ISSN 2220-5438

Reprint from

# Moscow Journal of Combinatorics and Number Theory

Moscow Journal of Combinatorics and Number Theory Volume 7 • Issue 2 2017

URSS

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Volume 7 • Issue 2 2017

Moscow Journal
of Combinatorics
and Number Theory
2017, vol. 7, iss. 2,
pp. 59–73, [pp. 147–161]

# On certain moments of Hardy's function $Z(t)$ over short intervals

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**Abstract:** Let as usual $Z(t) = \zeta(\frac{1}{2} + it)\chi^{-1/2}(\frac{1}{2} + it)$ denote Hardy's function, where $\zeta(s) = \chi(s)\zeta(1-s)$. Assuming the Riemann hypothesis upper and lower bounds for some integrals involving $Z(t)$ and $Z'(t)$ are proved. It is also proved that

$$H(\log T)^{k^2} \ll_{k,\alpha} \sum_{T<\gamma\leqslant T+H} \max_{\gamma\leqslant\tau_\gamma\leqslant\gamma^+} |\zeta(\tfrac{1}{2} + i\tau_\gamma)|^{2k} \ll_{k,\alpha} H(\log T)^{k^2}.$$

Here $k > 1$ is a fixed integer, $\gamma, \gamma^+$ denote ordinates of consecutive complex zeros of $\zeta(s)$ and $T^\alpha \leqslant H \leqslant T$, where $\alpha$ is a fixed constant such that $0 < \alpha \leqslant 1$. This sharpens and generalizes a result of M. B. Milinovich [17].

**Keywords:** Riemann zeta-function, Riemann Hypothesis, Hardy's function, moments, short intervals

**AMS Subject Classification:** 11M06

**Received:** 13.10.2016; **revised:** 05.12.2016

## 1. Introduction

Let the Riemann zeta-function be, as usual,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad (\operatorname{Re} s > 1).$$

For $\operatorname{Re} s \leqslant 1$ one defines $\zeta(s)$ by analytic continuation (see the monographs of H. M. Edwards [3], the author [11] and E. C. Titchmarsh [20] for the properties of $\zeta(s)$). Here the Riemann Hypothesis (RH), that all complex zeros of $\zeta(s)$ satisfy $\mathrm{Re}\, s = \frac{1}{2}$, is assumed *throughout the paper*. The Riemann zeta-function satisfies the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s) \quad (\forall\, s) \in \mathbb{C}), \quad \chi(s) := \frac{\Gamma(\frac{1}{2}(1-s))}{\Gamma(\frac{1}{2}s)}\pi^{s-1/2}, \tag{1.1}$$

where $\Gamma(s)$ is the familiar gamma-function. One then defines Hardy's function $Z(t)$ as

$$Z(t) := \zeta(\tfrac{1}{2}+it)\big(\chi(\tfrac{1}{2}+it)\big)^{-1/2}, \tag{1.2}$$

which is real for $t$ real and $|\zeta(\frac{1}{2}+it)| = |Z(t)|$. Thus the real zeros of $Z(t)$ correspond to the zeros of $\zeta(s)$ of the form $\frac{1}{2}+it$, which makes Hardy's function an invaluable tool in the study of zeros of $\zeta(s)$ on the critical line $\mathrm{Re}\, s = \frac{1}{2}$. For an extensive account on $Z(t)$ the reader is referred to the author's monograph [16].

Several papers deal with the estimation of the sum

$$\mathcal{M}_k(T) := \frac{1}{N(T)}\sum_{0<\gamma\leqslant T}\max_{\gamma\leqslant\tau_\gamma\leqslant\gamma^+}|\zeta(\tfrac{1}{2}+i\tau_\gamma)|^{2k} \equiv \frac{1}{N(T)}\sum_{0<\gamma\leqslant T}\max_{\gamma\leqslant\tau_\gamma\leqslant\gamma^+}|Z(\tau_\gamma)|^{2k}. \tag{1.3}$$

Here $k \in \mathbb{N}$ is fixed, and $\gamma, \gamma^+$ denote ordinates of consecutive complex zeros of $\zeta(s)$, ordered according to their size. Also, as usual,

$$N(T) = \sum_{0<\gamma\leqslant T} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T) \tag{1.4}$$

counts (with multiplicities) the number of zeros of $\zeta(s)$ whose ordinates $\gamma$ satisfy $0 < \gamma \leqslant T$.

