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On infinite spectra of first order properties of random graphs

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Abstract: Spectrum of k is the set of all α such that the random graph $G(n, n^{-\alpha})$ does not obey zero-one k-law. It is known that for k large enough spectrum of k is infinite. In this paper, we get new bounds on the minimal and the maximal limit points of the spectrum. Moreover, we prove that the minimal k such that the spectrum of k is infinite is either 4, or 5.

Keywords: Random graph, first order properties, zero-one law, spectrum

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1. Introduction

The asymptotic behavior of probabilities of first-order properties the Erdős—Rényi random graph $G(n, p)$ can satisfy has been widely studied in [1]– [3], [7]– [14], [22]. Let $n \in \mathbb{N}$, $0 \leq p \leq 1$. Consider the set $\Omega_n = \{G = (V_n, E)\}$ of all the undirected graphs without loops and multiple edges with the set of vertices $V_n = \{1, 2, ..., n\}$. An *Erdős—Rényi random graph* [1, 7, 11, 22] is a random element $G(n, p)$ on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, such that it maps Ω to Ω_n and its distribution $\mathsf{P}_{n,p}$ on $\mathcal{F}_n = 2^{\Omega_n}$ is defined by

$$\mathsf{P}_{n,p}(G) = p^{|E|}(1-p)^{C_n^2 - |E|}.$$

Let us denote the event "$G(n, p)$ satisfies a property L" by $\{G(n, p) \models L\}$.

$G(n,p)$ is said to *obey the Zero-One Law* if for any first order property L (see [15]) the probability $\mathsf{P}(G(n,p) \models L)$ tends either to 0, or to 1. It was proved in [9] that if $p = n^{-\alpha+o(1)}$, $\alpha \in \mathbb{R}_+ \setminus \mathbb{Q}$, then $G(n,p)$ obeys the Zero-One Law. To avoid trivialities, we shall restrict ourselves to $0 < \alpha < 1$ (the case $p = O(1/n)$ was studied in [9]). If $\alpha \in \mathbb{Q} \cap (0,1)$, then $G(n, n^{-\alpha})$ does not obey the Zero-One Law (see, e. g., [22]).

In [16]– [22] the Zero-One k-Law was studied ($G(n,p)$ is said to obey *Zero-One k-Law*, if for any property L which is expressed by a first-order formula of quantifier depth at most k (see [15]) the probability $\mathsf{P}(G(n,p) \models L)$ either tends to 0, or tends to 1). Let us remind that the *quantifier depth* of a first-order formula is the maximum number of nested quantifiers. We denote by $\mathcal{L}_k$ the set of all graph properties which can be expressed by first order formulae of quantifier depth at most k. We also set $\mathcal{L} = \bigcup_{k\in\mathbb{N}} \mathcal{L}_k$, so that $\mathcal{L}$ is the set of all first order graph properties.

In 2012 we proved (see [20, 21]) that if $k \geq 3$ and $\alpha \in (0, 1/(k-2))$, then $G(n, n^{-\alpha})$ obeys the Zero-One k-Law. In [20,21] we also proved that $G(n, n^{-1/(k-2)})$ does not obey the Zero-One k-Law. In 2014 we proved (see [16]) that if $k > 3$ and $\alpha = 1 - \frac{1}{2^{k-1}+\beta}$, $\beta \in (0,\infty) \setminus \mathcal{Q}$, where $\mathcal{Q}$ is the set of all positive rational numbers with numerators at most 2^{k-1}, then $G(n, n^{-\alpha})$ obeys the Zero-One k-Law. In [16] it was also proved that $G(n, n^{-\alpha})$ does not obey the Zero-One k-Law if $\alpha = 1 - \frac{1}{2^{k-1}+\beta}$, where $\beta \in \{0, 1, \ldots, 2^{k-1}-2\}$. Finally, in [19] it was proved that $G(n, n^{-\alpha})$ obeys the Zero-One k-Law if $\alpha \in \{1 - \frac{1}{2^k-1}, 1 - \frac{1}{2^k}\}$. Thus, $1 - \frac{1}{2^k-2}$ is the maximum of $\alpha \in (0,1)$ for which $G(n, n^{-\alpha})$ does not obey the Zero-One k-Law.

In the present paper we prove (see Section 2) that there are finitely many $\alpha \in (1 - \frac{1}{2^{k-1}}, 1)$ such that $G(n, n^{-\alpha})$ does not obey the Zero-One k-Law.

If $G(n, n^{-\alpha})$ does not obey the Zero-One k-Law for some $\alpha \in (0,1)$ and $k \in \mathbb{N}$, then we say that α is in the *spectrum of* k. We remind that in [14] two kinds of spectra of a first-order property $L \in \mathcal{L}$ were considered. The first one deals with $p = p(n) = n^{-\alpha}$. It is denoted by $S^1(L)$ and is defined as the set of all $\alpha \in (0,1)$ for which it is *not* true that $\lim_{n\to\infty} \mathsf{P}(G(n, p(n)) \models L)$ exists and is either zero, or one. The second deals with $p = p(n) = n^{-\alpha+o(1)}$. It is denoted by $S^2(L)$ and is defined as the set of all $\alpha \in (0,1)$ for which it is *not* true that there exist $\delta \in \{0,1\}$ and $\epsilon > 0$ such that $\lim_{n\to\infty} \mathsf{P}(G(n,p(n)) \models L) = \delta$, whenever $n^{-\alpha-\epsilon} < p(n) < n^{-\alpha+\epsilon}$. For each $k \in \mathbb{N}$ let us denote the unions of $S^1(L)$ and $S^2(L)$ over all $L \in \mathcal{L}_k$ by S^1_k and S^2_k respectively.

It was proved in [10] that the sets S^1_k and S^2_k are infinite when k is large enough. Up to tautological equivalence, there are (see, e. g., [15]) finitely many first order sentences of a given quantifier depth. Thus, for each $j \in \{1,2\}$ the set S^j_k is infinite if and only if there is at least one L of quantifier depth at most k such that $S^j(L)$ is infinite. Therefore, each time we prove that S^j_k is infinite, we simply search for a property with an infinite spectrum.

It is also known [13] that all the limit points of S^1_k and S^2_k are approached only from above.

It was proved in [14] that the minimum k_1 and k_2 such that the sets $S^1_{k_1}$ and $S^2_{k_2}$ are infinite are in the sets $\{4,\dots,12\}$ and $\{4,\dots,10\}$ respectively. We also estimate in [14] the minimum and the maximum limit points of S^1_k, S^2_k. Let us denote the sets of limit points of S^1_k and S^2_k by $(S^1_k)'$ and $(S^2_k)'$ respectively. We have

$$\min(S^1_k)' \in \left[\frac{1}{k-2}, \frac{1}{k-11}\right], \text{ if } k \geq 15, \quad \min(S^2_k)' \in \left[\frac{1}{k-1}, \frac{1}{k-7}\right], \text{ if } k \geq 10,$$

$$\max(S^j_k)' \in \left[1 - \frac{1}{2^{k-13}}, 1 - \frac{1}{2^{k-1}}\right], \quad \text{if } k \geq 16,\ j \in \{1,2\}.$$

In the next Section, we state our new results. We prove them in Section 4. Some statements concerning the distribution of small subgraphs in the random graph, which we use in our proofs, are formulated in Section 3.

2. New results

THEOREM 2.1. *For every $k \geq 5$ we have* $\frac{1}{\lfloor k/2 \rfloor} \in (S^1_k)'$.

Theorem 2.1 provides a better upper bound for the minimum limit point of S^1_k for any $k \leq 20$ and a better upper bound for the minimum limit point of S^2_k for all $k \leq 12$. Besides that, Theorem 2.1 and the Zero-One k-Law from [20, 21] imply the following statement.

COROLLARY 2.1. *The minimum k such that the set S_k^1 (S_k^2) is infinite equals 4 or 5.*

We also obtain a better lower bound for the maximum limit points of the spectra (for small k as well).

THEOREM 2.2. *For every $k \geq 8$ we have $1 - \frac{1}{2^{k-5}} \in (S_k^1)'$.*

The fact that there are no points of S_k^1 in the interval $\left(1 - \frac{1}{2^{k-1}}, 1\right)$ is a corollary to the following result.

THEOREM 2.3. *Let k and b be arbitrary positive integers, $k > 3$. Let $\frac{a}{b}$ be a positive rational number in reduced form. Set $\nu = \max\{1, 2^{k-1} - b\}$. Let $a \in \{\nu, \nu + 1, \dots, 2^{k-1}\}$, $\alpha = 1 - \frac{1}{2^{k-1} + a/b}$. Then the random graph $G(n, n^{-\alpha})$ obeys the Zero-One k-Law.*

3. Small subgraphs in the random graph

For an arbitrary graph $G = (E, V)$, set $e(G) = |E|$, $v(G) = |V|$, $\rho(G) = \frac{e(G)}{v(G)}$, $\rho^{\max}(G) = \max\limits_{H \subseteq G} \rho(H)$. We remind that $\rho(G)$ is called the *density* of G. Denote the number of copies of G in $G(n, p)$ by N_G. Denote the property of containing a copy of G by L_G.

THEOREM 3.1 [2, 8]. *If $p = o\left(n^{-1/\rho^{\max}(G)}\right)$ then $\lim\limits_{n\to\infty} \mathsf{P}(G(n,p) \models L_G) = 0$. If $n^{-1/\rho^{\max}(G)} = o(p)$ then $\lim\limits_{n\to\infty} \mathsf{P}(G(n,p) \models L_G) = 1$.*

In other words, the function $n^{-1/\rho^{\max}(G)}$ is *a threshold* (see [1, 7]) for the property L_G.

Let G be a *strictly balanced graph* (i. e. its density is greater than the density of any of its proper subgraphs) with $a(G)$ automorphisms.

THEOREM 3.2 [2]. *If $p = n^{-1/\rho(G)}$ then $N_G \xrightarrow{d} \mathrm{Pois}\left(\frac{1}{a(G)}\right)$.*

Consider arbitrary graphs G and H such that $H \subset G$, $V(H) = \{x_1, ..., x_m\}$, $V(G) = \{x_1, ..., x_l\}$, $E(G) \setminus (E(H) \cup E(G \setminus H)) \neq \varnothing$. Set $e(G, H) = e(G) - e(H)$,

$$v(G,H) = v(G) - v(H),\ \rho(G,H) = \frac{e(G,H)}{v(G,H)},\ \rho^{\max}(G,H) = \max_{H\subset K\subseteq G} \rho(K,H).$$

Let also $e^{\min}(G,H)$ be the minimum of $e(K,H)$ over all graphs K such that $H \subset K \subseteq G$, $\rho(K,H) = \rho^{\max}(G,H)$, $E(K) \setminus (E(H) \cup E(K \setminus H)) \neq \varnothing$. Consider graphs $\widetilde{H}$, $\widetilde{G}$, where $V(\widetilde{H}) = \{\widetilde{x}_1, ..., \widetilde{x}_m\}$, $V(\widetilde{G}) = \{\widetilde{x}_1, ..., \widetilde{x}_l\}$, $\widetilde{H} \subset \widetilde{G}$. A graph $\widetilde{G}$ is called a $(G, (x_1, \dots, x_m))$*-extension of the ordered tuple* $(\widetilde{x}_1, \dots, \widetilde{x}_m)$ if

$$\{x_{i_1}, x_{i_2}\} \in E(G) \setminus E(H) \Rightarrow \{\widetilde{x}_{i_1}, \widetilde{x}_{i_2}\} \in E(\widetilde{G}) \setminus E(\widetilde{H}).$$

The extension is called *strict* if

$$\{x_{i_1}, x_{i_2}\} \in E(G) \setminus E(H) \Leftrightarrow \{\widetilde{x}_{i_1}, \widetilde{x}_{i_2}\} \in E(\widetilde{G}) \setminus E(\widetilde{H}).$$

We use $L_{(G,H)}$ to denote the property of containing a $(G, (x_1, \dots, x_m))$-extension of any ordered tuple of m vertices.

THEOREM 3.3 [12]. *There are* $0 < \varepsilon < K$ *such that the following statements hold*:

$$\text{if } p \leqslant \varepsilon n^{-1/\rho^{\max}(G,H)} (\ln n)^{1/e^{\min}(G,H)},\ \text{then } \lim_{n\to\infty} \mathsf{P}(G(n,p) \models L_{(G,H)}) = 0;$$
$$\text{if } p \geqslant K n^{-1/\rho^{\max}(G,H)} (\ln n)^{1/e^{\min}(G,H)},\ \text{then } \lim_{n\to\infty} \mathsf{P}(G(n,p) \models L_{(G,H)}) = 1.$$

Obviously, if the pair (G,H) is *balanced* (i. e. the maximum density $\rho^{\max}(G,H)$ equals $\rho(G,H)$), then the quantity $\rho^{\max}(G,H)$ in the statement of Theorem 3.3 can be replaced by $\rho(G,H)$. Same as for graphs, the pair (G,H) is called *strictly balanced* if $\rho(G,H) > \rho(K,H)$ for any graph K such that $H \subset K \subset G$.

Let us fix $\alpha \in (0,1)$ and set

$$v(G,H) = |V(G) \setminus V(H)|,\ \ e(G,H) = |E(G) \setminus E(H)|,$$

$$f_\alpha(G,H) = v(G,H) - \alpha e(G,H).$$

If for any graph S such that $H \subset S \subseteq G$ the inequality $f_\alpha(S,H) > 0$ holds, then the pair (G,H) is called α*-safe* (see [7, 22]). Finally, let us introduce the notion of a maximal pair. Let $\widetilde{H} \subset \widetilde{G} \subset \Gamma$ and $T \subset K$, where $V(T) = \{v_1, \dots, v_t\}$, $t \leq |V(\widetilde{G})|$. The pair $(\widetilde{G}, \widetilde{H})$ is called (K,T)*-maximal in* Γ, if any ordered tuple $\mathbf{t}$ of t vertices from $V(\widetilde{G})$ with at least one vertex from $V(\widetilde{G}) \setminus V(\widetilde{H})$ does not have a strict $(K,(v_1,\ldots,v_t))$-extension $\widetilde{K}$ in Γ such that the following properties hold: the intersection of $V(\widetilde{K})$ and $V(\widetilde{G})$ contains vertices from $\mathbf{t}$ only and any vertex from $V(\widetilde{K})$ which is not in $\mathbf{t}$ and any vertex from $V(\widetilde{G})$ which is not in $\mathbf{t}$ are not adjacent. Similarly, a graph $\widetilde{G}$ is called (K,T)*-maximal in* Γ, if any ordered tuple $\mathbf{t}$ of t vertices from $V(\widetilde{G})$ does not have a strict $(K,(v_1,\ldots,v_t))$-extension $\widetilde{K}$ in Γ such that the following properties hold: the intersection of the sets $V(\widetilde{K})$, $V(\widetilde{G})$ contains vertices from $\mathbf{t}$ only and any vertex from $V(\widetilde{K})$ which is not in $\mathbf{t}$ and any vertex from $V(\widetilde{G})$ which is not in $\mathbf{t}$ are not adjacent.

