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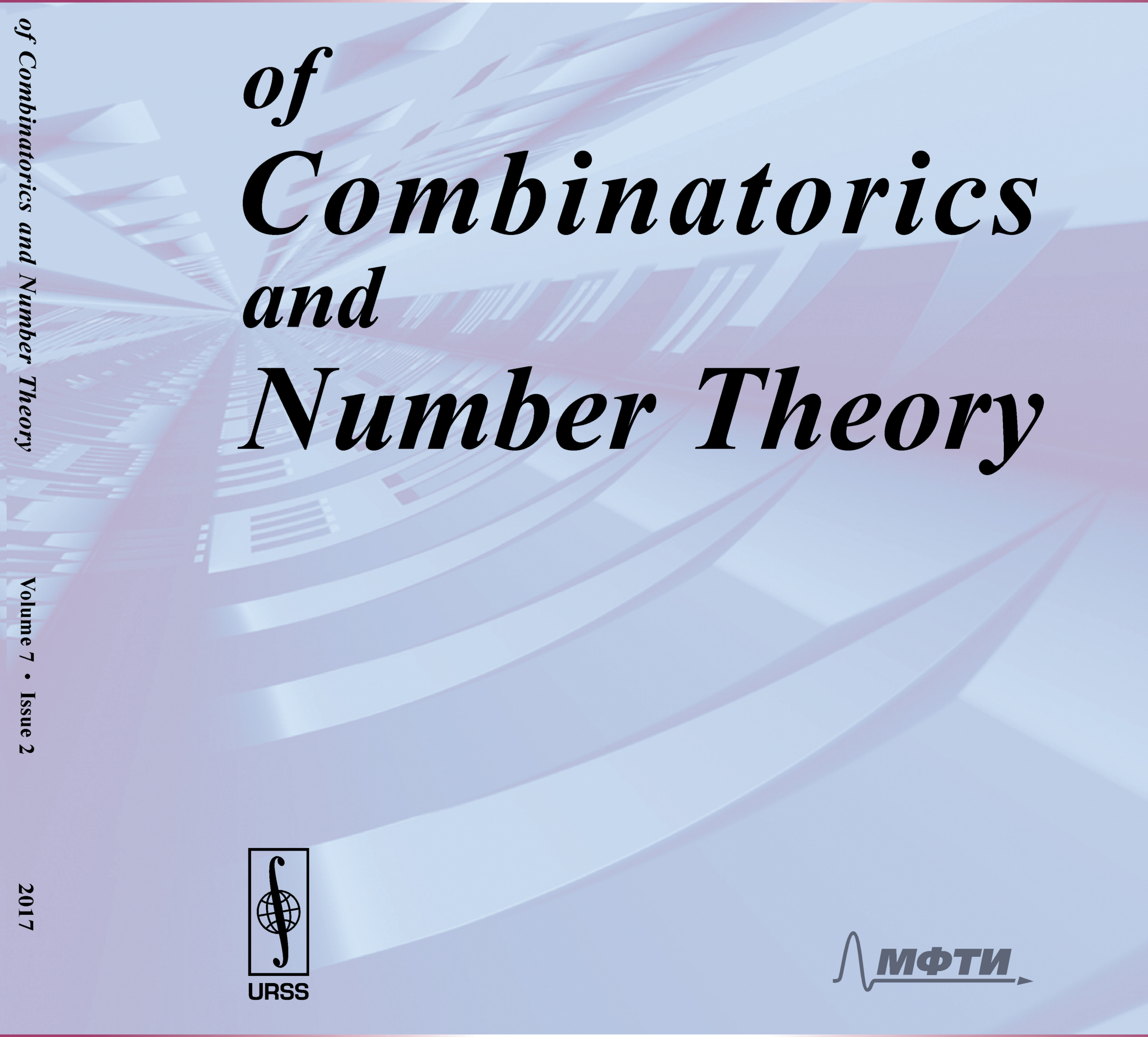

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Voronoi′s proof of his conjecture for primitive parallelotopes

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Abstract: We give in modern terms Voronoi's proof of his conjecture for primitive parallelotopes. This proof uses a simple algorithm for finding canonical lengths of normal vectors of a primitive parallelotope.

Keywords: Voronoi's conjecture, primitive parallelotope, canonical normal vectors

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1. Parallelotopes

A parallelotope is a polytope P whose parallel translations along vectors of a latice Λ form a *tiling* $\mathcal{T}$, i. e. they fill the space without gaps and intersections by inner points. Let F be a facet, i. e. a face of codimension 1, of the parallelotope $P = P(0)$ with its center in origin $0 \in \Lambda(P)$. The parallelotope $P(0)$ is adjacent by the facet F to a parallelotope $P(q_F)$ with its center $q_F \in \Lambda$. The vector q_F is called *facet* vector. All facet vectors generate integrally the lattice Λ. It is well known that facets of P are centrally symmetric. The center of the facet F is the point $\frac{1}{2}q_F$.