In [2] B. Conrey and A. Ghosh proved, under the RH,

$$\mathcal{M}_1(T) = \frac{e^2-5}{2}\log T + O(1). \tag{1.5}$$

Actually, they prove a somewhat stronger result than (1.5), namely

$$\sum_{T<\gamma\leqslant T+H}\max_{\gamma\leqslant\tau_\gamma\leqslant\gamma^+}|\zeta(\tfrac{1}{2}+i\tau_\gamma)|^2 = \frac{e^2-5}{4\pi}H\log^2 T + O(H\log T) \tag{1.6}$$

with $H = T^{3/4}$. This follows from their proof on noting that (1.4) implies

$$N(T+H) - N(T) \sim \frac{H}{2\pi}\log T \qquad (T \to \infty).$$

B. Conrey [1] obtained, also under the RH,

$$\frac{\sqrt{21}}{45\pi}(1+o(1))\log^4 T \leqslant \mathcal{M}_2(T) \leqslant \frac{1+o(1)}{\pi\sqrt{15}}\log^4 T \quad (T \to \infty), \tag{1.7}$$

and R. R. Hall [4], [5] obtained some further improvements of (1.7). A general result, due to M. B. Milinovich [17], states that under the RH, for fixed $k \in \mathbb{N}$.

$$(\log T)^{k^2-\varepsilon} \ll_{k,\varepsilon} \mathcal{M}_k(T) \ll_{k,\varepsilon} (\log T)^{k^2+\varepsilon}. \tag{1.8}$$

Here $\ll_{k,\varepsilon}$ means that the constant implied by the $\ll$-symbol depends only on $k$ and $\varepsilon$, an arbitrarily small positive number, not necessarily the same one at each occurrence. The bounds in (1.8), when $k = 1, 2$, are implied by (1.6) and (1.7), respectively.

## 2. Statement of results

Milinovich [17] derives (1.8) from upper and lower bounds involving certain integrals with $Z(t)$ and $Z'(t)$, which seem to be of independent interest. He investigated integrals over the "long" interval $[0, T]$, but here we are interested in the integrals over the "short" intervals $[T, T+H]$, where $H = H(T)$ may be much smaller than $T$. We shall prove here the following theorems.

THEOREM 1. *Let $k \geqslant 2$ be a fixed integer. Under the RH we have, for $T^\alpha \leqslant H = H(T) \leqslant T$, $0 < \alpha \leqslant 1$ a fixed constant,*

$$\int_T^{T+H} (Z'(t))^2 Z^{2k-2}(t)\,\mathrm{d}t \gg_{k,\alpha} H(\log T)^{k^2+2}. \tag{2.1}$$

THEOREM 2. *Let $k \in \mathbb{N}$ be fixed. Under the RH we have, for $T^\alpha \leqslant H = H(T) \leqslant T$, $0 < \alpha \leqslant 1$ a fixed constant,*

$$\int_T^{T+H} |\zeta'(\tfrac{1}{2}+it)|^{2k}\,\mathrm{d}t \ll_{k,\alpha} H(\log T)^{k^2+2}, \tag{2.2}$$

*and*

$$(2.3)\qquad \int_T^{T+H} \bigl(Z'(t)\bigr)^{2k} \mathrm{d}t \ll_{k,\alpha} H(\log T)^{k^2+2}.$$

These bounds differ from the analogous results of [17] in two aspects. Firstly, Milinovich has the integrals over $[0,T]$, which corresponds to the case $H = T$ in our theorems. Indeed, if (2.1)–(2.3) hold with $H = T$, then replacing $T$ by $T/2, T/2^2, \dots$ etc. and adding up all the results we obtain (2.1)–(2.3) with the interval of integration $[0,T]$. Secondly, in (2.1) Milinovich obtained $k^2+2-\varepsilon$ as the exponent of the logarithm, and in (2.2) and (2.3) he had the exponents $k^2+2+\varepsilon$ of the logarithm. He remarks on page 1122 that one can get rid of the $\varepsilon$'s in his bounds provided that one has

$$(2.4)\qquad \int_0^T |\zeta(\tfrac12+it)|^{2k}\mathrm{d}t \ll_k T(\log T)^{k^2}$$

and

$$(2.5)\qquad \int_0^T |\zeta'(\tfrac12+it)|^{2k}\mathrm{d}t \ll_k T(\log T)^{k^2+2k}.$$