Consider the random graph $G(n,p)$, arbitrary vertices $\widetilde{x}_1,...,\widetilde{x}_m \in V_n$, and a random variable $N^{(K,T)}_{(G,H)}(\widetilde{x}_1,...,\widetilde{x}_m)$ which assigns to each graph $\mathcal{G}$ from Ω_n the number of strict $(G,(x_1,\ldots,x_m))$-extensions $\widetilde{G}$ of $(\widetilde{x}_1,...,\widetilde{x}_m)$ in $\mathcal{G}$ such that the pair $(\widetilde{G},\widetilde{G}|_{\{\widetilde{x}_1,\ldots,\widetilde{x}_m\}})$ is (K,T)-maximal in $\mathcal{G}$ (and $N_{(G,H)}(\widetilde{x}_1,...,\widetilde{x}_m)$ is the number of all $(G,(x_1,\ldots,x_m))$-extensions of $(\widetilde{x}_1,...,\widetilde{x}_m)$ in $\mathcal{G}$). The following result on the asymptotic behavior of this variable was proved in [14].

THEOREM 3.4 [14]. *Let* $0<\alpha_1<\alpha_2<1$. *Suppose* (G,H) *is* α_2*-safe*, $f_{\alpha_1}(K,T)<$ <0, *and* $v(T)\leq v(G)$. *Let* $p\in[n^{-\alpha_2},n^{-\alpha_1}]$. *Then a.a.s. for any* $\widetilde{x}_1,\ldots,\widetilde{x}_m$ *we have* $N^{(K,T)}_{(G,H)}(\widetilde{x}_1,...,\widetilde{x}_m)>0$.

If $\widetilde{H}=(\varnothing,\varnothing)$ and $(\widetilde{G},\widetilde{H})$ is (K,T)-maximal in Γ, then $\widetilde{G}$ is (K,T)-maximal in Γ. Therefore, we can state a particular case of Theorem 3.4 which considers (K,T)-maximal graphs. Let $N^{(K,T)}_G$ be a random variable that assigns to each $\mathcal{G}$ from Ω_n the number of (K,T)-maximal copies of G in $\mathcal{G}$.

COROLLARY 3.1. *Let* $0<\alpha_1<\alpha_2<1$. *Let* G *be a strictly balanced graph with* $\rho(G)<1/\alpha_2$ *and* $f_{\alpha_1}(K,T)<0$. *If* $p\in[n^{-\alpha_2},n^{-\alpha_1}]$, *then a.a.s. we have* $N^{(K,T)}_G>0$.

We say that two pairs $(G,(x_1,\ldots,x_m))$ and $(\widetilde{G},(\widetilde{x}_1,\ldots,\widetilde{x}_m))$, where $\{x_1,\ldots,x_m\}\subset V(G)$ and $\{\widetilde{x}_1,\ldots,\widetilde{x}_m\}\subset \widetilde{V}(G)$, are *isomorphic*, if $\widetilde{G}$ is a strict $(G,(x_1,\ldots,x_m))$-extension of $(\widetilde{x}_1,\ldots,\widetilde{x}_m)$.

In our proofs we use a lemma on the existence of a copy of a strictly balanced graph without extensions, which is stated below. A method for obtaining such results is introduced in [3]. Here, we use this method to prove the lemma.

Let H be a strictly balanced graph, and let (G,H) be a strictly balanced pair. Suppose that $\rho(H)=\rho(G,H)=1/\alpha$. Suppose also that $V(H)=\{h_1,\ldots,h_v\}$, where $v=v(H)$. Let W_n be a set of maximum cardinality which contains ordered v-tuples of vertices from V_n which satisfy the following property: for any two ordered tuples $w_1 = (x_{i_1}, \ldots, x_{i_v})$, $w_2 = (x_{i_{\sigma(1)}}, \ldots, x_{i_{\sigma(v)}}) \in W_n$ which coincide as sets, the permutation of $(h_1, \ldots, h_v)$ induced by σ does not preserve the edges of H (i. e. the mapping $\phi : V(H) \to V(H)$ such that $\phi(h_i) = h_{\sigma(i)}$, $i \in \{1, \ldots, v\}$, is not an automorphism of H). Obviously, $|W| = \dfrac{n!}{(n-v)!a(H)}$. For each $w \in W_n$ we denote the set of the elements of w by $\overline{w}$. For each $w = (x_{i_1}, \ldots, x_{i_v}) \in W_n$ let us consider the event A_w that some spanning subgraph in $G(n, n^{-\alpha})|_{\overline{w}}$ is isomorphic to H and the corresponding isomorphism maps x_{i_j} to h_j for each $j \in \{1, \ldots, v\}$.

Lemma 3.1. *There exists a subsequence $\{n_i\}_{i\in\mathbb{N}}$ of the sequence of positive integers such that the following property holds. With positive asymptotic probability less than* 1, *there is at least one copy of H in $G(n_i, n_i^{-\alpha})$, and for any $w \in W_{n_i}$ either $\overline{A_w}$ holds, or there is no $(G, (h_1, \ldots, h_v))$-extension of w in $G(n_i, n_i^{-\alpha})$.*

Proof. Denote $N_H^-(w) = \sum_{\widetilde{w}} I(A_{\widetilde{w}})$, where the summation is taken over all $\widetilde{w} \in W_n$ which do not intersect w. Denote $N_H^+(w) = \sum_{\widetilde{w}} \xi_{\widetilde{w}}$, where the summation is taken over all $\widetilde{w} \in W_n$ which intersect w and satisfy the condition $\overline{\widetilde{w}} \cap \overline{w} \neq \overline{w}$. The random variable $\xi_{\widetilde{w}}$ is defined in the following way. For any $\mathcal{G} \in \Omega_n$, the equality $\xi_{\widetilde{w}}(\mathcal{G}) = 1$ holds if and only if $\mathcal{G}$ with edges between any two vertices from $\overline{w} \cap \overline{\widetilde{w}}$ satisfies A_w (otherwise, $\xi_{\widetilde{w}}(\mathcal{G}) = 0$). Set $N_H(w) = N_H^-(w) + N_H^+(w)$.

Denote by μ_n the probability of the event that there is at least one copy of H in $G(n, n^{-\alpha})$ and for any ordered v-tuple w of vertices from V_n either A_w holds, or there is no $(G, (h_1, \ldots, h_v))$-extension of w. Then

$$\mathsf{P}(N_H > 0) \geq \mu_n = \mathsf{P}(N_G = 0) - \mathsf{P}(N_H = 0) \geq \mathsf{P}(N_H = 1, N_G = 0).$$

Theorem 3.2 implies $\lim\limits_{n\to\infty} \mathsf{P}(N_H > 0) = 1 - e^{-1/a(H)}$. Finally,

$$\mathsf{P}(N_H = 1, N_G = 0) = \sum_{w\in W} \mathsf{P}(N_H = 1, N_G = 0 | A_w)\mathsf{P}(A_w) =$$

$$= \sum_{w\in W} \mathsf{P}(N_H(w) = 0, N_{(G,H)}(w) = 0 | A_w)\mathsf{P}(A_w) =$$

$$\sum_{w\in W} \mathsf{P}(N_H(w) = 0, N_{(G,H)}(w) = 0)\mathsf{P}(A_w) =$$

$$= \mathsf{P}(N_H(w_0) = 0, N_{(G,H)}(w_0) = 0) \sum_{w \in W} \mathsf{P}(I_w) \sim$$

$$\sim \frac{1}{a(H)} \mathsf{P}(N_H^-(w_0) = 0, N_{(G,H)}(w_0) = 0),$$

where $w_0 \in W$ is an arbitrary ordered tuple. The asymptotic equality holds because, as implied by Theorem 3.1, a.a.s. there is no subgraph in $G(n, n^{-\alpha})$ with at most $2v$ vertices and density greater than $1/\alpha$. The probability $\mathsf{P}(N_H^-(w_0) = 0, N_{(G,H)}(w_0) = = 0)$ converges to some positive number less than 1 (see [3]), which proves the lemma. □

4. Proofs

First of all, let us introduce some notation.

Let $\mathcal{G}$ be an arbitrary graph. Let r, s be arbitrary positive integers. Given arbitrary vertices $x_1, \ldots, x_s$ of $\mathcal{G}$, we denote the set of all common r-neighbors of $x_1, \ldots, x_s$ in $\mathcal{G}$ by $N_r(x_1, \ldots, x_s)$ (we omit $\mathcal{G}$ when there is no risk of confusion). An *r-neighbor* of a vertex x is a vertex y such that the minimum length of a path connecting x and y equals r (*the length of a path* is the number of edges in it). Set $N(x_1, \ldots, x_s) := N_1(x_1, \ldots, x_s)$.

Let x, y be arbitrary vertices of $\mathcal{G}$ and let A, B be arbitrary subgraphs of $\mathcal{G}$. Denote by $d_{\mathcal{G}}(x, y)$ the length of a minimal path in $\mathcal{G}$ connecting x and y (*a minimal path* is a path with the minimum length among all the paths considered). We say that a path connecting x and some vertex of A is *a minimal path connecting x and A in $\mathcal{G}$*, if its length equals $\min_{y \in V(A)} d_{\mathcal{G}}(x, y)$. Let us set $d_{\mathcal{G}}(x, A) = d_{\mathcal{G}}(A, x) = \min_{v \in V(A)} d_{\mathcal{G}}(x, v)$, $d_{\mathcal{G}}(A, B) = \min_{v \in V(A)} d_{\mathcal{G}}(v, B)$.

4.1. Proof of Theorem 2.1

Let $k \geq 5$, $m \in \mathbb{N}$, $m \geqslant \lfloor k/2 \rfloor$, $\alpha = \dfrac{1}{\lfloor k/2 \rfloor} + \dfrac{1}{\lfloor k/2 \rfloor (m + \lfloor k/2 \rfloor - 1)}$, and $p = n^{-\alpha}$.

Consider the set $\widetilde{\Omega}_n$ of all graphs $\mathcal{G}$ from Ω_n satisfying the following properties.

1. For each strictly balanced pair (G, H) such that $V(H) = \{h_1, \ldots, h_v\}$, $\rho(G, H) < 1/\alpha$, $v \leq m + \lfloor k/2 \rfloor - 1$, $v(G) \leq 2(m + \lfloor k/2 \rfloor - 1)$, every ordered vertex v-tuple has a $(G, (h_1, \ldots, h_v))$-extension in $\mathcal{G}$.

2. For each G with $v(G) \leq 2(m + \lfloor k/2 \rfloor + 1)$ and $\rho^{\max}(G) > 1/\alpha$, there is no copy of G in $\mathcal{G}$.

   Theorem 3.1 and Theorem 3.3 imply that $\mathsf{P}(G(n,p) \in \widetilde{\Omega}_n) \to 1$ as $n \to \infty$. Let L be the first-order property expressed by the formula

$$\exists\, x_1 \ldots \exists\, x_{\lfloor k/2 \rfloor}\ \varphi(x_1, \ldots, x_{\lfloor k/2 \rfloor})$$

of quantifier depth $\max(2\lfloor k/2 \rfloor, \lfloor k/2 \rfloor + 3) \leq k$, where $\varphi(x_1, \ldots, x_{\lfloor k/2 \rfloor}) =$

$$[K(x_1, \ldots, x_{\lfloor k/2 \rfloor}) \wedge (\exists\, y_1 \ldots \exists\, y_{\lfloor k/2 \rfloor}\ [(y_1 \in N(x_1, \ldots, x_{\lfloor k/2 \rfloor})) \wedge \ldots \wedge (y_{\lfloor k/2 \rfloor} \in$$
$$\in N(x_1, \ldots, x_{\lfloor k/2 \rfloor})) \wedge$$

$$K(y_1, \ldots, y_{\lfloor k/2 \rfloor})]) \wedge (\neg(\exists\, z\ [R_z^2 \wedge \ldots \wedge R_z^{\lfloor k/2 \rfloor} \wedge (\forall\, y\ ((y \in N(x_1, \ldots, x_{\lfloor k/2 \rfloor})) \Rightarrow$$
$$\Rightarrow R_z^{1,2}(y)))]))].$$

Here we use the following notation:

$$K(x_1,\ldots,x_{\lfloor k/2 \rfloor}) = ((x_1 \sim x_2) \wedge (x_1 \sim x_3) \wedge \ldots \wedge (x_1 \sim x_{\lfloor k/2 \rfloor}) \wedge \ldots \wedge (x_{\lfloor k/2 \rfloor - 1} \sim x_{\lfloor k/2 \rfloor})),$$
$$(y \in N(x_1,\ldots,x_{\lfloor k/2 \rfloor})) = ((y \sim x_1) \wedge \ldots \wedge (y \sim x_{\lfloor k/2 \rfloor}));$$
$$R_z^{i,j}(a) = (\exists v\, [(v \in N(z,a,x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{j-1},x_{j+1},\ldots,x_{\lfloor k/2 \rfloor})) \wedge$$
$$\wedge (v \nsim x_i) \wedge (v \nsim x_j)]),$$

for each $1 \leq i < j \leq \lfloor k/2 \rfloor$, and

$$R_z^i = (\exists\, v\ [v \in N(z, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{\lfloor k/2 \rfloor})]),$$

for each $2 \leq i \leq \lfloor k/2 \rfloor$.

