Reprint from

ISSN 2220-5438

# Moscow Journal of Combinatorics and Number Theory

URSS

МФТИ

Volume 7 • Issue 3

2017

Moscow Journal of Combinatorics and Number Theory Volume 7 • Issue 3 2017 URSS

Moscow Journal
of Combinatorics
and Number Theory
2017, vol. 7, iss. 3,
pp. 66–72, [pp. 252–258]

# On the Littlewood conjecture in $\mathbb{Z}/p\mathbb{Z}$

Tomasz Schoen (Poznań)

**Abstract:** We show that for every set $A \subseteq \mathbb{Z}/p\mathbb{Z}$ we have $\|\widehat{1}_A\|_1 := \frac{1}{p}\sum_{r=0}^{p-1} |\sum_{a\in A} e^{2\pi ira/p}| \gg (\log|A|)^{1/16-o(1)}$.

**Keywords:** The Littlewood conjecture, exponential sums, sumsets

**AMS Subject Classification:** 11B75, 11P99

**Received:** 07.11.2016; **revised:** 23.02.2017

## 1. Introduction

The following conjecture, known as the Littlewood conjecture, was posed in [6]: for every finite set of integers $A$ we have

$$\int_0^1 |\sum_{a\in A} e^{2\pi iax}|\, dx \gg \log|A|.$$

This conjecture attracted attention of many mathematicians (see for example [1], [3], [4], [11]) and finally it was confirmed independently by McGehee, Pigno and Smith [10] and Konyagin [7]. Green and Konyagin asked an analogues question in discrete case for subsets of $\mathbb{Z}/p\mathbb{Z}$: is it true that

$$\|\widehat{1}_A\|_1 := \frac{1}{p}\sum_{r=0}^{p-1} |\sum_{a\in A} e^{2\pi ira/p}| \gg \log|A|.$$

They proved [5] that

$$\|\widehat{1}_A\|_1 \gg \delta\Big(\frac{\log p}{\log\log p}\Big)^{1/3},$$

then this estimate was improved by Sanders [12]

$$\|\widehat{1}_A\|_1 \gg \Big(\frac{\log p}{(\log\log p)^3}\Big)^{1/2}$$

for sets with positive density. Konyagin and Shkredov [8] solved the problem for sparse sets with size $|A| \leqslant e^{(\log p)^{1/3-o(1)}}$. Furthermore, Konyagin and Shkredov [8], [9] proved the following results, which we will use later.

Theorem 1.1. *Let $A$ be a subset of $\mathbb{Z}/p\mathbb{Z}$. If $e^{(\log p)^{1/3}} \leqslant |A| \leqslant p/3$ then*

$$\|\widehat{1}_A\|_1 \gg (\log(1/\delta))^{1/3-o(1)}, \tag{1.1}$$

*if $\delta \geqslant (\log p)^{-1/4}(\log\log p)^{1/2}$ then*

$$\|\widehat{1}_A\|_1 \gg \delta^{3/2}(\log p)^{1/2-o(1)},$$

*if $\delta < (\log p)^{-1/4}(\log\log p)^{1/2}$ then*

$$\|\widehat{1}_A\|_1 \gg \delta^{1/2}(\log p)^{1/4-o(1)},$$

The above results provides polylogarithmic lower bounds for sparse and very dense sets, however they gives very poor estimates for sets with density

$$e^{-(\log p)^{\varepsilon}} < \delta < (\log p)^{-1/2+\varepsilon}.$$

Our aim is to prove the following theorem.