The estimates (2.4) and (2.5) do hold indeed. Namely K. Soundararajan [19] proved (2.4) with the exponent of $\log T$ in (2.4) equal to $k^2+\varepsilon$. The author [15] improved and sharpened Soundararajan's bound by showing that

$$(2.6)\qquad \int_T^{T+H} |\zeta(\tfrac12+it)|^{2k}\mathrm{d}t \ll_{k,\alpha} H(\log T)^{k^2\left(1+O(1/\log_3 T)\right)} \quad \text{(RH)}.$$

Here $T^\alpha \leqslant H \leqslant T$ where $0<\alpha\leqslant 1$ is a fixed number, and

$$\log_3 T = \log\log\log T = \log(\log_2 T).$$

Note that [15] appeared before [19] because of the backlog of "Ann. Math." The key result in [15], which is a proper generalization of the corresponding result in [19], is

THEOREM A. *Let $H = T^\theta$ where $0 < \theta \leqslant 1$ is a fixed number, and let $\mu(T, H, V)$ denote the measure of points $t$ from $[T-H,\ T+H]$ such that*

$$\log|\zeta(\tfrac{1}{2}+it)| \geqslant V, \quad 10\sqrt{\log_2 T} \leqslant V \leqslant \frac{3\log 2T}{8\log_2(2T)}.$$

*Then, under the RH, for* $10\sqrt{\log_2 T} \leqslant V \leqslant \log_2 T$ *we have*

$$\mu(T,H,V) \ll H\frac{V}{\sqrt{\log_2 T}}\exp\left(-\frac{V^2}{\log_2 T}\left(1-\frac{7}{2\theta\log_3 T}\right)\right),$$

*for* $\log_2 T \leqslant V \leqslant \frac{1}{2}\theta\log_2 T\log_3 T$ *we have*

$$\mu(T,H,V) \ll H\exp\left(-\frac{V^2}{\log_2 T}\left(1-\frac{7V}{4\theta\log_2 T\log_3 T}\right)^2\right),$$

*and for* $\frac{1}{2}\theta\log_2 T\log_3 T \leqslant V \leqslant \frac{3\log 2T}{8\log_2(2T)}$ *we have*

$$\mu(T,H,V) \ll H\exp(-\tfrac{1}{20}\theta V\log V).$$

Later A. Harper [9] (RH) improved the upper bound in [19] by establishing (2.4) for the long interval $[0,T]$. As remarked in [9] on p. 4, the method of [15] leading to (2.6), i. e., Theorem A, can be combined with that of [9] to produce the sharp upper bound over the short interval $[T, T+H]$, namely

$$\int_T^{T+H}|\zeta(\tfrac{1}{2}+it)|^{2k}\mathrm{d}t \ll_{k,\alpha} H(\log T)^{k^2} \quad \text{(RH)}. \tag{2.7}$$

The bound in (2.7) is the key ingredient in the proof of our results. It is, up to the values of the $\ll$-constants, best possible, since long ago it was shown by R. Balasubramanian and K. Ramachandra (see the latter's monograph [18], in particular the remark on p. 45) that, if $k \geqslant 1$ is a fixed integer, then for $C(\varepsilon, k)\log\log T \leqslant H \leqslant T/2$ we have

$$\int_T^{T+H}|\zeta(\tfrac{1}{2}+it)|^{2k}\mathrm{d}t \geqslant (C_k' - \varepsilon)H(\log H)^{k^2}, \tag{2.8}$$

where

$$C_k' = \frac{1}{2\Gamma(k^2+1)} \prod_p \left\{ (1-p^{-1})^{k^2} \sum_{m=0}^{\infty} \Bigl( \frac{\Gamma(k+m)}{\Gamma(k)m!} \Bigr)^2 p^{-m} \right\}.$$

We note that the lower bound in (2.8) is *unconditional*, with a very wide range for $H$. As for (2.5), it will be shown later that a corresponding result holds over $[T, T+H]$.