Suppose that $\mathcal{G} \in \widetilde{\Omega}_n$ satisfies L. Consider vertices $x_1, \ldots, x_{\lfloor k/2 \rfloor}$ such that $\varphi(x_1, \ldots, x_{\lfloor k/2 \rfloor})$ is true. Set $X = \mathcal{G}|_{\{x_1,\ldots,x_{\lfloor k/2 \rfloor}\} \cup N(x_1,\ldots,x_{\lfloor k/2 \rfloor})}$, $\chi = |V(X)| - \lfloor k/2 \rfloor$, where $N(x_1, \ldots, x_{\lfloor k/2 \rfloor}) = \{x^1, \ldots, x^{\chi}\}$. Let us prove that $\chi \geq m$. Suppose that $\chi < m$. By the definition of $\widetilde{\Omega}_n$ there are vertices $z, v_1, \ldots, v_{\chi + \lfloor k/2 \rfloor - 1}$ in $\mathcal{G}$ such that for every $i \in \{1, \ldots, \chi\}$ we have $v_i \in N(x^i, z, x_3, \ldots, x_{\lfloor k/2 \rfloor})$, and for every $i \in \{\chi + 1, \ldots, \chi + \lfloor k/2 \rfloor - 1\}$ we have $v_i \in N(z, x_1, \ldots, x_{i-\chi}, x_{i-\chi+2}, \ldots, x_{\lfloor k/2 \rfloor})$.

Indeed, in this case the pair $(\mathcal{G}|_{\{x_1,\ldots,x_{\lfloor k/2\rfloor},v_1,\ldots,v_{\chi+\lfloor k/2\rfloor-1},z\}\cup N(x_1,\ldots,x_{\lfloor k/2\rfloor})}, X)$ is strictly balanced with the density

$$\frac{\lfloor k/2\rfloor(\chi+\lfloor k/2\rfloor-1)}{\chi+\lfloor k/2\rfloor}=\frac{1}{1/\lfloor k/2\rfloor+1/(\lfloor k/2\rfloor(\chi+\lfloor k/2\rfloor-1))}<\frac{1}{\alpha}.$$

This contradicts L. Therefore, $\chi\geq m$. Now let us prove that $\chi=m$. Suppose that $\chi>m$. Remove some of the vertices from $N(x_1,\ldots,x_{\lfloor k/2\rfloor})$ preserving $m+1$ vertices (but $\lfloor k/2\rfloor$ pairwise adjacent vertices are still in the set). Denote by $\widetilde{X}$ the subgraph of X induced by the union of the remaining vertices with $x_1,\ldots,x_{\lfloor k/2\rfloor}$. Then

$$\rho(\widetilde{X})\geq\frac{\lfloor k/2\rfloor(m+1)+\lfloor k/2\rfloor(\lfloor k/2\rfloor-1)}{m+1+\lfloor k/2\rfloor}>1/\alpha.$$

This contradicts Property 2 in the definition of $\widetilde{\Omega}_n$.

So, $\chi=m$. Let z be a vertex such that $R_z^{1,2}$ is true for all vertices in $N(x_1,\ldots,x_{\lfloor k/2\rfloor})$ and R_z^i is true for every $i\in\{2,\ldots,\lfloor k/2\rfloor\}$. Then there are vertices $v_1,\ldots,v_j$ in $\mathcal{G}$ such that $z\in N(v_1,\ldots,v_j)$ and the set $\{x_1,\ldots,x_{\lfloor k/2\rfloor-1}\}\cup$ $\cup\, N(x_1,\ldots,x_{\lfloor k/2\rfloor})$ can be divided into j subsets $N_1,\ldots,N_j$ so that for each $i\in$ $\in\{1,\ldots,j\}$ and each $y\in N_i$ we have $y\sim v_i$, where v_i is adjacent to $\lfloor k/2\rfloor-2$ vertices from $\{x_1,\ldots,x_{\lfloor k/2\rfloor}\}\setminus\{y\}$. Set $Y=\mathcal{G}|_{\{x_1,\ldots,x_{\lfloor k/2\rfloor},v_1,\ldots,v_j,z\}\cup N(x_1,\ldots,x_{\lfloor k/2\rfloor})}$. Then

$$1/\rho(Y)\leq\frac{\lfloor k/2\rfloor+j+1+m}{\lfloor k/2\rfloor(m+\lfloor k/2\rfloor-1)+m+\lfloor k/2\rfloor-1+j(\lfloor k/2\rfloor-1)}.$$

Notice that the inequality $j<m+\lfloor k/2\rfloor-1$ implies $1/\rho(Y)<\alpha$. Thus, it follows from the definition of $\widetilde{\Omega}_n$ that $j\geq m+\lfloor k/2\rfloor-1$. As $j\leq m+\lfloor k/2\rfloor-1$, the equality $j=m+\lfloor k/2\rfloor-1$ holds. Therefore, $1/\rho(Y)\leq\dfrac{2(m+\lfloor k/2\rfloor)}{2\lfloor k/2\rfloor(m+\lfloor k/2\rfloor-1)}=\dfrac{1}{\lfloor k/2\rfloor}+$ $+\dfrac{1}{\lfloor k/2\rfloor(m+\lfloor k/2\rfloor-1)}=\alpha$, $1/\rho(X)\leq\dfrac{m+\lfloor k/2\rfloor}{\lfloor k/2\rfloor(m+\lfloor k/2\rfloor-1)}=\alpha$. Property 2 in the definition of $\widetilde{\Omega}_n$ implies the equalities $\rho(X)=\rho(Y)=1/\alpha$. Since there is no vertex z in $\mathcal{G}$ satisfying the above properties, the graph $\mathcal{G}$ does not contain a copy of Y which in turn contains X.

In the remaining part of the proof we will keep on using X and Y to denote the obtained graphs (the first one is strictly balanced, the second one is balanced, and the pair (Y, X) is strictly balanced) of density $1/\alpha$. We also denote by $\widetilde{L}$ the obtained property of $\mathcal{G}$ (the existence of a copy of X such that no copy of Y contains it). So, we have proved that if $\mathcal{G} \in \widetilde{\Omega}_n$ and $\mathcal{G}$ satisfies L, then $\mathcal{G}$ satisfies $\widetilde{L}$.

Suppose that $\mathcal{G} \in \widetilde{\Omega}_n$ and $\mathcal{G}$ satisfies $\widetilde{L}$. Then, obviously, $\mathcal{G}$ satisfies L as well.

By Lemma 3.1 there exists a partial limit $\lim_{i\to\infty} \mathsf{P}(G(n_i, n_i^{-\alpha}) \models \widetilde{L}) = c$, which is neither 0, nor 1. Moreover,

$$\mathsf{P}(G(n_i, n_i^{-\alpha}) \models L) \sim \mathsf{P}(G(n_i, n_i^{-\alpha}) \in \widetilde{\Omega}_{n_i}, G(n_i, n_i^{-\alpha}) \models L) =$$

$$= \mathsf{P}(G(n_i, n_i^{-\alpha}) \in \widetilde{\Omega}_{n_i}, G(n_i, n_i^{-\alpha}) \models \widetilde{L}) \sim \mathsf{P}(G(n_i, n_i^{-\alpha}) \models \widetilde{L}) = c. \qquad (4.1)$$

Noticing that $\frac{1}{\lfloor k/2 \rfloor} + \frac{1}{\lfloor k/2 \rfloor (m + \lfloor k/2 \rfloor - 1)} \to \frac{1}{\lfloor k/2 \rfloor}$ as $m \to \infty$, we complete the proof.

4.2. Proof of Theorem 2.2

Let $m \geq 2$ be an arbitrary integer, $\alpha = 1 - \frac{1}{2^{k-5}} + \frac{1}{2^{k-5}m}$, and $p = n^{-\alpha}$.

Consider the set $\widetilde{\Omega}_n$ of all graphs $\mathcal{G}$ from Ω_n satisfying the following the properties.

1. For every strictly balanced pair (G, H) such that $V(H) = \{h_1, \ldots, h_v\}$, $\rho(G, H) < 1/\alpha$, $v \leq (2^{k-5} - 1)(m - 1) + 2$, $v(G) \leq 2(2^{k-5} - 1)(m - 1) + 3$, each ordered vertex v-tuple has a $(G, (h_1, \ldots, h_v))$-extension in $\mathcal{G}$.
2. For any G with $v(G) \leq 2(2^{k-5} - 1)(m + 1) + 2$ and $\rho^{\max} > 1/\alpha$ there is no copy of G in $\mathcal{G}$.

Theorem 3.1 and Theorem 3.3 imply that $\mathsf{P}(G(n, p) \in \widetilde{\Omega}_n) \to 1$ as $n \to \infty$.

The property of vertices x and y *to be at the distance* i (i. e., the length of a minimal path which connects x and y equals i) is expressed by the formula

$$D_i^*(x, y) = D_i(x, y) \wedge \left(\neg \left(\bigvee_{j=1}^{i-1} D_j(x, y) \right) \right),$$

where $D_i(x, y)$ is the following formula of quantifier depth $\lceil \log_2 i \rceil$:

$$D_i(x, y) = \exists\, v\, (D_{i/2}(x, v) \wedge D_{i/2}(y, v)), \quad \text{if } i \text{ is even},$$

$$D_i(x, y) = \exists\, v\, (D_{(i-1)/2}(x, v) \wedge D_{(i+1)/2}(y, v)), \quad \text{if } i \text{ is odd},$$

and $D_1(x,y)=(x\sim y)$, $D_0(x,y)=(x=y)$. We also set $D^*_{i,j}(x,y,z)=D^*_i(x,z)\wedge D^*_j(z,y)$.

Let L be the first-order property expressed by the formula $\exists\, a\exists\, b\ \varphi(a,b)$ of quantifier depth k, where

$$\varphi(a,b)=(S(a,b)\wedge[\forall\, u\,(D^*_{2^{k-6},2^{k-6}}(a,b,u)\Rightarrow R(a,u))]\wedge \\ \wedge[\neg(\exists\, z\ ((z\neq a)\wedge(\forall\, u\,(D^*_{2^{k-6},2^{k-6}}(a,b,u)\Rightarrow D^*_{2^{k-5}}(u,z)))))]).$$

The predicate

$$S(a,b)=(D^*_{2^{k-5}}(a,b)\wedge \\ \wedge(\neg(\exists\, u_1\exists\, u_2\exists\, x\ \ [(u_1\neq u_2)\wedge D^*_{2^{k-6},2^{k-6}}(u_1,u_2,b)\wedge D^*_{2^{k-6},2^{k-6}}(u_1,u_2,a)\wedge \\ \wedge\psi(a,b,u_1,u_2,x)]))),$$

where

$$\psi(a,b,u_1,u_2,x)=\left(\neg\left(\left(\bigvee_{s=2^{k-6}}^{2^{k-5}}\bigvee_{i=1}^{s}\bigvee_{j=2^{k-6}-i}^{2^{k-5}-i}(D^*_{i,s-i}(a,u_1,x)\wedge D^*_j(x,u_2))\right)\vee \right.\right. \\ \left.\left.\vee\left(\bigvee_{i=1}^{2^{k-6}}(D^*_{i,2^{k-6}-i}(u_1,b,x)\wedge D^*_i(u_2,x))\right)\right)\right),$$

is true when there are no two distinct paths of length at most 2^{k-5} connecting the vertex a and two distinct vertices from $N_{2^{k-6}}(a,b)$ (notice that any two distinct vertices from $N_{2^{k-6}}(a,b)$ do not have common neighbors) and there are no two distinct intersecting paths of length 2^{k-6} connecting b and two distinct vertices from $N_{2^{k-6}}(a,b)$. If the predicate

$$R(a,u)=\left(\exists\, x_1\exists\, x_2\ \ \left[D^*_{2^{k-6},2^{k-6}}(a,u,x_1)\wedge D^*_{2^{k-7},2^{k-7}}(a,u,x_2)\wedge \right.\right. \\ \left.\left.\wedge(\neg D^*_{2^{k-7}}(x_1,x_2))\wedge\xi(a,x_1,x_2)\wedge\xi(u,x_1,x_2)\right]\right),$$

where

$$\xi(a,x_1,x_2)=\left(\neg\left(\exists\, y\left(\bigvee_{i=1}^{2^{k-7}-1}(D^*_{i,2^{k-6}-i}(a,x_1,y)\wedge D^*_{2^{k-7}-i}(y,x_2))\right)\right)\right),$$

is true, then there are two non-intersecting paths of length 2^{k-6} and 2^{k-5} connecting a and u.

Suppose that a graph $\mathcal{G} \in \widetilde{\Omega}_n$ satisfies L. Let a, b be vertices such that $\varphi(a, b)$ is true. Let X be the union of all the paths of length 2^{k-5} which connect a and b in $\mathcal{G}$. Let χ be the number of all such paths and let $N_{2^{k-6}}(a, b) = \{x^1, \ldots, x^\chi\}$. Let us prove that $\chi \geq m$. Suppose $\chi < m$. By the definition of $\widetilde{\Omega}_n$ there is a vertex z in $\mathcal{G}$ such that $D^*_{2^{k-5}}(x^i, z)$ holds for each $i \in \{1, \ldots, \chi\}$, and there are χ paths $P_1, \ldots, P_\chi$ of length 2^{k-5} connecting z and $x^1, \ldots, x^\chi$ respectively such that for any distinct $i, j \in \{1, \ldots, \chi\}$ we have $V(P_i) \cap V(P_j) = \{z\}$. But if those paths do exist, then the pair $(X \cup P_1 \cup \ldots \cup P_\chi, X)$ is strictly balanced and its density equals

$$\frac{2^{k-5}\chi}{(2^{k-5}-1)\chi + 1} = \frac{1}{1 - 1/2^{k-5} + 1/(\chi 2^{k-5})} < \frac{1}{1 - 1/2^{k-5} + 1/(m 2^{k-5})} = \frac{1}{\alpha},$$

which contradicts L. Therefore, $\chi \geq m$. Finally, let us prove that $\chi = m$. Suppose $\chi > m$. Let us remove from X some of the paths of length 2^{k-5} connecting a and b (keeping a and b themselves) in such a way that $m+1$ paths remain. Now let us add to the remaining graph the paths from $\mathcal{G}$ of length 2^{k-5} which connect a and the vertices from $N_{2^{k-6}}(a, b)$ (one path for each vertex) and satisfy the following condition: the intersection of any two of those paths equals $\{a\}$ and the intersection of any of those paths with any path from X contains only a and a vertex from $N_{2^{k-6}}(a, b)$. Denote the final graph by $\widetilde{X}$. Then $\rho(\widetilde{X}) = \dfrac{2^{k-4}(m+1)}{2(2^{k-5}-1)(m+1)+2} > 1/\alpha$, which contradicts Property 2 in the definition of $\widetilde{\Omega}_n$.