Theorem 1.2. *Let $A$ be a subset of $\mathbb{Z}/p\mathbb{Z}$. Then*

$$\|\widehat{1}_A\|_1 \gg (\log|A|)^{1/16-o(1)},$$

*as $|A| \to \infty$.*

We will use the following notation. Let $G$ be a finite abelian group. For a function $f : G \to \mathbb{C}$ we set

$$||f||_{L^q} = \Big(\frac{1}{|G|}\sum |f(x)|^q\Big)^{1/q},$$

$$||f||_{\ell^q} = (\sum |f(x)|^q)^{1/q},$$

and if $\gamma \in \widehat{G}$ is a character of G then we define the Fourier coefficient by

$$\widehat{f}(\gamma) = \frac{1}{|G|}\sum_x f(x)\overline{\gamma(x)}.$$

The convolution of two functions $f, g : G \to \mathbb{C}$ is defined by

$$(f * g)(t) = \sum_{x\in G} f(x)g(t-x),$$

for $t \in G$. Furthermore, we will write $1_A$ for the indicator function of a set $A$ and $||\widehat{1}_A||_1$ for $||\widehat{1}_A||_{\ell^1}$.

Throughout the paper we will assume, if necessary, that the order of the considered group is large enough.

## 2. Auxiliary Lemmas

We apply a quasi-periodic lemma due to Croot, Laba and Sisask [2]. To formulate this result we need some basic properties of Bohr sets contained in a finite abelian group $G$. For a set of characters $\Gamma \subseteq \widehat{G}$ and $0 < \varepsilon \leqslant 2$ we define

$$B(\Gamma, \varepsilon) = \{x \in G : |\gamma(x) - 1| \leqslant \varepsilon \text{ for all } \gamma \in \Gamma\}.$$

The size of $\Gamma$ is the *rank* of $B$ and $\varepsilon$ is its *radius*. Furthermore, we will use also a lower bound for the size of a Bohr set, see [13].

LEMMA 2.3. *Suppose that* $B(\Gamma, \varepsilon)$ *is a Bohr set in a finite abelian group* $G$. *Then* $B(\Gamma, \varepsilon) \geqslant (\varepsilon/2\pi)^{|\Gamma|}|G|$.

Lemma 2.4. *Let $q \geqslant 2$ and $0 < \varepsilon < 1$ be parameters. Let $G$ be a finite abelian group and let $f : G \to \mathbb{C}$. Then there exists a Bohr set $B$ of rank $\ll q/\varepsilon^2$ and radius $\gg \varepsilon$ such that for each $t \in B$*

$$||f(x+t) - f(x)||_{L^q} \leqslant \varepsilon ||\widehat{f}||_1.$$

Corollary 2.5. *Suppose that $A$ is a subset of finite abelian group $G$, $|A| = \delta|G|$, $\delta \leqslant 1/2$, and that $\max_{t\neq 0}(1_A * 1_{-A})(t) < (1-\beta)|A|$ for some $0 < \beta \leqslant 1$. Then*

$$||\widehat{1}_A||_1 \gg \Big(\frac{\log|G|}{\log(1/\beta\delta)\log\log|G|}\Big)^{1/2}.$$

Proof. We apply Lemma 2.4 with $f = 1_A$ and $\varepsilon, q$ to be determined later.

Let $B$ be a Bohr set given by the lemma. Observe that for each $t \in G$

$$||1_A(x+t) - 1_A(x)||^q_{L^q} = 2|G|^{-1}(|A| - (1_A * 1_{-A})(t)),$$

hence, for every $t \in B$

$$(1_A * 1_{-A})(t) \geqslant |A| - \frac{1}{2}\varepsilon^q ||\widehat{1}_A||_1^q |G|.$$

By our assumption $(1_A * 1_{-A})(t) < (1-\beta)|A|$ for every $t \neq 0$, so if $t \in B \setminus \{0\}$ then $\varepsilon^q ||\widehat{1}_A||^q_1 |G| \geqslant \beta|A|$, hence

$$||\widehat{1}_A||_1 \geqslant (\beta\delta)^{1/q}\varepsilon^{-1}.$$

We take $q = 2\log(1/\beta\delta)$ and $\varepsilon = \Big(\frac{c\log|G|}{q\log\log|G|}\Big)^{-1/2}$, where $c > 0$ is an appropriate constant. To finish the proof it is enough to show that for such choice of parameters $B$ contains a nonzero element. By Lemma 2.4 there exists a positive constant $C$ such that by Lemma 2.3 we have