THEOREM 3. *Let* $1 < k \in \mathbb{N}$ *be fixed*, $\gamma, \gamma^+$ *denote ordinates of consecutive complex zeros of* $\zeta(s)$ *and* $T^\alpha \leqslant H = H(T) \leqslant T$, *where* $\alpha$ *is a fixed constant such that* $0 < \alpha \leqslant 1$. *Under the RH we have then*

$$H(\log T)^{k^2} \ll_{k,\alpha} \sum_{T<\gamma\leqslant T+H} \max_{\gamma\leqslant\tau_\gamma\leqslant\gamma^+} |\zeta(\tfrac12 + i\tau_\gamma)|^{2k} \ll_{k,\alpha} H(\log T)^{k^2}. \tag{2.9}$$

**Remark.** *The case* $k = 1$ *was treated in* [2] (*see* (1.6)) *and is not covered by Theorem* 3. *This is because Theorem* 1 *does not cover the case* $k = 1$. *The method of its proof* (*see* (3.4)) *does not work in obtaining a lower bound for*

$$\int_T^{T+H} |Z'(t)Z(t)| \mathrm{d}t.$$

## 3. Proof of Theorem 1

As was also done in [17], we follow Conrey and Ghosh [2], and introduce the analytic function

$$Z_1(s) := \zeta'(s) - \frac{\chi'(s)}{2\chi(s)}\zeta(s). \tag{3.1}$$

Its usefulness comes from the fact that differentiation of (1.2) gives

$$Z'(t) = i\left\{ \zeta'(\tfrac12 + it) - \tfrac12 \frac{\chi'(\frac12 + it)}{\chi(\frac12 + it)} \zeta(\tfrac12 + it) \right\} \chi^{-1/2}(\tfrac12 + it). \tag{3.2}$$

This gives

$$|Z'(t)| \;=\; |Z_1(\tfrac12 + it)|. \tag{3.3}$$

Using (3.3) and $|Z(t)| = |\zeta(\frac{1}{2}+it)|$ one may write

$$\int_T^{T+H} (Z'(t))^2 Z^{2k-2}(t)\mathrm{d}t = \int_T^{T+H} \left|Z_1(\tfrac{1}{2}+it)\zeta(\tfrac{1}{2}+it)^{k-1}\right|^2 \mathrm{d}t. \tag{3.4}$$

The basic idea is to use the inequality

$$\begin{aligned}&\Big|\int_T^{T+H} Z_1(\tfrac{1}{2}+it)\zeta(\tfrac{1}{2}+it)^{k-1}\overline{A}(t)\mathrm{d}t\Big|^2\\ &\leqslant \int_T^{T+H} (Z'(t))^2 Z^{2k-2}(t)\mathrm{d}t \cdot \int_T^{T+H} |A(t)|^2\mathrm{d}t.\end{aligned} \tag{3.5}$$

This comes on using (3.4) and the Cauchy-Schwarz inequality for integrals with a suitably chosen function $A(t)$. Following [17] we set

$$A(t) := \mathcal{A}(\tfrac{1}{2}+it), \quad \mathcal{A}(s) = \mathcal{A}(s;k,\xi) := \sum_{n\leqslant\xi} d_k(n)n^{-s},$$

where $d_k(n)$ (generated by $\zeta^k(s)$ for $\mathrm{Re}\, s > 1$) is the (generalized) divisor function which represents the number of ways $n$ can be written as a product of $k$ fixed factors (see e. g., Chapter 13 of [11] for more properties). The parameter $\xi$ is given by $\xi = T^\theta, 0 < \theta < 1$. The function $A(t)$ has the property that in mean square it behaves like $(\log\xi)^{k^2}$ (see Chapter 13 of [11]). Therefore, by the mean value theorem for Dirichlet polynomials (see e. g., Theorem 5.2 of [11]), we have

$$\begin{aligned}\int_T^{T+H} |A(t)|^2\mathrm{d}t &= H\sum_{n\leqslant\xi} d_k^2(n)n^{-1} + O\Big(\sum_{n\leqslant\xi} d_k^2(n)\Big)\\ &= H(C_k+o(1))(\log\xi)^{k^2} + O(\xi(\log\xi)^{k^2-1})\end{aligned} \tag{3.6}$$