Thus, $\chi = m$. Let $z \neq a$ be a vertex such that $D^*_{2^{k-5}}(\cdot, z)$ is true for all vertices from $N_{2^{k-6}}(a, b)$. Then there are paths $P_1, \ldots, P_m$ in $\mathcal{G}$ of length 2^{k-5} which connect z with $x^1, \ldots, x^m$ respectively. Suppose that for some $i \in \{1, \ldots, m-1\}$ we have $P_{i+1} \subseteq P_1 \cup \ldots \cup P_i$. Set

$$P_{i+1} = (\{x^{i+1}, v_1, \ldots, v_{2^{k-5}-1}, z\}, \{\{x^{i+1}, v_1\}, \{v_1, v_2\}, \ldots, \{v_{2^{k-5}-1}, z\}\}).$$

Then v_1 is in $V(P_j)$ for some $j \in \{1, \ldots, i\}$. Obviously, $v_1 \neq x^j$ (otherwise $D_{2^{k-5}-1}(x^j, z)$ is true). Suppose $v_1 \nsim x^j$ in $\mathcal{G}$. Then $D_s(z, v_1)$ is true for some natural $s < 2^{k-5} - 1$. Since $v_1 \sim x^{i+1}$, the proposition $D_{s+1}(x^{i+1}, z)$ is true as well. This contradicts the truth of $D^*_{2^{k-5}}(x^{i+1}, z)$. Therefore, v_1 is a common neighbor of x^{i+1} and x^j. This contradicts the truth of $S(a, b)$. Hence $P_{i+1} \nsubseteq P_1 \cup \ldots \cup P_i$

for any $i \in \{1, \dots, m-1\}$. Let us replace X with its union with the paths from $\mathcal{G}$ of length 2^{k-5} which connect a and the vertices from $N_{2^{k-6}}(a,b)$ (one path for each vertex) and satisfy the following condition: the intersection of any two of those paths equals $\{a\}$ and the intersection of any of those paths with any path from X contains only a and a vertex from $N_{2^{k-6}}(a,b)$. Consider the sequence of graphs $X_0 = X$, $X_1 = X \cup P_1$, $X_2 = X \cup P_1 \cup P_2$, ..., $X_m = X \cup P_1 \cup \ldots \cup P_m$. Set $Y := X_m$. For each $i \in \{0, \dots, m-1\}$ the graph X_{i+1} is obtained from X_i by adding $n_i \leq 2^{k-5} - 1$ vertices and $e_i \geq n_i + 1$ edges. Therefore,

$$1/\rho(Y) \leq \frac{2(2^{k-5}-1)m + 2 + n_1 + \ldots + n_m + 1}{2^{k-4}m + n_1 + \ldots + n_m + m} \leq \alpha,$$

where the equalities hold if and only if $n_i = 2^{k-5} - 1$ and $e_i = 2^{k-5}$ for each $i \in \{0, \dots, m-1\}$. Therefore, by the definition of $\widetilde{\Omega}_n$ those equalities do hold and $1/\rho(Y) = 1/\rho(X) = \alpha$. Since there is no vertex z in $\mathcal{G}$ which satisfies the above properties, $\mathcal{G}$ does not contain a copy of Y which contains X.

Same as in Theorem 2.1, we use X and Y below to denote the two obtained graphs of density $1/\alpha$ (obviously, the graph X and the pair (Y, X) are strictly balanced). We also denote by $\widetilde{L}$ the obtained property of $\mathcal{G}$ (the existence of a copy of X which is not contained in any copy of Y). We proved that if $\mathcal{G} \in \widetilde{\Omega}_n$ and $\mathcal{G}$ satisfies L, then $\mathcal{G}$ satisfies $\widetilde{L}$ as well.

Finally, suppose that $\mathcal{G} \in \widetilde{\Omega}_n$ and $\mathcal{G}$ satisfies $\widetilde{L}$. Then, obviously, $\mathcal{G}$ satisfies L as well.

By Lemma 3.1 there exists a partial limit $\lim_{i\to\infty} \mathsf{P}(G(n_i, n_i^{-\alpha}) \models \widetilde{L}) = c$, which is neither 0, nor 1. Taking into account (4.1) and noticing that $1 - \frac{1}{2^{k-5}} + \frac{1}{2^{k-5}m} \to$ $\to 1 - \frac{1}{2^{k-5}}$ as $m \to \infty$, we complete the proof.

4.3. Proof of Theorem 2.3

We start the proof with formulating the theorem of Ehrenfeucht in Section 4.3.1. This theorem is the main tool in proofs of zero-one laws. Then in Section 4.3.2 we define some supplementary constructions (cyclic extensions), after which in Section 4.3.3 we describe asymptotic properties of the random graph which imply the existence of a winning strategy of Duplicator. This strategy is described in Sections 4.3.4–4.3.8.

4.3.1. Ehrenfeucht game

In this section we state a particular case of Ehrenfeucht theorem (see [4]), which holds for graphs. First, let us define Ehrenfeucht game $\mathrm{EHR}(G,H,i)$ on graphs G,H and i rounds (see, e. g., [7, 22]). Let $V(G) = \{x_1,...,x_n\}$ and $V(H) = \{y_1,...,y_m\}$. In the ν-th round ($1 \leq \nu \leq i$) Spoiler chooses a vertex in any graph (he chooses either $x_{j_\nu} \in V(G)$, or $y_{j'_\nu} \in V(H)$). Then Duplicator chooses any vertex in the other graph. If Spoiler chooses in the μ-th round, say, a vertex $x_{j_\mu} \in V(G)$, $j_\mu = j_\nu$ ($\nu < \mu$), then Duplicator must choose the vertex $y_{j'_\nu} \in V(H)$. If in this round Spoiler chooses, say, a vertex $x_{j_\mu} \in V(G)$, $j_\mu \notin \{j_1,...,j_{\mu-1}\}$, then Duplicator must choose a vertex $y_{j'_\mu} \in V(H)$ such that $j'_\mu \notin \{j'_1,...,j'_{\mu-1}\}$. If he can not do this, Spoiler wins. At the end of the last round we have vertices $x_{j_1},...,x_{j_i} \in V(G)$ and $y_{j'_1},...,y_{j'_i} \in V(H)$ chosen. If some of them coincide, then we leave out the copies and consider only distinct vertices: $x_{h_1},...,x_{h_l}$; $y_{h'_1},...,y_{h'_l}$, $l \leq i$. Duplicator wins if and only if the corresponding subgraphs are isomorphic up to the order of the vertices:

$$G|_{\{x_{h_1},...,x_{h_l}\}} \cong H|_{\{y_{h'_1},...,y_{h'_l}\}}.$$

THEOREM 4.1 [4]. *For any graphs G,H and any $i \in \mathbb{N}$, Duplicator has a winning strategy in the game* $\mathrm{EHR}(G,H,i)$ *if and only if for any property L which is expressed by a first-order formula of quantifier depth at most i either both graphs satisfy L, or they both do not satisfy it.*

It can be easily shown that this theorem has the following corollary related to zero-one laws (see, e. g., [22]).

THEOREM 4.2. *$G(n,p)$ obeys the zero-one k-law if and only if*

$$\lim_{n,m\to\infty} \mathsf{P}(\textit{Duplicator has a winning strategy in } \mathrm{EHR}(G(n,p(n)),G(m,p(m)),k)) = 1.$$

4.3.2. Constructions

Let $m \geq 2$ be an arbitrary integer. Consider a pair of graphs (G,H) such that $G \supset H$. We say that G is a *cyclic m-extension of H* if one of the following properties holds.

- We have $m \geq 3$ and there is a vertex x_1 of G such that

$$V(G) \setminus V(H) = \{y^1_1,...,y^1_{t_1},y^2_1,...,y^2_{t_2}\},$$

$E(G) \setminus E(H) =$

$= \{\{x_1, y_1^1\}, \{y_1^1, y_2^1\}, ..., \{y_{t_1-1}^1, y_{t_1}^1\}, \{y_{t_1}^1, y_1^2\}, \{y_1^2, y_2^2\}, ..., \{y_{t_2-1}^2, y_{t_2}^2\}, \{y_{t_2}^2, y_{t_1}^1\}\},$

where $t_1 + t_2 \leq m - 1$, $t_1 \geq 0$, $t_2 \geq 2$ (if $t_1 = 0$, then x_1 is adjacent to $y_1^2, y_{t_2}^2$). If so, G is said to be an extension of the *first type*.

- We have $m \geq 2$ and there are two distinct vertices x_1, x_2 of G such that for some $t \leq m - 1$

$$G = (V(H) \sqcup \{y_1, ..., y_t\}, E(H) \sqcup \{\{x_1, y_1\}, \{y_1, y_2\}, ..., \{y_{t-1}, y_t\}, \{y_t, x_2\}\}).$$

If so, G is said to be an extension of the *second type*.

Let $H \subset G$ be two subgraphs in a graph Γ. The pair (G, H) is said to be *cyclically m-maximal in* Γ if there are no cyclic m-extensions of G in Γ which are not cyclic m-extensions of H.

4.3.3. Properties which imply the existence of a winning strategy for Duplicator

Let k and b be arbitrary positive integers, $k > 3$. Let $\frac{a}{b}$ be a positive rational number in reduced form, $\alpha = 1 - \frac{1}{2^{k-1} + a/b}$, $p = n^{-\alpha}$. Suppose also that $a \in \{\max\{1, 2^{k-1} - b\}, \ldots, 2^{k-1}\}$.

Define $\mathcal{S}$ to be the set of all the graphs G which satisfy the following three properties.

1) There are no strictly balanced subgraphs in G with at most $2^{2k}b$ vertices and density greater than $1/\alpha$.
2) Let $\mathcal{H}$ be the set of α-safe pairs (H_1, H_2) such that $v(H_1) \leq 2^{2k}b + k2^k$. Let $\mathcal{K}$ be the set of pairs (K_1, K_2) such that $v(K_1) \leq 2^k$, $v(K_2) \leq 2$ and $f_\alpha(K_1, K_2) < 0$. Then for any pair $(H_1, H_2) \in \mathcal{H}$, $V(H_2) = \{v_1, \ldots, v_h\}$, and any subgraph $G_2 \subset G$, $V(G_2) = \{x_1, \ldots, x_h\}$, there is a strict $(H_1, (v_1, \ldots, v_h))$-extension G_1 of the ordered tuple $(x_1, \ldots, x_h)$ such that the pair (G_1, G_2) is (K_1, K_2)-maximal in G for any pair $(K_1, K_2) \in \mathcal{K}$.
3) Let $\mathcal{H}$ be the set of pairs (H_1, H_2) such that $v(H_1) \leq 2^k$, $v(H_2) \leq 2$ and $f_\alpha(H_1, H_2) < 0$. Then for any strictly balanced graph H with at most $2^{2k}b$ vertices and $\rho(H) < 1/\alpha$ there is a copy of H in G which is (H_1, H_2)-maximal in G for any $(H_1, H_2) \in \mathcal{H}$.

By Theorem 3.1, Theorem 3.4 and Corollary 3.1 we have $\mathsf{P}(G(n,p) \in \mathcal{S}) \to 1$ as $n \to \infty$. Therefore, by Theorem 4.2 the statement of Theorem 2.3 follows from the existence of a winning strategy for Duplicator in $\mathrm{EHR}(G, H, k)$ for all pairs (G, H) such that $G, H \in \mathcal{S}$.

4.3.4. Winning strategy for Duplicator

Let $G, H \in \mathcal{S}$. Let X_r, Y_r be the graphs chosen in the r-th round by Spoiler and Duplicator respectively. Then the sets $\{X_r, Y_r\}$ and $\{G, H\}$ coincide for all $r \in \{1, \ldots, k\}$. We denote by $x_r^1, \ldots, x_r^r$ and $y_r^1, \ldots, y_r^r$ the vertices chosen in the first r rounds in X_r and Y_r respectively. Let us describe Duplicator's strategy by induction. The strategy is divided into two parts. We denote the first and the second strategy by S and SF respectively. In the first round Duplicator always uses the strategy S and follows it until entering a round when the chosen subgraphs allow embarking upon the strategy SF introduced in [18] (we do not describe this strategy in the present paper because its detailed description can be found in [19], Section 4.8).

Before describing the strategies we should introduce another important notion. Suppose the players have finished r rounds, $r \in \{1, \ldots, k\}$. Suppose $l \in \{1, \ldots, r\}$ and consider arbitrary graphs $\widetilde{X}_r^1, \ldots, \widetilde{X}_r^l \subset X_r$, $\widetilde{Y}_r^1, \ldots, \widetilde{Y}_r^l \subset Y_r$ having no common vertices and satisfying the following properties.

I We have $x_r^1, \ldots, x_r^r \in V(\widetilde{X}_r^1 \cup \ldots \cup \widetilde{X}_r^l)$ and $y_r^1, \ldots, y_r^r \in V(\widetilde{Y}_r^1 \cup \ldots \cup \widetilde{Y}_r^l)$.

II For any distinct $j_1, j_2 \in \{1, \ldots, l\}$ we have

$$d_{X_r}(\widetilde{X}_r^{j_1}, \widetilde{X}_r^{j_2}) > 2^{k-r}, d_{Y_r}(\widetilde{Y}_r^{j_1}, \widetilde{Y}_r^{j_2}) > 2^{k-r}.$$

III For any $j \in \{1, \ldots, l\}$ there is no cyclic 2^{k-r}-extension of $\widetilde{X}_r^j$ (resp. $\widetilde{Y}_r^j$) in X_r (resp. Y_r).

IV Cardinalities of the sets $V(\widetilde{X}_r^1 \cup \ldots \cup \widetilde{X}_r^l)$ and $V(\widetilde{Y}_r^1 \cup \ldots \cup \widetilde{Y}_r^l)$ are at most $2^{2k}b + 2^{k-1}r$.

V For each $j \in \{1, \ldots, l\}$ the graphs $\widetilde{X}_r^j$ and $\widetilde{Y}_r^j$ are isomorphic. Moreover, there is an isomorphism (on the unions of those graphs) which takes x_r^i to y_r^i, $i \in \{1, \ldots, r\}$.

Two ordered tuples $\widetilde{X}_r^1, \ldots, \widetilde{X}_r^l$ and $\widetilde{Y}_r^1, \ldots, \widetilde{Y}_r^l$ satisfying the above properties are said to be *(k, r, l)-regular equivalent in* (X_r, Y_r). Let us denote the isomorphism from Property V by $\varphi(k, r, l)$ (generally, it is not unique, but we may take any of such isomorphisms).

Notice that the $(k, 1, 1)$-regular equivalence of $\widetilde{X}_1^1$ and $\widetilde{Y}_1^1$ is defined by Properties I, III, IV and V. And the (k, k, l)-regular equivalence of $\widetilde{X}_k^1, \ldots, \widetilde{X}_k^l$ and $\widetilde{Y}_k^1, \ldots, \widetilde{Y}_k^l$ is defined by Properties I, II, IV and V.