A special case of a parallelotope is a *Dirichlet-Voronoi cell* of a point, say 0, of an n-dimensional lattice Λ with respect to an Euclidean metric $\varphi(x) = \langle x, Dx\rangle$. Here D is a symmetric positive definite $n \times n$ matrix, and $\langle p, x\rangle$ is scalar product of vectors $p, x \in \mathbb{R}^n$. The Dirichlet-Voronoi cell $P_\varphi(0)$ is defined as follows

$$P_\varphi(0) = \{x \in \mathbb{R}^n : \varphi(x) \le \varphi(x - \lambda) \text{ for all } \lambda \in \Lambda\}.$$

The inequality $\varphi(x) \le \varphi(x - \lambda)$ is equivalent to the inequality $\langle D\lambda, x\rangle \le \le \frac{1}{2}\langle\lambda, D\lambda\rangle = \frac{1}{2}\varphi(\lambda)$. If we set $p = D\lambda$, i. e. $\lambda = D^{-1}p$, then we obtain $\frac{1}{2}\varphi(\lambda) = f(p)$, where $f(p) = \langle p, Ap\rangle$ and $A = \frac{1}{2}D^{-1}$ is a positive definite symmetric matrix. Hence the Dirichlet-Voronoi cell $P_\varphi(0) = P$ has the following description by linear in x inequalities

$$P = \{x \in \mathbb{R}^n : \langle p, x\rangle \le f(p) \text{ for all } p \in L\},$$

where $L = D\Lambda = \frac{1}{2}A^{-1}\Lambda$ is the lattice generated by *normal* vectors of P. So, if P is a Dirichlet-Voronoi cell, then there are two lattices Λ and L related to P. The lattices Λ and L are generated by facet and normal vectors, respectively.

More than one hundred years ago G. F. Voronoi conjectured in his famous second memoir [1] that each parallelotope is the Dirichlet-Voronoi cell of a lattice for some Euclidean metric.

Note that each *polytope* P is described by linear inequalities as follows

$$P = \{x \in \mathbb{R}^n : \langle p, x\rangle \le f(p) \text{ for all } p \in \mathcal{P}\}, \tag{1}$$

where $\mathcal{P}$ is a set of vectors that contains the subset $\mathcal{P}_s$ of *normal* vectors of the polytope P. In particular, each *parallelotope* has the description (1). Obviously, lengths of normal vectors in the description (1) of a parallelotope can be changed arbitrary by multiplying left and right sides of inequalities on a positive scalar.

2. Canonical normal vectors

Voronoi proved in [1] that a polytope P described by (1) is a parallelotope if one can choose lengths of vectors $p \in \mathcal{P}$ such that the set $\mathcal{P}$ of new vectors generates integrally a lattice L and f is a positive definite quadratic form defined on $\mathcal{P}$. In this case, Voronoi says that the parallelotope P is *defined* by the quadratic form f, and normal vectors are *canonical*. But Voronoi takes the set of all integral vectors as the set $\mathcal{P}$, i. e. he uses $\mathcal{P} = \mathbb{Z}^n$.

Usually, the notion of canonical normal vectors of P relates to faces of codimension 2 of the tiling generated by the parallelotope P. It is well known that each face G of codimension 2 is contained in 3 or 4 facets. Normal vectors of these facets lie in a 2-dimensional plane. This plane is orthogonal to the affine space spanned by G. The face G is called *primitive* if it is intersection of 3 facets of the tiling. It is obvious that one can take lengths and signs of normal vectors of the 3 facets such that their sum is 0. We say, that they form a triangle. We call normal vectors *canonical in second sense* if, for each primitive face of codimension 2, corresponding normal vectors form a triangle.

It is proved in [2] that the both above notions of canonical normal vectors are equivalent. More exactly, if normal vectors are canonical in the second sense, then there is a positive definite symmetric matrix D such that $p = Dq_{F(p)}$ for each facet $F(p)$ with normal vector p and facet vector $q_{F(p)}$. Since facet vectors q_F generate the lattice Λ, the canonical normal vectors generate the lattice $L = D\Lambda$.

An n-dimensional parallelotope is called *primitive* if each its vertex belongs exactly to $n+1$ paralleotopes of its tiling. All faces of codimension 2 of a primitive parallelotope are primitive. Voronoi proved in §§ 22–36 of [1] that a primitive parallelotope has canonical normal vectors. In other words, he proved in these sections his conjecture for primitive parallelotopes. But, in order to obtain an explicit map of a primitive parallelotope into a Dirichlet-Voronoi cell, Voronoi constructs a generatrissa.

In this note we rewrite his proof of existence canonical normal vectors for primitive parallelotopes.