for $2 \leqslant \xi \leqslant T$ and a positive constant $C_k$, which may be made explicit. It remains to estimate from below

$$
\begin{aligned}
&\int_T^{T+H} Z_1(\tfrac{1}{2}+it)\zeta(\tfrac{1}{2}+it)^{k-1}\overline{A}(t)\mathrm{d}t \\
&= \frac{1}{i}\int_{1/2+iT}^{1/2+iT+iH} Z_1(s)\zeta^k(s)\mathcal{A}(1-s)\mathrm{d}s \\
&= \int_{a+iT}^{a+iT+iH} \frac{1}{i}Z_1(s)\zeta^k(s)\mathcal{A}(1-s)\mathrm{d}s + O_\varepsilon(T^\varepsilon\xi).
\end{aligned}
\tag{3.7}
$$

Here we used Cauchy's theorem and set $a := 1 + 1/\log T$. We also used standard consequences of the RH (see Chapter 12 of [20]):

$$\zeta(s) \ll_{\varepsilon,\sigma} |t|^\varepsilon, \quad \zeta'(s) \ll_{\varepsilon,\sigma} |t|^\varepsilon \qquad (s = \sigma + it, \sigma \geqslant \tfrac{1}{2}),$$

as well as the unconditional, elementary bound $d_k(n) \ll_{\varepsilon,k} n^\varepsilon$ and

$$\frac{\chi'(\sigma+it)}{\chi(\sigma+it)} = -\log\frac{t}{2\pi} + O\Big(\frac{1}{t}\Big). \tag{3.8}$$

One obtains (3.8) by logarithmic differentiation of (1.1) and the use of Stirling's formula for the gamma-function. It is valid for $\frac{1}{2} \leqslant \operatorname{Re} s \leqslant 2$, and the $O$-term in (3.8) in fact admits a full asymptotic expansion in terms of negative exponents of $t$, and the left-hand side of (3.8) can be further differentiated. The integral on the right-hand side of (3.7) is written as $J_1 + J_2$, where

$$
\begin{aligned}
J_1 &:= \frac{1}{i}\int_{a+iT}^{a+iT+iH} \zeta'(s)\zeta^{k-1}\mathcal{A}(1-s)\mathrm{d}s, \\
J_2 &:= -\frac{1}{2i}\int_{a+iT}^{a+iT+iH} \frac{\chi'(s)}{\chi(s)}\zeta^k(s)\mathcal{A}(1-s)\mathrm{d}s.
\end{aligned}
$$

Similarly as in [17] one shows that

$$J_1 = -H \sum_{n \leqslant \xi} \widetilde{d}_k(n) d_k(n) n^{-1} + O(T^{\varepsilon}\xi)$$

with

$$\widetilde{d}_k(n) := \sum_{\delta | n} d_{k-1}(\delta) \log \frac{n}{\delta} \leqslant d_k(n) \log n,$$

and

$$J_2 = \tfrac{1}{2} H \left( \log \frac{T}{2\pi \mathrm{e}} \right) \sum_{n \leqslant \xi} d_k^2(n) n^{-1} + O(T^{\varepsilon}\xi).$$

This yields

$$\begin{aligned}
&\int_T^{T+H} Z_1(\tfrac{1}{2} + it)\zeta(\tfrac{1}{2} + it)^{k-1} \overline{A}(t) \mathrm{d}t = J_1 + J_2 + O(T^{\varepsilon}\xi) \\
&= -H \sum_{n \leqslant \xi} \widetilde{d}_k(n) d_k(n) n^{-1} + O(T^{\varepsilon}\xi) + \tfrac{1}{2} H \log \frac{T}{2\pi}(1 + o(1)) \sum_{n \leqslant \xi} d_k^2(n) n^{-1} \\
&\geqslant -H \sum_{n \leqslant \xi} d_k^2(n) n^{-1} \log n + \tfrac{1}{2} H \log \frac{T}{2\pi}(1 + o(1)) \sum_{n \leqslant \xi} d_k^2(n) n^{-1} + O(T^{\varepsilon}\xi) \\
&\geqslant \left\{ \tfrac{1}{2} H(1 + o(1)) \log \frac{T}{2\pi} - H \log \xi \right\} \sum_{n \leqslant \xi} d_k^2(n) n^{-1} + O(T^{\varepsilon}\xi) \\
&\geqslant A_k H \log T \cdot (\log \xi)^{k^2}
\end{aligned}$$

for $\xi = T^{\theta}, \theta = \frac{1}{2}\alpha$ and $\varepsilon$ sufficiently small. From (3.5) and (3.6) we finally gather that

$$H^2 \log^2 T (\log T)^{2k^2} \ll_{k,\alpha} \int_T^{T+H} (Z'(t))^2 Z^{2k-2}(t) \mathrm{d}t \cdot H(\log T)^{k^2},$$

and (2.1) of Theorem 1 follows.