Two graphs $\widetilde{X}_r^1$ and $\widetilde{Y}_r^1$ are said to be (k, r)-*equivalent in* (X_r, Y_r) if

a) Properties I, IV and V hold for $l = 1$,

b) there is no cyclic $(2^{k-r} - 1)$-extension of $\widetilde{X}_r^1$ (resp. $\widetilde{Y}_r^1$) in X_r (resp. Y_r),

c) there is no second type cyclic 2^{k-r}-extension of $X_r|_{\{x_r^1, \ldots, x_r^r\}}$ (resp. $Y_r|_{\{y_r^1, \ldots, y_r^r\}}$) in X_r (resp. Y_r),

d) there is at most one cyclic 2^{k-r}-extension of $\widetilde{X}_r^1$ (resp. $\widetilde{Y}_r^1$) in X_r (resp. Y_r).

The main idea of Duplicator's strategy can be described as follows. Duplicator should play in such a way that for some $r \in \{1, \ldots, k-1\}$ and $l \in \{1, \ldots, r\}$ he gets (k, r, l)-regular equivalent ordered tuples of subgraphs in (X_r, Y_r). In the first round Duplicator must use the strategy S_1 described in the next section. After the r-th round, $r \in \{1, \ldots, k-3\}$, if (k, r, l)-regular equivalent ordered tuples have not been constructed yet, then, as we show, Duplicator can either find (k, r)-equivalent graphs (and then, in the $(r+1)$-th round, he must use the strategy S_{r+1} described in Section 4.3.6), or he must use the strategy S_{r+1}^1 described in Section 4.3.7. After S_{r+1}^1 Duplicator never turns back to S_{r+j}, $j \geq 2$. The strategy SF is described in [19] (Section 4.8) and is used by Duplicator in the $(r+1)$-th round, $r \geq 2$, if and only if after the r-th round for some $l \in \{1, \ldots, r\}$ (k, r, l)-regular equivalent ordered tuples of graphs in (X_r, Y_r) are constructed. It is proved in [19] that Duplicator wins when he uses the strategy SF.

4.3.5. Strategy S_1

Consider the first round and two possibilities to choose the first vertex by Spoiler.

Suppose there is no cyclic 2^{k-1}-extension of $(\{x_1^1\}, \varnothing)$ in X_1. Then Duplicator chooses a vertex $y_1^1 \in V(Y_1)$ such that there are no cyclic 2^{k-1}-extensions of $(\{y_1^1\}, \varnothing)$ in Y_1 (such a vertex exists because $Y_1 \in \mathcal{S}$ and, therefore, Y_1 satisfies 3)). Set $\widetilde{X}_1^1 = (\{x_1^1\}, \varnothing)$ and $\widetilde{Y}_1^1 = (\{y_1^1\}, \varnothing)$. Property III of $(k, 1, 1)$-regular equivalence of $\widetilde{X}_1^1$ and $\widetilde{Y}_1^1$ in (X_1, Y_1) is already proved. Obviously, Properties I, IV, and V hold as well. For the second round Duplicator uses the strategy SF.

Suppose there is at least one cyclic 2^{k-1}-extension $\widetilde{X}_1^1$ of $(\{x_1^1\},\varnothing)$ in X_1. Let us prove that there is a sequence of graphs $G_1, G_2, \ldots, G_s$ such that

a) for each $i \in \{1,\ldots,s-1\}$ G_{i+1} is a cyclic 2^{k-1}-extension of G_i in X_1 and G_1 is a cyclic 2^{k-1}-extension of $(\{x_1^1\},\varnothing)$,
b) we have either $X_1|_{V(G_s)} = G_s$, or $\rho(X_1|_{V(G_s)}) < 1/\alpha$,
c) there are no cyclic 2^{k-1}-extensions of G_s in X_1,
d) if for some $i \in \{1,\ldots,s-1\}$ G_{i+1} is a cyclic 2^{k-1}-extension of G_i, but it is not a cyclic $(2^{k-1}-1)$-extension of G_i, then there is a $\mu \in \{1,\ldots,s-1\}$ such that $G_{\mu+1}$ is a cyclic 2^{k-1}-extension of G_μ, but it is not a cyclic $(2^{k-1}-1)$-extension of G_μ, while there is no cyclic 2^{k-1}-extensions of G_μ in $X_1 \setminus (G_{\mu+1} \setminus G_\mu)$.

Let us prove the existence of such a sequence.

Obviously, there exists a sequence $G_1 \subset G_2 \ldots \subset G_i$ with the following two properties. The first one is that G_1 is a cyclic 2^{k-1}-extension of $(\{x_1^1\},\varnothing)$, and G_j is a cyclic 2^{k-1}-extension of G_{j-1} for each $j \in \{2,\ldots,i\}$. The second one is that i is the smallest value of j (in case the set of such j is not empty) such that G_j is a cyclic 2^{k-1}-extension of G_{j-1}, but it is not a cyclic $(2^{k-1}-1)$-extension of G_{j-1} (here $G_0 = (\{x_1^1\},\varnothing)$). If there is no such j, then there are no cyclic 2^{k-1}-extensions of G_i in X_1 (obviously, i is correctly defined and $i \le 2^{k-1}b+1$, because the density of G_i is greater than $1/\alpha$, so, $i = 2^{k-1}b+2$ would contradict Property 1)). In the latter case the sequence $G_1,\ldots,G_s$ $(s=i)$ satisfying Properties a), c) and d), is already built. Supposing that G_i is not the "last" extension, let us consider an arbitrary cyclic 2^{k-1}-extension $\widehat{G}_i$ of G_{i-1} in $X_1 \setminus (G_i \setminus G_{i-1})$ (if such an extension exists). Let us add cyclic 2^{k-1}-extensions $\widehat{G}_{i+1}, \widehat{G}_{i+2}, \ldots$ of the previously constructed graphs one by one in a similar way until there are no cyclic 2^{k-1}-extensions of $\widehat{G}_{\widehat{s}}$ in $X_1 \setminus (G_i \setminus G_{i-1})$. Obviously, $\widehat{G}_{\widehat{s}} \cup G_i$ is a cyclic 2^{k-1}-extension of $\widehat{G}_{\widehat{s}}$, but it is not its cyclic $(2^{k-1}-1)$-extension. Moreover, there are no cyclic 2^{k-1}-extensions of $\widehat{G}_{\widehat{s}}$ in $X_1 \setminus ((\widehat{G}_{\widehat{s}} \cup G_i) \setminus \widehat{G}_{\widehat{s}})$. So, the first $\widehat{s}+1$ graphs of the sequence are constructed. Those are $G_1,\ldots,G_{i-1},\widehat{G}_i,\ldots,\widehat{G}_{\widehat{s}},\widehat{G}_{\widehat{s}} \cup G_i$. Let us add cyclic 2^{k-1}-extensions to $\widehat{G}_{\widehat{s}} \cup G_i$ (each next graph is an extension of the previous one) until there are no cyclic 2^{k-1}-extensions of the final graph in X_1. Obviously, we get a sequence of graphs (we denote it by $G_1, G_2, \ldots, G_s$) satisfying Properties a), c) and d) (in addition, the inequality $s \le 2^{k-1}b+1$ holds, because the density of G_s is greater than $1/\alpha$, so, $s = 2^{k-1}b+2$ would contradict Property 1)).

Suppose $e(X_1|_{V(G_s)}) > e(G_s)$. Suppose also that $e(X_1|_{V(G_s)}) - e(G_s) \ge 2$. Since $s \le 2^{k-1}b+1$, Property 1) implies the inequalities $\rho^{\max}(X_1|_{V(G_s)}) \le 1/\alpha <$

$< 1 + \dfrac{1}{2^{k-1}-1}$. Then

$$1 + \frac{1}{2^{k-1}-1} > \rho^{\max}(X_1|_{V(G_s)}) \geq \frac{2^{k-1}+2}{2^{k-1}} = 1 + \frac{1}{2^{k-2}}.$$

This contradicts the inequality $k > 3$. So, $e(X_1|_{V(G_s)}) - e(G_s) = 1$. For each $i \in \{1, \ldots, s\}$ let us set $e(G_i) - e(G_{i-1}) = e_i \leq 2^{k-1}$, where $G_0 = (\{x_1^1\}, \varnothing)$. Then

$$1/\rho(X_1|_{V(G_s)}) = \frac{e_1 + \ldots + e_s - s + 1}{e_1 + \ldots + e_s + 1} = 1 - \frac{1}{2^{k-1} + \frac{(e_1 - 2^{k-1}) + \ldots + (e_s - 2^{k-1}) + 1}{s}}.$$

Therefore, we have either $1/\rho(X_1|_{V(G_s)}) = 1 - \dfrac{1}{2^{k-1} + \frac{1}{s}}$, or $1/\rho(X_1|_{V(G_s)}) \leq 1 - \dfrac{1}{2^{k-1}} < \alpha$, where the latter inequality holds if at least one of e_i, $i \in \{1, \ldots, s\}$, does not exceed $2^{k-1} - 1$. In the latter case we arrive at a contradiction with Property 1) of X_1, as $s \leq 2^{k-1}b + 1$. Thus, $1/\rho(X_1|_{V(G_s)}) = 1 - \dfrac{1}{2^{k-1} + \frac{1}{s}}$ and $e_1 = \ldots = e_s = 2^{k-1}$. If $1/\rho(X_1|_{V(G_s)}) > \alpha$, then Property b) holds. On the other hand, the inequality $1/\rho(X_1|_{V(G_s)}) < \alpha$ contradicts Property 1) of X_1. Hence $1 - \dfrac{1}{2^{k-1} + a/b} = 1 - \dfrac{1}{2^{k-1} + 1/s}$. Since a/b is in reduced form, we have $a = 1$, $b = s$. This can hold only if $2^{k-1} - b \leq 1$. Hence $s \geq 7$. Let us denote by by u, v the vertices of the additional edge, which exists according to our proposition. Suppose that $u, v \in V(G_{s-1})$ and that $u \in V(G_{j_1+1}) \setminus V(G_{j_1})$, $v \in V(G_{j_2+1}) \setminus V(G_{j_2})$, where $0 \leq j_1 \leq j_2 \leq s-2$, $G_0 = (\varnothing, \varnothing)$. If $V(G_{j_2+1}) \setminus V(G_{j_2})$ contains more than one vertex, then there obviously exist graphs $\widetilde{G}_{j_2+1}, \ldots, \widetilde{G}_{s+1}$ such that for every $j \in \{j_2, \ldots, s\}$ the graph $\widetilde{G}_{j+1}$ is a cyclic 2^{k-1}-extension of $\widetilde{G}_j$, where $\widetilde{G}_{j_2} = G_{j_2}$, and for every $j \in \{j_2+2, \ldots, s+1\}$ the equality $\widetilde{G}_j = X_1|_{V(G_{j-1})}$ holds. If $v(G_{j_2+1}, G_{j_2}) = 1$, then

$$1 + \frac{1}{2^{k-1}-1} > \rho^{\max}(X_1|_{V(G_{j_2+1})}) \geq \frac{2^{k-1}+3}{2^{k-1}+1} = 1 + \frac{1}{2^{k-2}+1/2},$$

which contradicts the inequality $k > 3$.

Obviously, the sequence $G_1, \ldots, G_{i-1}, \widetilde{G}_i, \ldots, \widetilde{G}_{s+1}$ satisfies Properties a)–d) (here $\widetilde{G}_{s+1}$ is the cyclic 2^{k-1}-extension of $\widetilde{G}_s$ from Property d)). Finally, suppose that at least one of the vertices u, v (e. g., v) belongs to $V(G_s) \setminus V(G_{s-1})$. If $G_s \setminus (G_{s-1} \setminus G_{s-2})$ is a cyclic 2^{k-1}-extension of G_{s-2} and $u \notin V(G_{s-1}) \setminus V(G_{s-2})$, then Duplicator can set $G_{s-1} := G_s \setminus (G_{s-1} \setminus G_{s-2})$ and find himself in the situation considered above. If either $G_s \setminus (G_{s-1} \setminus G_{s-2})$ is a cyclic 2^{k-1}-extension of G_{s-2} and $u \in V(G_{s-1}) \setminus V(G_{s-2})$, or $G_s \setminus (G_{s-1} \setminus G_{s-2})$ is not a cyclic 2^{k-1}-extension of G_{s-2}, then there exist graphs $\widetilde{G}_s, \widetilde{G}_{s+1}$ such that $\widetilde{G}_s$ is a cyclic 2^{k-1}-extension of G_{s-1}, $\widetilde{G}_{s+1}$ is a cyclic 2^{k-1}-extension of $\widetilde{G}_s$, $\widetilde{G}_{s+1} = X_1|_{V(G_s)}$, and there are no cyclic 2^{k-1}-extension of G_{s-1} in $X_1 \setminus (\widetilde{G}_s \setminus G_{s-1})$. Therefore, the sequence $G_1, \ldots, G_{s-1}, \widetilde{G}_s, \widetilde{G}_{s+1}$ satisfies Properties a)–d).

So, let $G_1, G_2, \ldots, G_s$ be a sequence satisfying Properties a)–d). Let us prove that the graph $X_1|_{V(G_s)}$ is strictly balanced. Let $\widetilde{G}$ be an arbitrary proper subgraph in $X_1|_{V(G_s)}$. Denote $\widetilde{G}_1 = X_1|_{V(G_1)} \cap \widetilde{G}$. If $\widetilde{G}_1 \neq X_1|_{V(G_1)}$, then $v(\widetilde{G} \cup X_1|_{V(G_1)}, \widetilde{G}) \leq$ $\leq 2^{k-1} - 1$ and $e(\widetilde{G} \cup X_1|_{V(G_1)}, \widetilde{G}) \geq v(\widetilde{G} \cup X_1|_{V(G_1)}, \widetilde{G}) + 1$. Therefore, the density of $\widetilde{G} \cup X_1|_{V(G_1)}$ is at least

$$\frac{e(\widetilde{G}) + v(\widetilde{G} \cup X_1|_{V(G_1)}, \widetilde{G}) + 1}{v(\widetilde{G}) + v(\widetilde{G} \cup X_1|_{V(G_1)}, \widetilde{G})} > \min\left\{\frac{e(\widetilde{G})}{v(\widetilde{G})}, 1 + \frac{1}{v(\widetilde{G} \cup X_1|_{V(G_1)}, \widetilde{G})}\right\} = \rho(\widetilde{G}).$$

It can be proved in the same way that $\rho(X_1|_{V(G_s)}) \geq \rho(\widetilde{G} \cup X_1|_{V(G_{s-1})}) \geq \ldots \geq$ $\geq \rho(\widetilde{G} \cup X_1|_{V(G_1)}) \geq \rho(\widetilde{G})$, where at least one of the inequalities is strict, because $\widetilde{G}$ is a proper subgraph in $X_1|_{V(G_s)}$. Thus, $X_1|_{V(G_s)}$ is strictly balanced.

