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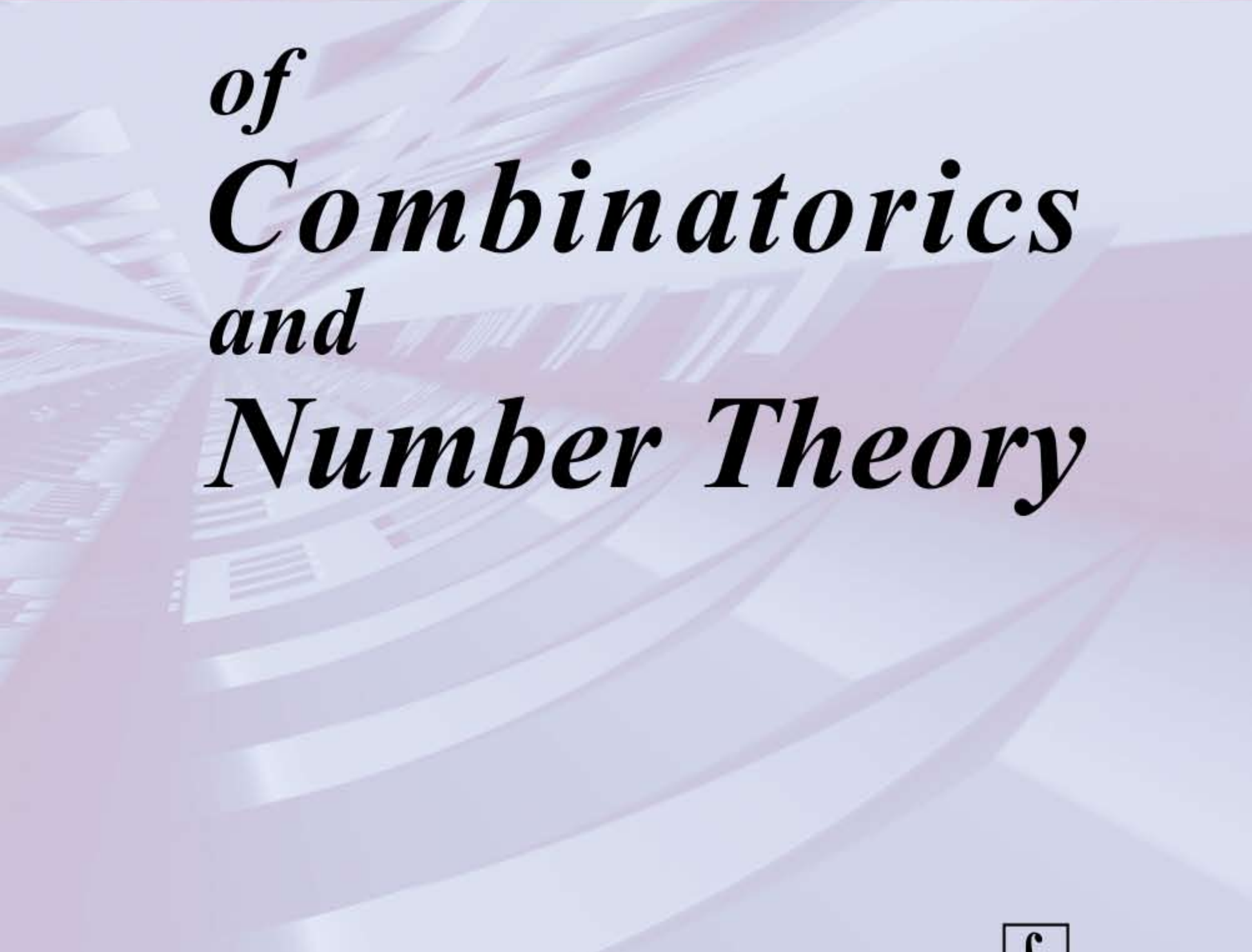

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The second Voronoi conjecture on parallelohedra for zonotopes

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Abstract: We prove the second Voronoi conjecture on parallelohedra for zonotopes. We show that for a given face-to-face tiling of the d-dimensional Euclidean space into parallel copies of a zonotope Z there are d vectors connecting the centers of zonotopes with a common facet, which form a basis of the corresponding lattice of the tiling.

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1. Parallelohedra

Definition. *A polytope* $P \subseteq \mathbb{R}^d$ *is called a parallelohedron or a parallelotope if the Euclidean space* $\mathbb{R}^d$ *can be tiled by non-overlapping parallel copies of* P.