## 4. Proof of Theorem 2

First note that, for $0 < R \leqslant \frac{1}{2}, T \leqslant t \leqslant 2T$, by Cauchy's integral formula we have

$$\zeta'(\tfrac{1}{2} + it) = \frac{1}{2\pi i} \int\limits_{|z|=R} \frac{\zeta(\frac{1}{2} + it + z)}{z^2} \mathrm{d}z.$$

This yields

$$\int\limits_T^{T+H} |\zeta'(\tfrac{1}{2} + it)|^{2k} \mathrm{d}t = \frac{1}{(2\pi)^{2k}} \int\limits_T^{T+H} \left| \int\limits_{|z|=R} \frac{\zeta(\frac{1}{2} + it + z)}{z^2} \mathrm{d}z \right|^{2k} \mathrm{d}t. \tag{4.1}$$

By Hölder's inequality for integrals the right-hand side of (4.1) does not exceed

$$\begin{aligned}
&\frac{1}{(2\pi)^{2k}} \int\limits_T^{T+H} \left\{ \int\limits_{|z|=R} |\zeta(\tfrac{1}{2} + it + z)|^{2k} |\mathrm{d}z| \right\} \cdot \left\{ \frac{|\mathrm{d}z|}{|z|^{4k/(2k-1)}} \right\}^{2k-1} \mathrm{d}t \\
&\leqslant \frac{1}{(2\pi)^{2k}} \int\limits_T^{T+H} \left\{ \int\limits_{|z|=R} |\zeta(\tfrac{1}{2} + it + z)|^{2k} |\mathrm{d}z| \right\} (2\pi R)^{2k-1} R^{-4k} \\
&\leqslant \frac{1}{R^{2k}} \max_{0 \leqslant \theta \leqslant 2\pi} \int\limits_T^{T+H} |\zeta(\tfrac{1}{2} + it + R\mathrm{e}^{i\theta})|^{2k} \mathrm{d}t.
\end{aligned}$$

Therefore

$$\int\limits_T^{T+H} |\zeta'(\tfrac{1}{2} + it)|^{2k} \mathrm{d}t \leqslant \frac{1}{R^{2k}} \max_{0 \leqslant \theta \leqslant 2\pi} \int\limits_T^{T+H} |\zeta(\tfrac{1}{2} + it + R\mathrm{e}^{i\theta})|^{2k} \mathrm{d}t. \tag{4.2}$$

As in [17], we could have obtained an inequality for the $\ell$-th derivative of $\zeta(\frac{1}{2} + it)$, but this is not necessary for our purposes.

Henceforth let $R = 1/\log T$ in (4.2). The integral on the right-hand side of (4.2) equals

$$\int\limits_T^{T+H} \left|\zeta\big(\tfrac{1}{2} + R\cos\alpha + i(t + R\sin\alpha)\big)\right|^{2k} \mathrm{d}t. \tag{4.3}$$

Recall that, under the RH (see [20]),

$$\zeta(\sigma+it) \ll \exp\left(C\frac{\log t}{\log\log t}\right) \qquad (\tfrac{1}{2}\leqslant\sigma\leqslant 1, C>0, |t|\geqslant 2). \tag{4.4}$$

When $\cos\alpha \geqslant 0$ in (4.3), we use (4.4) to obtain that the integral in (4.3) is equal to

$$\int_T^{T+H} |\zeta(\tfrac{1}{2}+R\cos\alpha+it)|^{2k}\mathrm{d}t + o(H). \tag{4.5}$$