If $\rho(X_1|_{V(G_s)}) < 1/\alpha$, then set $\widetilde{X}_1^1 = X_1|_{V(G_s)}$. By definition Y_1 contains a subgraph $\widetilde{Y}_1^1$ isomorphic to $\widetilde{X}_1^1$ which is (K,T)-maximal for any pair (K,T) such that $v(K) \leq 2^k$, $v(T) \leq 2$, and $f_\alpha(K,T) < 0$. Let $\varphi : \widetilde{X}_1^1 \to \widetilde{Y}_1^1$ be an isomorphism. Then Duplicator chooses $y_1^1 := \varphi(x_1^1)$. By construction $\widetilde{X}_1^1$ and $\widetilde{Y}_1^1$ do not have cyclic 2^{k-1}-extensions in X_1 and Y_1 respectively. Therefore, $\widetilde{X}_1^1$ and $\widetilde{Y}_1^1$ are $(k,1,1)$-regular equivalent in (X_1, Y_1). In the second round Duplicator uses the strategy SF.

Suppose $\rho(X_1|_{V(G_s)}) = 1/\alpha$. Then $\rho(G_s) = 1/\alpha$ as well. Set $G_0 = (\{x_1^1\}, \varnothing)$ and $e_i = e(G_i, G_{i-1})$ for each $i \in \{1, \ldots, s\}$. Then

$$1 + \frac{1}{2^{k-1} + a/b - 1} = \frac{e_1 + \ldots + e_s}{e_1 + \ldots + e_s - s + 1} = 1 + \frac{1}{\frac{e_1 + \ldots + e_s}{s-1} - 1}.$$

Since a/b is in reduced form, we have $s \geq b + 1$. Obviously, the inequality $a \geq \max\{1, 2^{k-1} - b\}$ implies the existence of $\mu \in \{0, \ldots, s-1\}$ such that $G_{\mu+1}$

is not a cyclic $(2^{k-1}-1)$-extension of G_μ. Indeed, otherwise

$$\rho(G_s) \geq \frac{(2^{k-1}-1)s}{(2^{k-1}-2)s+1} = 1 + \frac{1}{2^{k-1}-2+\frac{2^{k-1}-1}{s-1}} \geq 1 + \frac{1}{2^{k-1}-2+\frac{2^{k-1}-1}{b}} = $$
$$= 1 + \frac{1}{2^{k-1}+\frac{2^{k-1}-2b-1}{b}} > 1/\alpha.$$

Since G_s is strictly balanced, we have $\rho^{\max}(G_\mu) < 1/\alpha$. Furthermore, since $Y_1 \in \mathcal{S}$, there is a subgraph $\widetilde{Y}_1^1$ of Y_1 isomorphic to $\widetilde{X}_1^1 := G_\mu$ which is (K,T)-maximal for any pair (K,T) such that $v(K) \leq 2^k$, $v(T) \leq 2$, and $f_\alpha(K,T) < 0$. Let $\varphi : \widetilde{X}_1^1 \to \widetilde{Y}_1^1$ be an isomorphism. Then Duplicator chooses $y_1^1 := \varphi(x_1^1)$. By construction $\widetilde{X}_1^1$ and $\widetilde{Y}_1^1$ are $(k,1)$-equivalent in (X_1,Y_1). For this reason in the second round Duplicator uses the strategy S_2.

4.3.6. Strategy S_{r+1}

Suppose that after the r-th round, $r \in \{1,\ldots,k-2\}$, there exist graphs $\widetilde{X}_r^1, \widetilde{Y}_r^1$ which are (k,r)-equivalent in (X_r,Y_r). Let $\varphi : \widetilde{X}_r^1 \to \widetilde{Y}_r^1$ be an automorphism.

In the $(r+1)$-st round Spoiler chooses a vertex x_{r+1}^{r+1}. If $X_{r+1} = X_r$, then set $\widetilde{X}_{r+1}^1 = \widetilde{X}_r^1$, $\widetilde{Y}_{r+1}^1 = \widetilde{Y}_r^1$. Otherwise, set $\widetilde{X}_{r+1}^1 = \widetilde{Y}_r^1$, $\widetilde{Y}_{r+1}^1 = \widetilde{X}_r^1$.

Let $x_{r+1}^{r+1} \in V(\widetilde{X}_{r+1}^1)$. If $X_{r+1} = X_r$, then Duplicator chooses $y_{r+1}^{r+1} = \varphi(x_{r+1}^{r+1})$, and if $X_{r+1} = Y_r$, he chooses $y_{r+1}^{r+1} = \varphi^{-1}(x_{r+1}^{r+1})$. Since there are no cyclic 2^{k-r-1}-extensions of $\widetilde{X}_r^1$ and $\widetilde{Y}_r^1$ in X_r and Y_r respectively (by the definition of the (k,r)-equivalence), $\widetilde{X}_{r+1}^1, \widetilde{X}_{r+1}^1$ are $(k,r+1,1)$-regular equivalent in (X_{r+1},Y_{r+1}). Therefore, in the $(r+2)$-nd round Duplicator uses the strategy SF.

Let $x_{r+1}^{r+1} \notin V(\widetilde{X}_{r+1}^1)$. Consider two cases: $r < k-2$ and $r = k-2$.

Let $r < k-2$. If $d_{X_{r+1}}(\widetilde{X}_{r+1}^1, x_{r+1}^{r+1}) > 2^{k-r-1}$ and there are no cyclic 2^{k-r-1}-extensions of $(\{x_{r+1}^{r+1}\},\varnothing)$ in X_{r+1}, then set $\widetilde{X}_{r+1}^2 = (\{x_{r+1}^{r+1}\},\varnothing)$. By Property 2) Y_{r+1} has a vertex y_{r+1}^{r+1} such that $d_{Y_{r+1}}(\widetilde{Y}_{r+1}^1, y_{r+1}^{r+1}) = 2^{k-r-1}+1$ and there are no cyclic 2^{k-r-1}-extensions of $(\{y_{r+1}^{r+1}\},\varnothing)$ in Y_{r+1}. Set $\widetilde{Y}_{r+1}^2 = (\{y_{r+1}^{r+1}\},\varnothing)$. If there is exactly one cyclic 2^{k-r-1}-extension of $(\{x_{r+1}^{r+1}\},\varnothing)$, then we denote it by $\widetilde{X}_{r+1}^2$. Let $d_{X_{r+1}}(\widetilde{X}_{r+1}^1, \widetilde{X}_{r+1}^2) > 2^{k-r-1}$. By Property 2) $\widetilde{Y}_{r+1}^1$ has a vertex y_{r+1}^{r+1} and a subgraph $\widetilde{Y}_{r+1}^2$ such that $d_{Y_2}(\widetilde{Y}_{r+1}^1, \widetilde{Y}_{r+1}^2) = 2^{k-r-1}+1$, the pairs $(\widetilde{Y}_{r+1}^2, (\{y_{r+1}^{r+1}\},\varnothing))$ and $(\widetilde{X}_{r+1}^2, (\{x_{r+1}^{r+1}\},\varnothing))$ are isomorphic, and there are no cyclic 2^{k-r-1}-extensions of $\widetilde{Y}_{r+1}^2$ in Y_{r+1}. The property of (k,r)-equivalence of the graphs $\widetilde{X}_r^1, \widetilde{Y}_r^1$ in (X_r,Y_r) implies non-existence of cyclic 2^{k-r-1}-extensions of $\widetilde{X}_r^1$ and $\widetilde{Y}_r^1$ in X_r and Y_r respectively. Obviously, in all the considered cases the ordered tuples $\widetilde{X}^1_{r+1}, \widetilde{X}^2_{r+1}$ and $\widetilde{Y}^1_{r+1}, \widetilde{Y}^2_{r+1}$ are $(k, r+1, 2)$-regular equivalent in (X_{r+1}, Y_{r+1}). Thus, in the $(r+2)$-nd round Duplicator can use the strategy SF. Let $d_{X_{r+1}}(\widetilde{X}^1_{r+1}, \widetilde{X}^2_{r+1}) \leq 2^{k-r-1}$. The property of (k,r)-equivalence of $\widetilde{X}^1_r$ and $\widetilde{Y}^1_r$ in (X_r, Y_r) implies the relation $d_{X_{r+1}}(\widetilde{X}^1_{r+1}, \widetilde{X}^2_{r+1}) = 2^{k-r-1}$ and non-existence of cyclic $(2^{k-r-1}-1)$-extensions of $(\{x^{r+1}_{r+1}\}, \varnothing)$ in X_{r+1}. In this case by Property 2) of Y_{r+1} Duplicator can choose a vertex y^{r+1}_{r+1} such that there is an isomorphism $L_X \cup \widetilde{X}^1_{r+1} \to L_Y \cup \widetilde{Y}^1_{r+1}$ taking $x^1_{r+1}, \ldots, x^{r+1}_{r+1}$ to $y^1_{r+1}, \ldots, y^{r+1}_{r+1}$ respectively, where L_X is a minimal path in X_{r+1} connecting x^{r+1}_{r+1} and $\widetilde{X}^1_{r+1}$, L_Y is a minimal path in Y_{r+1} connecting y^{r+1}_{r+1} and $\widetilde{Y}^1_{r+1}$, and the pair $(L_Y \cup \widetilde{Y}^1_{r+1}, \widetilde{Y}^1_{r+1})$ is cyclically 2^{k-r-1}-maximal in Y_{r+1}. Set $\widetilde{X}^1_{r+1} := \widetilde{X}^1_{r+1} \cup L_X$, $\widetilde{Y}^1_{r+1} := \widetilde{Y}^1_{r+1} \cup L_Y$. Next, Duplicator uses the strategy S^1_{r+2}. Finally, let us prove that $(\{x^{r+1}_{r+1}\}, \varnothing)$ has no more than one cyclic 2^{k-r-1}-extension. Indeed, if there are two such extensions A and $\widetilde{A}$, then

$$1/\rho(A \cup \widetilde{A}) \leq \frac{2^{k-r-1} + 2^{k-r-1} - 1}{2^{k-r-1} + 2^{k-r-1}} = 1 - \frac{1}{2^{k-r}} < \alpha.$$

This contradicts Property 1), since $v(A \cup \widetilde{A}) \leq 2^{k-r} - 1$.

Suppose $d_{X_{r+1}}(\widetilde{X}^1_{r+1}, x^{r+1}_{r+1}) \leq 2^{k-r-1}$. Consider a minimal path L_X in X_{r+1} connecting x^{r+1}_{r+1} and $\widetilde{X}^1_{r+1}$. By Property 2) Y_{r+1} has a vertex y^{r+1}_{r+1} such that

a) $d_{Y_{r+1}}(\widetilde{Y}^1_{r+1}, y^{r+1}_{r+1}) = d_{X_{r+1}}(\widetilde{X}^1_{r+1}, x^{r+1}_{r+1})$,

b) there is an isomorphism $L_X \cup \widetilde{X}^1_{r+1} \to L_Y \cup \widetilde{Y}^1_{r+1}$ taking $x^1_{r+1}, \ldots, x^{r+1}_{r+1}$ to $y^1_{r+1}, \ldots, y^{r+1}_{r+1}$ respectively,

c) $(L_Y \cup \widetilde{Y}^1_{r+1}, \widetilde{Y}^1_{r+1})$ is cyclically 2^{k-r-1}-maximal,

where L_Y is a minimal path connecting y^{r+1}_{r+1} and $\widetilde{Y}^1_{r+1}$ in Y_{r+1}. Obviously, there are no cyclic 2^{k-r-1}-extensions of $\widetilde{Y}^1_{r+1}$ in Y_{r+1}. Set $\widetilde{Y}^1_{r+1} := \widetilde{Y}^1_{r+1} \cup L_Y$. Supposing that there are no cyclic 2^{k-r-1}-extensions of $L_X \cup \widetilde{X}^1_{r+1}$ in X_{r+1}, let us set $\widetilde{X}^1_{r+1} := L_X \cup \widetilde{X}^1_{r+1}$. Then, obviously, $\widetilde{X}^1_{r+1}$ and $\widetilde{Y}^1_{r+1}$ are $(k, r+1, 1)$-regular equivalent in (X_{r+1}, Y_{r+1}). Therefore, in the next round Duplicator can use the strategy SF. If there is a cyclic 2^{k-r-1}-extension of $L_X \cup \widetilde{X}^1_{r+1}$ in X_{r+1}, then $d_{X_{r+1}}(x^{r+1}_{r+1}, \widetilde{X}^1_{r+1}) = 2^{k-r-1}$ and there are no cyclic $(2^{k-r-1}-1)$-extensions of $L_X \cup \widetilde{X}^1_{r+1}$ in X_{r+1}. In this case L_X can be chosen from a set with at most two paths. If there is only one such path, then either a cyclic 2^{k-r-1}-extension of $L_X \cup \widetilde{X}^1_{r+1}$ is an extension of the first type, or one of the endpoints of L_X does not coincide with any of $x^1_{r+1}, \ldots, x^{r+1}_{r+1}$. If there are two such paths, then we consider two cases. First, we suppose that a cyclic 2^{k-r}-extension of $\widetilde{X}^1_{r+1}$ is an extension of the first type. Then we choose L_X to be any of those two paths. Second, we suppose that a 2^{k-r}-extension of $\widetilde{X}^1_{r+1}$ is an extension of the second type. Then (k,r)-equivalence of $\widetilde{X}^1_{r+1}$ and $\widetilde{Y}^1_{r+1}$ in (X_{r+1},Y_{r+1}) implies that at least one of the paths does not contain $x^1_{r+1},\ldots,x^r_{r+1}$, and that path can be taken as L_X. Obviously, $\widetilde{X}^1_{r+1}:=L_X\cup\widetilde{X}^1_{r+1}$ and $\widetilde{Y}^1_{r+1}$ are $(k,r+1)$-equivalent in (X_{r+1},Y_{r+1}). In the next round Duplicator ases the strategy S_{r+2}.