We start by mentioning two classical results in the theory of parallelohedra. The first one is the Minkowski theorem [4], which claims that a parallelohedron P is centrally symmetric, that any facet of P is centrally symmetric, and that the projection of P along any of its ridges (faces of codimension 2) is either a parallelogram or a centrally symmetric hexagon. The second classical result is the Venkov theorem, which claims that the three Minkowski conditions are sufficient for P to be a parallelohedron.

In general, two parallelohedra with a common boundary point share only a portion of a face of both, as can be seen in brick walls where bricks from consecutive courses share only half a facet.

Definition. *If the intersection of any two polytopes in the tiling* $\mathcal{T}$ *of* $\mathbb{R}^d$ *by non-overlapping polytopes is a face of each of them (this face can be empty), then the tiling* $\mathcal{T}$ *is called a face-to-face tiling.*

McMullen showed [3] that if a parallelohedron P admits a tiling that is not face-to-face, then it also admits a tiling that is face-to-face. In this article we will only consider the face-to-face tiling $\mathcal{T}(P)$ generated by a parallelohedron P. It is apparent that such face-to-face tilings are unique up to a translation. It is also apparent that the centers of the tiles of such a face-to-face tiling form a d-dimensional lattice $\Lambda(P)$.

On the other hand, if Λ is an arbitrary d-dimensional lattice, the Dirichlet—Voronoi construction associates with Λ a parallelotope; for a fixed point $\lambda \in \Lambda$ let P_λ be the set of points that are at least as close to λ as to any other point of Λ. The polytope P_λ, called the *Dirichlet—Voronoi polytope* for Λ, is a parallelohedron. This follows from the fact that P_λ and P_μ corresponding to distinct lattice points differ only by a translation, and that the collection of all such polytopes fit together face-to-face to the whole $\mathbb{R}^d$.

The first conjecture of Voronoi links the notions of parallelohedra and Dirichlet—Voronoi polytopes for lattices.

Conjecture (G. Voronoi [6]). For any parallelohedron P there is an affine transformation $\mathcal{A}$, and a lattice Λ, such that $\mathcal{A}(P)$ is the Dirichlet—Voronoi polytope for Λ.

Let $\mathcal{F}(P)$ be the set of the facet vectors for P, i. e. the set of all the vectors between the center of P and the centers of all the other parallelohedra in $\mathcal{T}(P)$ sharing a facet with P. The set $\mathcal{F}(P)$ generates the lattice $\Lambda(P)$, since each vector in $\Lambda(P)$ can be represented as a sum of vectors from $\mathcal{F}(P)$.

Conjecture (G. Voronoi [6]). We can choose d vectors in $\mathcal{F}(P)$ to form a basis of $\Lambda(P)$.

In this paper we will prove the second conjecture of Voronoi in the case of space-filling zonotopes.

2. Delone tilings and Dirichlet—Voronoi tilings

We can generalize the construction of Dirichlet—Voronoi polytope on the case of a general positive definite quadratic form.

Definition. *Let $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a positive definite quadratic form in the d-dimensional Euclidean space and let O be a point of some fixed lattice Λ. The Dirichlet—Voronoi polytope $P_\varphi(\Lambda)$ for Λ with respect to φ is the polytope consisting of all the points X of $\mathbb{R}^d$ such that $\varphi(\overrightarrow{OX})$ is not greater than $\varphi(\overrightarrow{O'X})$ for any other point $O' \in \Lambda$.*

The constructed polytope $P_\varphi(\Lambda)$ is also a parallelohedron and the corresponding tiling is called a *Dirichlet—Voronoi tiling* $\mathcal{V}_\varphi(\Lambda)$. Taking as φ the usual metric form $\varphi(\mathbf{x}) = \mathbf{x}^T\mathbf{x}$ gives us the usual Dirichlet—Voronoi polytope DV_Λ.

Applying an affine transformation $\mathcal{A}$ to a given lattice Λ and a polytope $P_\varphi(\Lambda)$ with $\varphi(\mathbf{x}) = \mathbf{x}^T Q\mathbf{x}$ we come to the lattice $\mathcal{A}\Lambda$ and to the polytope $P_{\varphi_\mathcal{A}}(\mathcal{A}\Lambda)$ with respect to the quadratic form $\varphi_\mathcal{A}(\mathbf{x}) = (\mathcal{A}^{-1}\mathbf{x})^T Q(\mathcal{A}^{-1}\mathbf{x})$. So, in order to prove the first conjecture of Voronoi for a given polytope P with a lattice Λ it suffices to show that there exists a quadratic form φ such that $P = P_\varphi(\Lambda)$.