When $\pi/2 \leqslant \theta \leqslant 3\pi/2$ in (4.5), that is, when $\cos\alpha \leqslant 0$, we use the functional equation (1.1). In this case we have, with $\sigma = \mathrm{Re}\, s = \frac{1}{2} + R\cos\alpha$, $R = 1/\log T$,

$$\chi(s) \ll |t|^{1/2-\sigma} \ll T^{-\cos\alpha/\log T} \ll 1,$$

thus we reduce the estimation of our integral to the case when $\cos\alpha \geqslant 0$. For this we shall use a convexity result which shows that essentially the integral in question is bounded by the $2k$-th moment of $|\zeta(\frac{1}{2}+it)|$ over a short interval. More precisely, let $1/2 \leqslant \sigma \leqslant 3/4$, $k>0$, $t\geqslant 2$,

$$J_k(\sigma) := \int_{-\infty}^{\infty} |\zeta(\sigma+it)|^{2k} w_k(t)\mathrm{d}t, \quad w_k(t) := \int_T^{T+H} \mathrm{e}^{-2k(t-\tau)^2}\mathrm{d}\tau. \tag{4.6}$$

Then

$$J_k(\sigma) \ll T^{\sigma-1/2}\Big(J_k(\tfrac{1}{2})\Big)^{3/2-\sigma} + \mathrm{e}^{-kT^2/4}. \tag{4.7}$$

The bound in (4.7) is the analogue of Lemma 4.2 of [17] for short intervals. This in turn is a result of D. R. Heath-Brown [10], which is also expounded in [12], pp. 321–323. In the original version the interval of integration in the kernel function function $w_k(t)$ was $[T, 2T]$. However, the change made in (4.6) does not affect the proof, and one obtains (4.7). Now note that $w_k(t) \gg 1$ for $t \in [T, T+H]$, so that

$$\int_T^{T+H} |\zeta(\sigma+it)|^{2k}\mathrm{d}t \ll J_k(\sigma). \tag{4.8}$$

We have the bound $w_k(t) \ll \exp(-2kH^2)$ when $t \leqslant T - H$ or $t \geqslant T + 2H$. On the other hand $w_k(t) \ll \exp(-kt^2)$ for $t < 0$ or $t > 3T$. Thus combining (4.7) and (4.8) it follows that

$$
\begin{aligned}
\int_T^{T+H} |\zeta(\sigma + it)|^{2k} \mathrm{d}t &= \int_{-\infty}^{0} + \int_{0}^{T-H} + \int_{T-H}^{T+2H} + \int_{T+2H}^{\infty} \\
&\ll 1 + \int_{T-H}^{T+2H} |\zeta(\tfrac{1}{2} + it)|^{2k} \mathrm{d}t \\
&\ll_{k,\alpha} H(\log T)^{k^2},
\end{aligned} \tag{4.9}
$$

where in the last step (2.7) was used. Inserting (4.9) in (4.2) the bound in (2.2) follows. The estimate (2.3) easily follows from (2.2), (2.9) and (3.2). Theorem 2 is proved.

## 5. Proof of Theorem 3

It is in the folklore that $Z(t)$, for $t \geqslant 14$, cannot have a negative local maximum or a positive local minimum under the RH. For this, see [3], or [13], [14], [6]. In other words, the zeros of $Z(t)$ and $Z'(t)$ are interlacing. Thus if $\gamma, \gamma^+$ are consecutive zeros of $Z(t)$, there is a unique point $\lambda_\gamma \in [\gamma, \gamma^+]$ for which $Z'(\lambda_\gamma) = 0$ (this is trivially true if $\gamma = \gamma^+$, that is, if $\gamma$ is a multiple zero of $Z(t)$). Therefore

$$
\max_{\gamma \leqslant \tau_\gamma \leqslant \gamma^+} |\zeta(\tfrac{1}{2} + i\tau_\gamma)|^{2k} = Z^{2k}(\lambda_\gamma). \tag{5.1}
$$

Then, since $Z(t)$ is positive in $(\gamma, \lambda_\gamma)$ and negative in $(\lambda_\gamma, \gamma^+)$, or conversely, we have

$$
\begin{aligned}
\int_\gamma^{\gamma^+} |Z'(t)Z^{2k-1}(t)| \mathrm{d}t &= \left| \int_\gamma^{\lambda_\gamma} Z'(t)Z^{2k-1}(t)\mathrm{d}t - \int_{\lambda_\gamma}^{\gamma} Z'(t)Z^{2k-1}(t)\mathrm{d}t \right| \\
&= \frac{1}{k} Z^{2k}(\lambda_\gamma).
\end{aligned} \tag{5.2}
$$