Finally, let $r=k-2$. If $d_{X_{k-1}}(\widetilde{X}^1_{k-1},x^{k-1}_{k-1})>2$, then set $\widetilde{X}^2_{k-1}=(\{x^{k-1}_{k-1}\},\varnothing)$. By Property 2) Y_{k-1} contains a vertex y^{k-1}_{k-1} such that $d_{Y_{k-1}}(\widetilde{Y}^1_{k-1},y^{k-1}_{k-1})=3$. Set $\widetilde{Y}^2_{k-1}=(\{y^{k-1}_{k-1}\},\varnothing)$. Since $\widetilde{X}^1_{k-2}$ and $\widetilde{Y}^1_{k-2}$ are $(k,k-2)$-equivalent in (X_{k-2},Y_{k-2}), they do not have cyclic 2-extensions in X_{k-2} and Y_{k-2} respectively. Thus, the ordered tuples $\widetilde{X}^1_{k-1},\widetilde{X}^2_{k-1}$ and $\widetilde{Y}^1_{k-1},\widetilde{Y}^2_{k-1}$ are $(k,k-1,2)$-regular equivalent in (X_{k-1},Y_{k-1}). Therefore, in the k-th round Duplicator can use the strategy SF.

Suppose $d_{X_{k-1}}(\widetilde{X}^1_{k-1},x^{k-1}_{k-1})\le 2$. Consider a minimal path L_X in X_{k-1} connecting x^{k-1}_{k-1} and $\widetilde{X}^1_{k-1}$. Suppose this path connects x^{k-1}_{k-1} and one of the $x^1_{k-1},\ldots,x^{k-2}_{k-1}$ (if there is a path of minimal length with this property). By Property 2) Y_{k-1} contains a vertex y^{k-1}_{k-1} such that

a) $d_{Y_{k-1}}(\widetilde{Y}^1_{k-1},y^{k-1}_{k-1})=d_{X_{k-1}}(\widetilde{X}^1_{k-1},x^{k-1}_{k-1})$,

b) there is an isomorphism $L_X\cup\widetilde{X}^1_{k-1}\to L_Y\cup\widetilde{Y}^1_{k-1}$ taking $x^1_{k-1},\ldots,x^{k-1}_{k-1}$ to $y^1_{k-1},\ldots,y^{k-1}_{k-1}$ respectively,

c) $(L_Y\cup\widetilde{Y}^1_{k-1},\widetilde{Y}^1_{k-1})$ is cyclically 2-maximal,

where L_Y is a minimal path in Y_{k-1} connecting y^{k-1}_{k-1} and $\widetilde{Y}^1_{k-1}$. Obviously, there are no cyclic 2-extensions of $\widetilde{Y}^1_{k-1}$ in Y_{k-1}. If there are no cyclic 2-extensions of $L_X\cup\widetilde{X}^1_{k-1}$ in X_{k-1}, then $\widetilde{X}^1_{k-1}\cup L_X$ and $\widetilde{Y}^1_{k-1}\cup L_Y$ are $(k,k-1,1)$-regular equivalent in (X_{k-1},Y_{k-1}). In the next round Duplicator uses the strategy SF. If there is a cyclic 2-extension of $L_X\cup\widetilde{X}^1_{k-1}$ in X_{k-1}, then $d_{X_{k-1}}(x^{k-1}_{k-1},\widetilde{X}^1_{k-1})=2$. Moreover, by the property of $(k,k-2)$-equivalence of $\widetilde{X}^1_{k-2}$ and $\widetilde{Y}^1_{k-2}$ in (X_{k-2},Y_{k-2}), the only path of length 2 not coinciding with L_X but connecting x^{k-1}_{k-1} with some vertex of $\widetilde{X}^1_{k-1}$ satisfies the following property: either its endpoint (distinct from x^{k-1}_{k-1}) is not one of $x^1_{k-1},\ldots,x^{k-2}_{k-1}$, or it equals one of the endpoints of L_X. Obviously, in the k-th round, if Spoiler chooses a vertex from one of the graphs $L_X\cup\widetilde{X}^1_{k-1},L_Y\cup\widetilde{Y}^1_{k-1}$, then Duplicator wins by choosing the image of x^k_k under an isomorphism of the graphs. If Spoiler chooses a vertex outside those graphs so that it is adjacent to no more than one of $x^1_k,\ldots,x^{k-1}_k$, then Duplicator has a winning strategy by Property 2) of Y_k. Obviously, there are no more than two vertices in $\{x^1_k,\ldots,x^{k-1}_k\}$ adjacent to x_k^k. Finally, if x_k^k is adjacent to two of $x_k^1, \dots, x_k^{k-1}$, then Duplicator chooses a vertex of degree 2 either from L_X, or from L_Y, and wins.

4.3.7. Strategy S^1_{r+1}

Suppose that after the r-th round, $r \in \{2, \dots, k-2\}$, there exist induced subgraphs $\widetilde{X}_r^1$ and $\widetilde{Y}_r^1$ of X_r and Y_r respectively such that $\widetilde{Y}_r^1$ is cyclically 2^{k-r}-maximal, $x_r^1, \dots, x_r^r \in V(\widetilde{X}_r^1)$, $y_r^1, \dots, y_r^r \in V(\widetilde{Y}_r^1)$, and there is an isomorphism $\varphi : \widetilde{X}_r^1 \to \widetilde{Y}_r^1$ taking $x_r^1, \dots, x_r^r$ to $y_r^1, \dots, y_r^r$ respectively. We have $\widetilde{X}_r^1 = \widetilde{X}_{r-1}^1 \cup L_X$, $\widetilde{Y}_r^1 = \widetilde{Y}_{r-1}^1 \cup L_Y$, where $\widetilde{X}_{r-1}^1, \widetilde{Y}_{r-1}^1$ have one common vertex with L_X and L_Y respectively; $\varphi|_{\widetilde{X}_{r-1}^1} : \widetilde{X}_{r-1}^1 \to \widetilde{Y}_{r-1}^1$ is an isomorphism; $x_r^1, \dots, x_r^{r-1}$ are in $V(\widetilde{X}_{r-1}^1)$; x_r^r and y_r^r are the endpoints of L_X and L_Y and they do not belong to $V(\widetilde{X}_{r-1}^1)$ and $V(\widetilde{Y}_{r-1}^1)$ respectively. Finally, there is only one cyclic 2^{k-r}-extension $C_X \cup \widetilde{X}_r^r$ of $\widetilde{X}_r^r$, where C_X is a path of length $l \in [2^{k-r-1}, 2^{k-r})$ connecting x_r^r with some non-terminal vertex x of L_X. Besides that, $l + e(L_X) = 2^{k-r+1}$ and $d_{X_r}(x, x_r^r) + l = 2^{k-r}$.

In the $(r+1)$-st round, $r \in \{1, \dots, k-2\}$, Spoiler chooses x_{r+1}^{r+1}. If $X_{r+1} = X_r$, then set $\widetilde{X}_{r+1}^1 = \widetilde{X}_r^1$, $\widetilde{Y}_{r+1}^1 = \widetilde{Y}_r^1$. Otherwise, set $\widetilde{X}_{r+1}^1 = \widetilde{Y}_r^1$, $\widetilde{Y}_{r+1}^1 = \widetilde{X}_r^1$ and rename $L_X := L_Y$, $L_Y := L_X$.

Let $x_{r+1}^{r+1} \in V(\widetilde{X}_{r+1}^1)$. If $X_{r+1} = X_r$, then Duplicator chooses $y_{r+1}^{r+1} = \varphi(x_{r+1}^{r+1})$. If $X_{r+1} = Y_r$, then he chooses $y_{r+1}^{r+1} = \varphi^{-1}(x_{r+1}^{r+1})$. There are no cyclic 2^{k-r-1}-extensions of $\widetilde{Y}_r^1$ in Y_r. There is a cyclic 2^{k-r-1}-extension of $\widetilde{X}_r^1$ if and only if $l = 2^{k-r-1}$ (moreover, the number of such extensions does not exceed one). Suppose that the latter equality holds.

Let $x_{r+1}^{r+1} \in \widetilde{X}_{r+1} \setminus L_X$. Set $\widetilde{X}_{r+1}^1 = \widetilde{X}_{r-1}^1$, $\widetilde{Y}_{r+1}^1 = \widetilde{Y}_{r-1}^1$, if $X_{r+1} = X_r$, and $\widetilde{X}_{r+1}^1 = \widetilde{Y}_{r-1}^1$, $\widetilde{Y}_{r+1}^1 = \widetilde{Y}_{r-1}^1$, otherwise. Set $\widetilde{X}_{r+1}^2 = (\{x_{r+1}^{r+1}\}, \varnothing)$, $\widetilde{Y}_{r+1}^2 = (\{y_{r+1}^{r+1}\}, \varnothing)$. Obviously, $d_{X_{r+1}}(\widetilde{X}_{r+1}^1, \widetilde{X}_{r+1}^2) = d_{Y_{r+1}}(\widetilde{Y}_{r+1}^1, \widetilde{Y}_{r+1}^2) = 2^{k-r} + 2^{k-r-1} > 2^{k-r-1}$ and, moreover, there are no cyclic 2^{k-r-1}-extensions of $\widetilde{X}_{r+1}^1$ or $\widetilde{X}_{r+1}^2$ in X_{r+1}, as well as there are no cyclic 2^{k-r-1}-extension of $\widetilde{Y}_{r+1}^1$ or $\widetilde{Y}_{r+1}^2$ in Y_{r+1}.

Let $x_{r+1}^{r+1} \in L_X$. Let d be the distance between x_{r+1}^{r+1} and the endpoint of L_X which belongs to $\widetilde{X}_{r+1}^1$. Denote a minimal path connecting those two vertices by $\widetilde{L}_X$.

Suppose $d < 2^{k-r}$. Rename $\widetilde{X}_{r+1}^1 := \widetilde{X}_{r-1}^1 \cup \widetilde{L}_X$, $\widetilde{Y}_{r+1}^1 := \varphi(\widetilde{X}_{r-1}^1 \cup \widetilde{L}_X)$, if $X_{r+1} = X_r$, and $\widetilde{X}_{r+1}^1 := \widetilde{Y}_{r-1}^1 \cup \widetilde{L}_X$, $\widetilde{Y}_{r+1}^1 := \varphi^{-1}(\widetilde{Y}_{r-1}^1 \cup L_X)$, otherwise. Set $\widetilde{X}_{r+1}^2 = (\{x_{r+1}^{r+1}\}, \varnothing)$, $\widetilde{Y}_{r+1}^2 = (\{y_{r+1}^{r+1}\}, \varnothing)$. Obviously, $d_{X_{r+1}}(\widetilde{X}_{r+1}^1, \widetilde{X}_{r+1}^2) = d_{Y_{r+1}}(\widetilde{Y}_{r+1}^1, \widetilde{Y}_{r+1}^2) > 2^{k-r-1}$. Moreover, there are no cyclic 2^{k-r-1}-extensions of $\widetilde{X}_{r+1}^1$ or $\widetilde{X}_{r+1}^2$ in X_{r+1}, as well as there are no cyclic 2^{k-r-1}-extensions of $\widetilde{Y}_{r+1}^1$ or $\widetilde{Y}_{r+1}^2$ in Y_{r+1}.

Suppose $d \geq 2^{k-r}$. If $X_{r+1} = X_r$, rename $\widetilde{X}^1_{r+1} := \widetilde{X}^1_{r-1}$, $\widetilde{Y}^1_{r+1} := Y^1_{r-1}$ and set $\widetilde{X}^2_{r+1} = \widetilde{L}_X$, $\widetilde{Y}^2_{r+1} = \varphi(\widetilde{L}_X)$. Otherwise, rename $\widetilde{X}^1_{r+1} := \widetilde{Y}^1_{r-1}$, $\widetilde{Y}^1_{r+1} := X^1_{r-1}$ and set $\widetilde{X}^2_{r+1} = \widetilde{L}_X$, $\widetilde{Y}^2_{r+1} = \varphi^{-1}(\widetilde{L}_X)$. Obviously, $d_{X_{r+1}}(\widetilde{X}^1_{r+1}, \widetilde{X}^2_{r+1}) = d_{Y_{r+1}}(\widetilde{Y}^1_{r+1}, \widetilde{Y}^2_{r+1}) \geq 2^{k-r} > 2^{k-r-1}$. Moreover, if $d > 2^{k-r}$, then there are no cyclic 2^{k-r-1}-extensions of $\widetilde{X}^1_{r+1}$ or $\widetilde{X}^2_{r+1}$ in X_{r+1}, as well as there are no cyclic 2^{k-r-1}-extensions of $\widetilde{Y}^1_{r+1}$ or $\widetilde{Y}^2_{r+1}$ in Y_{r+1}.

In each case the ordered tuples $\widetilde{X}^1_{r+1}, \widetilde{X}^2_{r+1}$ and $\widetilde{Y}^1_{r+1}, \widetilde{Y}^2_{r+1}$ are $(k, r+1, 2)$-regular equivalent in (X_{r+1}, Y_{r+1}). Thus, in the next round Duplicator can use the strategy SF.

If in the latter case we have $d = 2^{k-r}$, then in the $(r+2)$-nd round, after Spoiler chooses x^{r+2}_{r+2}, Duplicator uses the strategy described in Section 4.3.8.

Finally, let $l > 2^{k-r-1}$. Then $\widetilde{X}^1_{r+1}$ and $\widetilde{Y}^1_{r+1}$ are $(k, r+1, 1)$-regular equivalent. Thus, in the $(r+2)$-th round Duplicator can use the strategy SF.

If $x^{r+1}_{r+1} \notin V(\widetilde{X}^1_{r+1})$ but x^{r+1}_{r+1} is in the (only) cyclic 2^{k-r}-extension of $\widetilde{X}^1_{r+1}$, then denote a minimal path in X_{r+1} connecting x^r_{r+1} and x^{r+1}_{r+1} by $\widetilde{X}^2_{r+1}$. Rename $\widetilde{X}^1_{r+1} := \widetilde{X}^1_{r-1}$, $\widetilde{Y}^1_{r+1} := \widetilde{Y}^1_{r-1}$, if $X_{r+1} = X_r$, and $\widetilde{X}^1_{r+1} := \widetilde{Y}^1_{r-1}$, $\widetilde{Y}^1_{r+1} := \widetilde{X}^1_{r-1}$, otherwise. By Property 2) Y_{r+1} contains a vertex y^{r+1}_{r+1} and a path $\widetilde{Y}^2_{r+1}$ such that

a) $d_{Y_{r+1}}(y^r_{r+1}, \widetilde{Y}^1_{r+1}) = d_{Y_{r+1}}(\widetilde{Y}^2_{r+1}, \widetilde{Y}^1_{r+1}) > 2^{k-r-1}$,

b) $\widetilde{X}^2_{r+1}$ and $\widetilde{Y}^2_{r+1}$ are isomorphic,

c) the corresponding isomorphism takes x^{r+1}_{r+1} to y^{r+1}_{r+1},

d) there are no cyclic 2^{k-r-1}-extensions of $\widetilde{Y}^2_{r+1}$ in Y_{r+1}.