$$\begin{aligned} |B| &\geqslant \Big(\frac{\varepsilon}{2C\pi}\Big)^{Cq/\varepsilon^2}|G| = \Big(\frac{\log\log|G|}{2Cc\pi q\log|G|}\Big)^{\frac{Cc\log|G|}{2\log\log|G|}}|G| \\ &= |G|\exp\Big(-\frac{Cc\log|G|}{2\log\log|G|}(\log\log|G| + \log q - \log\log\log|G| + O(1))\Big) \\ &\geqslant |G|\exp(-(1+o(1))Cc\log|G|). \end{aligned}$$

Thus, taking $c = 1/(2C)$ we see that $|B| > 1$ and the proof is completed. □

Put $A_d = A \cap (A + d)$. The next lemma provides a straightforward dependence between $\|\widehat{1}_A\|_1$ and $\|\widehat{1}_{A_d}\|_1$, which is very important in our approach.

Lemma 2.6. *Let $A$ be a finite subset of $\mathbb{Z}/p\mathbb{Z}$. Then for every $d \in A - A$ we have*

$$\|\widehat{1}_A\|_1 \geqslant \|\widehat{1}_{A_d}\|_1^{1/2}$$

Proof. We have

$$\widehat{1}_{A_d}(r) = \frac{1}{p}\sum_x 1_A(x)1_A(x-d)e^{-2\pi irx/p},$$

and applying the Fourier inversion formula $1_A(x) = \sum_{r=0}^{p-1} \widehat{1}_A(r)e^{2\pi irx/p}$ we see that

$$\widehat{1}_{A_d}(r) = \sum_s \widehat{1}_A(s)\widehat{1}_A(r-s)e^{2\pi id(r-s)/p},$$

hence

$$\|\widehat{1}_{A_d}\|_1 \leqslant \sum_r\sum_s |\widehat{1}_A(s)||\widehat{1}_A(r-s)| = \|\widehat{1}_A\|_1^2$$

and the assertion follows. □

## 3. Proof of Theorem 1.2

In view of Theorem 1.1 we can restrict our attention to sets with density

$$e^{-(\log p)^{1/4}} < \delta < \frac{1}{2}(\log p)^{-1/4}.$$

By Cauchy-Davenport theorem (see [13]) we have

$$|A - A| \geqslant \min(2|A| - 1,\ p) \geqslant \frac{3}{2}|A|,$$

provided that $|A| \geqslant 2$, so there exists $d \in A - A$ such that

$$|A_d| = (1_A * 1_{-A})(d) \leqslant \frac{|A|^2}{\frac{3}{2}|A|} = \frac{2}{3}|A|.$$

We put $B = A \setminus A_d$ and observe that $|B| \geqslant \frac{1}{3}|A|$ and $d \notin B - B$. We consider two cases. First, let us assume that $\max_{t\neq 0}(1_B * 1_{-B})(t) < (1-\beta)|B|$, where $\beta = e^{-2(\log p)^{3/4}} \leqslant 1/3$. Then, by Corollary 2.5 we obtain

$$\|\widehat{1}_B\|_1 \gg (\log p)^{1/8-o(1)}.$$

Next let us assume that there exists $t \neq 0$ with $(1_B * 1_{-B})(t) \geqslant (1-\beta)|B|$. We show that then there is $s \in B - B$ such that $\beta|B| < |B_s| \leqslant 3\beta|B|$. Suppose to the contradiction. Let $x$ be any element satisfying

$$(1_B * 1_{-B})(x) \geqslant 3\beta|B|,$$

so there are representations $x = a_i - b_i$, where $a_i, b_i \in B$ and $1 \leqslant i \leqslant \lceil 3\beta|B| \rceil$. Notice that from $(1_B * 1_{-B})(t) \geqslant (1-\beta)|B|$ it follows that among $a_i's$ there are at least