Therefore (5.1) and (5.2) give, in view of (4.4),

$$(5.3)\qquad \sum_{T<\gamma\leqslant T+H}\max_{\gamma\leqslant\tau_\gamma\leqslant\gamma^+}|\zeta(\tfrac12+i\tau_\gamma)|^{2k} = k\int_T^{T+H}|Z'(t)Z^{2k-1}(t)|\mathrm{d}t + O_{k,\varepsilon}(T^\varepsilon).$$

Assume that $k\geqslant 2$, so that $2k-2\geqslant 2$. To bound the integral on the right-hand side of (5.3) from below, note that

$$|(Z')^2Z^{2k-2}| = |Z'|^{1/2}|Z|^{k-1/2}\cdot|Z'|^{3/2}\cdot|Z|^{k-3/2}.$$

Thus Hölder's inequality for integrals shows that

$$\int_T^{T+H}(Z'(t))^2Z^{2k-2}(t)\mathrm{d}t\leqslant$$

$$\left(\int_T^{T+H}(|Z'|^{1/2}|Z|^{k-1/2})^p\mathrm{d}t\right)^{\frac1p}\left(\int_T^{T+H}|Z'|^{3q/2}\mathrm{d}t\right)^{\frac1q}\left(\int_T^{T+H}|Z|^{r(k-3/2)}\mathrm{d}t\right)^{\frac1r}$$

with $p,q,r>0$, $1/p+1/q+1/r=1$. Take

$$\frac1p=\frac12,\quad \frac1q=\frac{3}{4k},\quad \frac1r=\frac12-\frac{3}{4k}.$$

Then the right-hand side is, on using (2.3) and (2.4),

$$\leqslant\left(\int_T^{T+H}|ZZ|^{2k-1}\mathrm{d}t\right)^{\frac12}\left(\int_T^{T+H}|Z'|^{2k}\mathrm{d}t\right)^{\frac{3}{4k}}\left(\int_T^{T+H}|Z|^{2k}\mathrm{d}t\right)^{\frac12-\frac{3}{4k}}$$

$$\ll_{k,\alpha}\left(\int_T^{T+H}|ZZ|^{2k-1}\mathrm{d}t\right)^{\frac12}\left(H(\log T)^{k^2+2k}\right)^{\frac{3}{4k}}\left(H(\log T)^{k^2}\right)^{\frac12-\frac{3}{4k}}.$$

This gives, on using (2.1),

$$H(\log T)^{k^2+2}\ll_{k,\alpha} I^{1/2}\left(H(\log T)^{k^2+2k}\right)^{\frac{3}{4k}}\left(H(\log T)^{k^2}\right)^{\frac12-\frac{3}{4k}},$$

which on simplifying yields

$$I := \int_T^{T+H} |Z'(t)Z^{2k-1}(t)|\,\mathrm{d}t \;\gg_{k,\alpha}\; H(\log T)^{k^2+1}.$$

In view of (5.3) this proves the lower bound in (2.9) of Theorem 3.

As for the upper bound, the integral in (5.3) does not exceed, by Hölder's inequality for integrals,

$$\begin{aligned}
&\left|\int_T^{T+H} (Z'(t))^{2k}\mathrm{d}t\right|^{1/(2k)} \left|\int_T^{T+H} |\zeta(\tfrac12+it)|^{2k}\mathrm{d}t\right|^{1-1/(2k)} \\
&\ll_{k,\alpha} \left\{H(\log T)^{k^2+2k}\right\}^{1/(2k)} \left\{H(\log T)^{k^2}\right\}^{1-1/(2k)} \\
&= H(\log T)^{k^2+1},
\end{aligned} \tag{5.4}$$

which finishes the proof of Theorem 3 when $k \geqslant 2$. Here we used (2.3) and (2.4), and we note that the bound in (5.4) holds also for $k = 1$. It is the lower bound in this case which is problematic.

## Acknowledgment

The author wishes to thank M. B. Milinovich for valuable remarks.

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