For any Dirichlet—Voronoi tiling of $\mathbb{R}^d$ into polytopes $P_\varphi(\Lambda)$ we can construct a dual Delone tiling. The Delone tiling $\mathcal{D}_\varphi(\Lambda)$ is defined by a lattice Λ and a positive definite quadratic form φ. For a given form φ consider the ellipsoid $\mathcal{E}_\varphi$ determined by the equation $\varphi(\mathbf{x}) = 1$. Consider an arbitrary homothetic copy $\mathcal{E}_0$ of $\mathcal{E}_\varphi$ such that $\mathcal{E}_0$ has some d-dimensional set of points from Λ on the boundary $\partial\mathcal{E}$ but does not have points from Λ inside $\mathcal{E}_0$. This ellipsoid $\mathcal{E}_0$ defines a convex polytope inscribed in $\mathcal{E}$ with vertices from Λ. The set of all such polytopes inscribed in «empty» ellipsoids we will call the *Delone tiling* $\mathcal{D}_\varphi(\Lambda)$.

For any vertex O of the Dirichlet—Voronoi tiling the values of φ on vectors connecting O with the centers of the polytopes from $\mathcal{V}_\varphi(\Lambda)$ that meet at O are equal, so O is the center of an empty ellipsoid that defines a polytope from the Delone tiling $\mathcal{D}_\varphi(\Lambda)$.

3. Zonotopes and dicings

Definition. *A polytope $P \subseteq \mathbb{R}^d$ is called a zonotope if P can be represented as the Minkowski sum of a finite number of segments; if they are described by a set of vectors $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then we denote the zonotope by $Z(V)$. The vectors $\mathbf{v}_i$ are called the zone vectors for $Z(V)$.*

Equivalently, zonotope can be defined as a projection of a cube C^n of some dimension $n \geqslant d$.

Erdahl in his work [2] proved the first conjecture of Voronoi for zonotopes. Later Deza and Grishukhin proved the first conjecture of Voronoi for zonotopes using oriented matroids [1]. In this paper we will formulate and use several notions and theorems concerning zonotopes and corresponding Delone tilings and Dirichlet—Voronoi tilings from Erdahl's paper [2].

Definition. *Consider n families of parallel hyperplanes in the d-dimensional Euclidean space that slice $\mathbb{R}^d$ into layers of constant width; the width can vary between families. This set of n families and the corresponding tiling of $\mathbb{R}^d$ is called dicing if there are d linearly independent normals among these families and if each point belonging to one hyperplane from each of d independent families also belongs to one hyperplane from all other families.*

Assume that $\mathbf{a}_0$ is a vertex of a dicing, and that $\mathbf{a}_0$ belongs to one hyperplane from each hyperplane family. Then the lengths of the corresponding normals $\mathbf{d}_1, \ldots, \mathbf{d}_n$ can be adjusted in such a way that the equations of the hyperplanes in each family take the form $\mathbf{d}_i \cdot (\mathbf{x} - \mathbf{a}_0) = a$ with $a \in \mathbb{Z}$. This representation of the hyperplane families is equally valid if we substitute $-\mathbf{d}_i$ for $\mathbf{d}_i$. Assuming that such an adjustment of normal lengths has been made, we give

Definition. *We denote the described dicing by $\mathfrak{D}(\mathbf{d}_1, \ldots, \mathbf{d}_n)$, and we call the set $\mathcal{D} = \{\pm\mathbf{d}_1, \ldots, \pm\mathbf{d}_n\}$ the normals for the dicing $\mathfrak{D}$.*

Consider an arbitrary vertex of a dicing $\mathfrak{D}$ (i. e. a point of intersection of hyperplanes from d families with independent normals) and consider all the edges of $\mathfrak{D}$ incident to this vertex. This set of edges is centrally symmetric.

Definition. *The set of edges $\mathcal{E} = \{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_k\}$ we just mentioned is called the edge set of a dicing $\mathfrak{D}$. It is clear that the construction of $\mathcal{E}$ does not depend on the choice of a vertex of the dicing.*

It also follows immediately from this construction that all the vertices of a dicing $\mathfrak{D}$ form a lattice $\Lambda(\mathfrak{D})$, that we will call the lattice of a dicing.