$$(1_B * 1_{-B})(x) - \beta|B| \geqslant 2\beta|B|$$

such that $a_i + t \in B$. Therefore, we infer that $(1_B * 1_{-B})(x+t) \geqslant 2\beta|B|$, but from our assumption it follows that

$$(1_B * 1_{-B})(x+t) \geqslant 3\beta|B|.$$

Since $(1_B * 1_{-B})(0) = |B| \geqslant 3\beta|B|$ we see that $B - B = \mathbb{Z}/p\mathbb{Z}$, which is a contradiction. If $\beta|B| < |B_s| \leqslant 3\beta|B|$ then by Lemma 2.6 and (1.1) we have

$$\|\widehat{1}_B\|_1 \geqslant \|\widehat{1}_{B_s}\|_1^{1/2} \gg (\log p)^{1/8-o(1)}.$$

To finish the proof it is enough to observe that $\widehat{1}_A(r) = \widehat{1}_B(r) + \widehat{1}_{A_d}(r)$, so that $|\widehat{1}_A(r)| \geqslant |\widehat{1}_B(r)| - |\widehat{1}_{A_d}(r)|$ and $\|\widehat{1}_A\|_1 \geqslant \|\widehat{1}_B\|_1 - \|\widehat{1}_{A_d}\|_1$. Again, by Lemma 2.6

$$\|\widehat{1}_A\|_1 \geqslant \max(\|\widehat{1}_B\|_1 - \|\widehat{1}_{A_d}\|_1, \|\widehat{1}_{A_d}\|_1^{1/2}) \gg (\log p)^{1/16-o(1)},$$

which completes the proof. □

## Acknowledgment

The author is supported by National Science Centre, Poland grant 2016/21/B/ST1/00307.

## Bibliography

1. ***P. J. Cohen,*** *On a conjecture of Littlewood and idempotent measures*, Amer. J. Math. **82** 1960, 191-Ц212.
2. ***E. Croot, I. Laba, O. Sisask,*** *Arithmetic progressions in sumsets and $L^p$-almost-periodicity*, Combin. Probab. Comput. **22** (2013), 351–365.
3. ***H. Davenport,*** *On a theorem of P. J. Cohen*, Mathematika **7** (1960), 93–97.
4. ***J. J. F. Fournier,*** *On a theorem of Paley and the Littlwood conjecture*, Arkiv för Mathematik **17** (1979), 199–216.
5. ***B. J. Green, S. V. Konyagin,*** *On the Littlewood problem modulo a prime*, Canad. J. Math. **61** (2009), 141–164.
6. ***G. H. Hardy, J. E. Littlewood,*** *A new proof of a theorem on rearrangements*, J. London Math. Soc. **23** (1948), 163–168.
7. ***S. V. Konyagin,*** *On a problem of Littlewood*, Izvestiya of Russian Academy of Sciences, **45** (1981), 243–265.
8. ***S. V. Konyagin, I. D. Shkredov,*** *A quantitative version of BeurlingÖ Helson theorem*, Functional Analysis and Its Applications, **49** (2015), 110–121.
9. ***S. V. Konyagin, I. D. Shkredov,*** *On Wiener norm of subsets of $\mathbb{Z}_p$ of medium size*, Journal of Mathematical Sciences, **218** (2016), 599–608.
10. ***O. C. McGehee, L. Pigno, B. Smith,*** *Hardy's inequality and the $L^1$ norm of exponential sums*, Annals of Math. **113** (1981), 613–618.
11. ***S. K. Pichorides,*** *On the $L^1$ norm of exponential sums*, Annales de l'institut Fourier **30** (1980), 79–89.
12. ***T. Sanders,*** *The Littlewood-Gowers problem*, J. Anal. Math. **101** (2007), 123–162.
13. ***T. Tao, V. Vu,*** Additive combinatorics, Cambridge University Press 2006.

TOMASZ SCHOEN

Faculty of Mathematics and Computer Science,
Adam Mickiewicz University,
Umultowska 87, 61–614 Poznań, Poland
schoen@amu.edu.pl