If $d_{X_{r+1}}(x^r_{r+1}, x^{r+1}_{r+1}) < 2^{k-r-1}$, then there obviously are no cyclic 2^{k-r-1}-extensions of $\widetilde{X}^2_{r+1}$ in X_{r+1}. The ordered tuples $\widetilde{X}^1_{r+1}, \widetilde{X}^2_{r+1}$ and $\widetilde{Y}^1_{r+1}, \widetilde{Y}^2_{r+1}$ are $(k, r+1, 2)$-regular equivalent in (X_{r+1}, Y_{r+1}). Thus, in the $(r+2)$-nd round Duplicator can use the strategy SF. If $d_{X_{r+1}}(x^r_{r+1}, x^{r+1}_{r+1}) = 2^{k-r-1}$, then in the $(r+2)$-nd round, after Spoiler chooses x^{r+2}_{r+2}, Duplicator can use the strategy described in Section 4.3.8.

Finally, suppose x^{r+1}_{r+1} belongs neither to $V(\widetilde{X}^1_{r+1})$, nor to any cyclic 2^{k-r}-extension of $\widetilde{X}^1_{r+1}$. If $d_{X_{r+1}}(x^{r+1}_{r+1}, \widetilde{X}^1_{r+1}) \leq 2^{k-r-1}$, then redefine L_X to be a minimal path connecting x^{r+1}_{r+1} and some vertex of $\widetilde{X}^1_{r+1}$. Obviously, there are no cyclic 2^{k-r-1}-extensions of $\widetilde{X}^1_{r+1} \cup L_X$ in X_{r+1}. By Property 2) Y_{r+1} contains a vertex y^{r+1}_{r+1} such that

a) $d_{Y_{r+1}}(\widetilde{Y}^1_{r+1}, y^{r+1}_{r+1}) = d_{X_{r+1}}(\widetilde{X}^1_{r+1}, x^{r+1}_{r+1})$,

b) there is an isomorphism $L_X \cup \widetilde{X}^1_{r+1} \to L_Y \cup \widetilde{Y}^1_{r+1}$ taking $x^1_{r+1}, \ldots, x^{r+1}_{r+1}$ to $y^1_{r+1}, \ldots, y^{r+1}_{r+1}$ respectively,

c) $(L_Y \cup \widetilde{Y}^1_{r+1}, \widetilde{Y}^1_{r+1})$ is cyclically 2^{k-r-1}-maximal,

where L_Y is a minimal path connecting y^{r+1}_{r+1} and $\widetilde{Y}^1_{r+1}$ in Y_{r+1}. Obviously, $\widetilde{X}^1_{r+1} := := \widetilde{X}^1_{r+1} \cup L_X$ and $\widetilde{Y}^1_{r+1} := \widetilde{Y}^1_{r+1} \cup L_Y$ are $(k, r+1, 1)$-regular equivalent in (X_{r+1}, Y_{r+1}). Therefore, in the $(r+2)$-nd round Duplicator can use the strategy SF. Finally, if $d_{X_{r+1}}(x^{r+1}_{r+1}, \widetilde{X}^1_{r+1}) > 2^{k-r-1}$, then denote by $\widetilde{X}^2_{r+1}$ the only cyclic 2^{k-r-1}-extension of $(\{x^{r+1}_{r+1}\}, \varnothing)$ (if there are no such extensions, then set $\widetilde{X}^2_{r+1} = (\{x^{r+1}_{r+1}\}, \varnothing))$. We have $d_{X_{r+1}}(\widetilde{X}^2_{r+1}, \widetilde{X}^1_{r+1}) > 2^{k-r-1}$. By Property 2) Y_{r+1} contains a vertex y^{r+1}_{r+1} and a subgraph $\widetilde{Y}^2_{r+1}$ such that

a) $d_{Y_{r+1}}(\widetilde{Y}^2_{r+1}, \widetilde{Y}^1_{r+1}) = 2^{k-r-1} + 1$,

b) there is an isomorphism $\widetilde{X}^2_{r+1} \to \widetilde{Y}^2_{r+1}$ taking x^{r+1}_{r+1} to y^{r+1}_{r+1},

c) there are no cyclic 2^{k-r-1}-extensions of $\widetilde{Y}^2_{r+1}$ in Y_{r+1}.

Obviously, the ordered tuples $\widetilde{X}^1_{r+2}, \widetilde{X}^2_{r+2}$ and $\widetilde{Y}^1_{r+2}, \widetilde{Y}^2_{r+2}$ are $(k, r+1, 2)$-regular equivalent in (X_{r+1}, Y_{r+1}). Therefore, in the $(r+2)$-nd round Duplicator can use the strategy SF.

4.3.8. The next round strategy

If $X_{r+2} = X_{r+1}$, then set $\widetilde{X}^1_{r+2} = \widetilde{X}^1_{r+1}$, $\widetilde{X}^2_{r+2} = \widetilde{X}^2_{r+1}$, $\widetilde{Y}^1_{r+2} = \widetilde{Y}^1_{r+1}$, $\widetilde{Y}^2_{r+2} = = \widetilde{Y}^2_{r+1}$. Otherwise, set $\widetilde{X}^1_{r+2} = \widetilde{Y}^1_{r+1}$, $\widetilde{X}^2_{r+2} = \widetilde{Y}^2_{r+1}$, $\widetilde{Y}^1_{r+2} = \widetilde{X}^1_{r+1}$, $\widetilde{Y}^2_{r+2} = \widetilde{X}^2_{r+1}$. Let $\varphi : \widetilde{X}^1_{r+2} \cup \widetilde{X}^2_{r+2} \to \widetilde{Y}^1_{r+2} \cup \widetilde{Y}^2_{r+2}$ be an isomorphism taking $x^1_{r+2}, \ldots, x^{r+1}_{r+2}$ to $y^1_{r+2}, \ldots, y^{r+1}_{r+2}$ respectively. If $x^{r+2}_{r+2} \in V(\widetilde{X}^1_{r+2})$, then Duplicator chooses $y^{r+2}_{r+2} = = \varphi(x^{r+2}_{r+2})$. If $r = k-2$, then Duplicator wins. If $r < k-2$, then, there obviously are no cyclic 2^{k-r-2}-extensions of $\widetilde{X}^1_{r+2}$ or $\widetilde{X}^2_{r+2}$ in X_{r+2}, as well as there are no 2^{k-r-2}-extensions of $\widetilde{Y}^1_{r+2}$ or $\widetilde{Y}^2_{r+2}$ in Y_{r+2}. Thus, the ordered tuples $\widetilde{X}^1_{r+2}, \widetilde{X}^2_{r+2}$ and $\widetilde{Y}^1_{r+2}, \widetilde{Y}^2_{r+2}$ are $(k, r+2, 2)$-regular equivalent in (X_{r+2}, Y_{r+2}). Therefore, in the $(r+3)$-rd round Duplicator can use the strategy SF.

Suppose x^{r+2}_{r+2} belongs to the only cyclic 2^{k-r-1}-extension of $\widetilde{X}^2_{r+2}$. Let $\widetilde{L}_X$ be a path of minimal length with endpoints coinciding with those of $\widetilde{X}^2_{r+2}$ such that x^{r+2}_{r+2} is in $V(\widetilde{L}_X)$. Obviously, there is an isomorphism $\widetilde{\varphi} : \widetilde{X}^1_{r+2} \cup L_X \to Y^1_{r+2} \cup Y^2_{r+2}$ taking $x^1_{r+2}, \ldots, x^{r+2}_{r+2}$ to $y^1_{r+2}, \ldots, y^{r+2}_{r+2}$. Thus, if $r = k-2$, then Duplicator wins. If $r < k-2$, then, obviously, the ordered tuples $\widetilde{X}^1_{r+2}, \widetilde{X}^2_{r+2} := \widetilde{L}_X$ and $\widetilde{Y}^1_{r+2}, \widetilde{Y}^2_{r+2}$ are $(k, r+2, 2)$-regular equivalent in (X_{r+2}, Y_{r+2}). Therefore, in the $(r+3)$-rd round Duplicator can use the strategy SF.

Suppose x^{r+2}_{r+2} does not belong to the cyclic 2^{k-r-1}-extension of $\widetilde{X}^2_{r+2}$ and $d_{X_{r+2}}(x^{r+2}_{r+2}, \widetilde{X}^1_{r+2} \cup \widetilde{X}^2_{r+2}) \leq 2^{k-r-2}$. Let $\widetilde{L}_X$ be a minimal path connecting x^{r+2}_{r+2} and $\widetilde{X}^1_{r+2} \cup \widetilde{X}^2_{r+2}$. Obviously, there are no cyclic 2^{k-r-2}-extensions of $\widetilde{X}^1_{r+2} \cup \widetilde{X}^2_{r+2} \cup \widetilde{L}_X$ in X_{r+2}. By property 2) Y_{r+2} contains a vertex y^{r+2}_{r+2} such that

a) $d_{Y_{r+2}}(\widetilde{Y}^1_{r+2} \cup \widetilde{Y}^2_{r+2}, y^{r+2}_{r+2}) = d_{X_{r+1}}(\widetilde{X}^1_{r+2} \cup \widetilde{X}^2_{r+2}, x^{r+2}_{r+2})$,

b) there is an isomorphism $\widetilde{L}_X \cup \widetilde{X}^1_{r+2} \cup \widetilde{X}^2_{r+2} \to \widetilde{L}_Y \cup \widetilde{Y}^1_{r+2} \cup \widetilde{Y}^2_{r+2}$ taking $x^1_{r+2}, \ldots, x^{r+2}_{r+2}$ to $y^1_{r+2}, \ldots, y^{r+2}_{r+2}$ respectively,

c) $(\widetilde{L}_Y \cup \widetilde{Y}^1_{r+2} \cup \widetilde{Y}^2_{r+2}, \widetilde{Y}^1_{r+2} \cup \widetilde{Y}^2_{r+2})$ is cyclically 2^{k-r-2}-maximal,

where $\widetilde{L}_Y$ is a minimal path connecting y^{r+2}_{r+2} and $\widetilde{Y}^1_{r+2} \cup \widetilde{Y}^2_{r+2}$ in Y_{r+2}. If $r = k-2$, then Duplicator wins. If $r < k-2$, then, obviously, $\widetilde{X}^1_{r+2} := \widetilde{X}^1_{r+2} \cup \widetilde{X}^2_{r+2} \cup \widetilde{L}_X$ and $\widetilde{Y}^1_{r+2} := \widetilde{Y}^1_{r+2} \cup \widetilde{Y}^2_{r+2} \cup \widetilde{L}_Y$ are $(k, r+2, 1)$-regular equivalent in (X_{r+2}, Y_{r+2}). Thus, in the $(r+3)$-rd round Duplicator can use the strategy SF. Finally, if x^{r+2}_{r+2} is not in the cyclic 2^{k-r-1}-extension of $\widetilde{X}^2_{r+2}$ and $d_{X_{r+2}}(x^{r+2}_{r+2}, \widetilde{X}^1_{r+2} \cup \widetilde{X}^2_{r+2}) > 2^{k-r-2}$, then denote by $\widetilde{X}^3_{r+2}$ the only cyclic 2^{k-r-2}-extension of $(\{x^{r+2}_{r+2}\}, \varnothing)$ (if there are no such extensions, then set $\widetilde{X}^3_{r+2} = (\{x^{r+2}_{r+2}\}, \varnothing)$). Obviously, $d_{X_{r+2}}(\widetilde{X}^3_{r+2}, \widetilde{X}^1_{r+2} \cup \widetilde{X}^2_{r+2}) > 2^{k-r-2}$. By Property 2) Y_{r+2} contains a vertex y^{r+2}_{r+2} and a subgraph $\widetilde{Y}^3_{r+2}$ such that

a) $d_{Y_{r+2}}(\widetilde{Y}^3_{r+2}, \widetilde{Y}^1_{r+2} \cup \widetilde{Y}^2_{r+2}) = 2^{k-r-2} + 1$,

b) there is an isomorphism $\widetilde{X}^3_{r+2} \to \widetilde{Y}^3_{r+2}$ taking x^{r+2}_{r+2} to y^{r+2}_{r+2},

c) there are no cyclic 2^{k-r-2}-extensions of $\widetilde{Y}^3_{r+2}$.

If $r = k-2$, then Duplicator wins. If $r < k-2$, then, obviously, the ordered tuples $\widetilde{X}^1_{r+2}, \widetilde{X}^2_{r+2}, \widetilde{X}^3_{r+2}$ and $\widetilde{Y}^1_{r+2}, \widetilde{Y}^2_{r+2}, Y^3_{r+2}$ are $(k, r+2, 3)$-regular equivalent in (X_{r+2}, Y_{r+2}). Therefore, in the $(r+3)$-rd round Duplicator can use the strategy SF.

5. Extended law

Theorem 2.3 can be extended in the following way.

THEOREM 5.1. *Let k and b be arbitrary natural numbers, $k > 3$. Let $\frac{a}{b}$ be a positive rational number in reduced form and let $\alpha = 1 - \frac{1}{2^{k-1} + a/b}$. Denote $\nu = \max\{1, 2^{k-1} - b\}$ and suppose $a \in \{\nu, \nu+1, \ldots, 2^{k-1}\}$. Then $\alpha \notin S^2_k$.*

The proof of Theorem 5.1 is almost the same as the proof of Theorem 8 from [14]. For this reason we do not give it here. The idea of that proof is based upon the following observation. Since Duplicator has a winning strategy in the game $\mathrm{EHR}(G, H, k)$ for all pairs (G, H) such that $G, H \in \mathcal{S}$ (see Section 4.3.3), by Theorem 4.2 it suffices to show that for every α from the statement of Theorem 5.1 there is an ε such that $\mathsf{P}(G(n,p) \in \mathcal{S}) \to 1$ as $n \to \infty$ for each $p \in [n^{-\alpha-\varepsilon}, n^{-\alpha+\varepsilon}]$ (see the proof of Theorem 8 from [14]).

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