3. Main result

Proposition 1. *Any primitive parallelotope has canonical normal vectors.*

Proof. Let V be the set of vertices of a primitive parallelotope P. There are $\frac{1}{2}n(n+1)$ facets of the tiling $\mathcal{T}$ that contain a vertex v. For $v \in V$, we construct a simplex $S(v)$ whose edges are parallel to all normal vectors of these facets as follows.

Let F_i, $i \in N = \{1, 2, ..., n\}$, be all n facets of P containing the vertex v. Let p_i be an outer normal vector to the facet F_i. Voronoi considers the following simplicial cone

$$A(v) = \{x \in \mathbb{R}^n : x = \sum_{i \in N} \lambda_i p_i, \lambda_i \geqslant 0\}$$

spanned by normal vectors p_i for all $i \in N$. Obviously, origin 0 is apex of $A(v)$.

Voronoi proves in § 4 of his second memoir [1] that the set of cones $A(v)$ for all $v \in V$ partitions the space. If vertices v and v' are adjacent in P, then the intersection $A(v) \cap A(v')$ is a facet of both these cones. A sequence of cones $\{A(v_1), A(v_2), ..., A(v_m)\}$ is called a *path* if intersections $A(v_i) \cap A(v_{i+1})$ are facets for all $i > m$. This path is called *circuit* if $v_m = v_1$. Since the space is simply connected, any two cones can be connected by a path.

For $i \in N$, let P_i be a parallelotope adjacent to P by the facet F_i. All the P_i, $i \in N$, share a common edge e_v with one end-vertex v. Besides, for all pairs $i, j \in N$, $i \neq j$, the intersection $F_{ij} = P_i \cap P_j$ is a facet with a normal vector p_{ij}. Since the facet F_{ij} contains the edge e_v, the normal vector p_{ij} is parallel to an affine hyperplane H_v that is orthogonal to the edge e_v and intersects it. Besides the vector p_{ij} is parallel to 2-dimensional plane H_{ij} that is orthogonal to the affine space of the primitive face $G_{ij} = F_i \cap F_j$. The plane H_{ij} is generated by the corresponding 2-face of the cone $A(v)$. Hence the normal vector p_{ij} is parallel to 1-dimensional intersection $H_{ij} \cap H_v$.

Since $\langle p_i, e_v \rangle > 0$ for all $i \in N$, the intersection $H_v \cap A(v)$ is an $(n-1)$-dimensional simplex S_v and its edges are parallel to normal vectors p_{ij}. We set $S(v) = conv(S_v \cup 0)$. If we fix a location of the hyperplane H_v, then we fix lengths of all edges of the simplex $S(v)$. These lengths determine canonical normal vectors.

For a given vertex $v_0 \in V$, let lengths of edges of $S(v_0)$ be fixed. Then lengths of edges of all simplices $S(v)$ are determined if the intersection $S(v) \cap S(v_0)$ is distinct from the vertex 0.

Consider a sequence of vertices v_k, $0 \leqslant k \leqslant m$, of the parallelotope P. Let this sequence be such that intersections $S(v_k) \cap S(v_{k+1})$ are facets for all k. Now, suppose that $S(v_0) \cap S(v_m) = F$, where F is a facet of both the simplices. Since simplex is uniquely scaled, lengths of edges of F obtained from $S(v_0)$ and $S(v_m)$ may differ by a common scalar. We have to prove that lengths of normal vectors spanning the facet F in both the simplices are equal. Since $S(v_0) \cap S(v_m) = F$, the sequence $v_0, v_1, ..., v_m = v_0$ is a circuit C. It is obvious that if all simplices of the circuit have a common face distinct from the vertex 0 then lengths of all edges of all the simlices are uniquely determined.

By well known today method, see, for example, papers [3], [4] and many others, Voronoi (in §§ 24–28 of [1]) reduces the circuit C to such small circuits that simplices of each small circuit have a common face. This method consists of a representation of the circuit of simplices C by a one-dimensional circuit c.

The circuit c intersects transversally faces of simplices of C. Obviously, there is a one-to-one correspondence between sequences of simplices $S(v)$ and cones $A(v)$. Since the space is simply connected, the one-dimensional circuit c can be shrunk into a point. More exactly, the circuit c can be represented as a superposition of *elementary* circuits. By definition, each elementary circuit belongs to a set of cones $A(v)$ having a common face distinct from 0.

The result follows. ☐

Remark. *Voronoi himself calls the simplex $S(v)$ by L-simplex related to the vertex v. He uses piecewise affine function $P(x)$ instead of the simplices S_v. The simplex S_v is a parallel shift of the kernel of linear part $P_{A(v)}(x)$ of $A(x)$ on the cone $A(v)$. By the same method, he proves that the function $P(x)$ is uniquely defined (up to a multiple).*

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