in \mathcal{P}_2(Q)$ satisfying the system of inequalities (80) at some point $(x_1, x_2) \in \Pi$ can be estimated as follows:

$$\#(b_1, b_0) \leq 2^5 c_5 c_{22}^2 \varepsilon^{-2} |b_2|^2 (\mu_1 J_k)^2.$$

Using this inequality we have:

$$m_1 \leq \frac{2^7 h_2 c_5 c_{22}^2 \cdot (\mu_1 J_k)^2}{\varepsilon^3 c_3} \cdot Q^{\lambda-\frac{1}{2}} \sum_{|b_2| < Q^{\lambda-\frac{1}{2}}} |b_2| \leq \frac{2^7 h_2 c_5 c_{22}^2 \cdot (\mu_1 J_k)^2}{\varepsilon^3 c_3} \cdot Q^{3\lambda-\frac{3}{2}} < < \frac{1}{4} c_3^{-1} \mu_1 J_k \cdot Q^\lambda < \frac{t_k}{2}. \quad (81)$$

for $\lambda < \frac{3}{4}$ and $Q > Q_0$. By (81), it follows that the number of $\left(\frac{1}{2}, \frac{1}{2}\right)$ -ordinary squares Π_j doesn't exceed

$$m_2 \geq t_k - \frac{1}{2}t_k > \frac{1}{2}t_k. \quad (82)$$

From Theorem 2 and the estimate (82) it now follows that in the case $\frac{1}{2} < \lambda < \frac{3}{4}$, the strip $L_J(Q, \lambda)$ contains at least $c_6 Q^{n+1-\lambda}$ algebraic points of degree at most n and height at most Q .

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