Particularly, Erdahl proved the following theorem on the connection between the sets $\mathcal{D}$ and $\mathcal{E}$ [2, Theorem 3.1].

Theorem 1 (R. Erdahl). *A given set $\mathcal{D}$ of n vectors with d linearly independent vectors can be a set of normals for a dicing if and only if there exists a set $\mathcal{E}$ such that:*

(E1) *Any pair of opposite vectors* $\pm\mathbf{e}_i \in \mathcal{E}$ *lies in some one-dimensional intersection* $\mathbf{d}_{i_1}^{\perp} \cap \ldots \cap \mathbf{d}_{i_{d-1}}^{\perp}$ *with independent vectors* $d_{i_j} \in \mathcal{D}$, *and conversely, for any* $d-1$ *linearly independent vectors from* $\mathcal{D}$ *there is a corresponding pair of opposite vectors in* $\mathcal{E}$;

(E2) *For any pair of vectors* $\mathbf{d} \in \mathcal{D}$ *and* $\mathbf{e} \in \mathcal{E}$ *the inner product* $\mathbf{d}^T\mathbf{e}$ *equals to* 0 *or* ± 1.

DEFINITION. *Matrices* $\mathbf{D}$ *and* $\mathbf{E}$ *with vectors from* $\mathcal{D}$ *and* $\mathcal{E}$ *written in their columns are called the matrix of normal vectors and the matrix of edge vectors of the dicing* $\mathfrak{D}$, *respectively.*

Under the affine transformation of $\mathbb{R}^d$ with matrix L matrices $\mathbf{D}$ and $\mathbf{E}$ turn into matrices $\mathbf{D}' = (L^{-1})^T\mathbf{D}$ and $\mathbf{E}' = L\mathbf{E}$. Moreover, there exists an affine transformation turning the entries of $\mathbf{D}$ and $\mathbf{E}$ into either 0, or ± 1 [2, Theorem 3.3].

THEOREM 2 (R. ERDAHL). *There exists an affine transformation giving a totally unimodular matrix* $\mathcal{D}'$, *i. e. the one with all its minors equal to* 0 *or* ± 1. *Moreover, this transformation can be chosen in such a way that both sets* $\mathbf{D}$ *and* $\mathbf{E}$ *contain* d *vectors of the standard basis of* $\mathbb{R}^d$, *i. e. the vectors* $(1, 0, \ldots, 0)^T, (0, 1, 0, \ldots, 0)^T, \ldots, (0, \ldots, 0, 1)^T$. *Besides that, this transformation turns all the entries of* $\mathbf{D}$ *and* $\mathbf{E}$ *into* 0 *or* ± 1.

We will take only a half of the vectors from the set $\mathcal{D}$, so that no opposite vectors are included. This new set $\mathcal{D}^+$ determines the same unique dicing $\mathfrak{D}$. Consider a quadratic form

$$\varphi(\mathbf{x}) = \sum_{\mathbf{d} \in \mathcal{D}^+} \omega_{\mathbf{d}} (\mathbf{d} \cdot \mathbf{x})^2$$

with some positive constants $\omega_{\mathbf{d}}$. The corresponding Dirichlet—Voronoi polytope is described in [2, Theorem 4.3].

THEOREM 3 (R. ERDAHL). *The Dirichlet—Voronoi polytope for the lattice* $\Lambda(\mathfrak{D})$ *with respect to the quadratic form* $\varphi(\mathbf{x})$ *is a zonotope with zone vectors*

$$\mathbf{z}_{\mathbf{d}} = \left(\sum_{\mathbf{d} \in \mathcal{D}^+} \omega_{\mathbf{d}} \mathbf{d}\mathbf{d}^T \right)^{-1} \omega_{\mathbf{d}} \mathbf{d}, \quad \mathbf{d} \in \mathcal{D}^+.$$

In the same paper Erdahl proved that if $\varphi(\mathbf{x})$ is the standard Euclidean metric, then the zone vectors of the corresponding zonotope are $\mathbf{z}_{\mathbf{d}} = \omega_{\mathbf{d}}\mathbf{d}$ [2, Sect. 6, p. 442].

Besides that, [2, Theorem 1.2] claims that the Dirichlet—Voronoi polytope of a lattice is a zonotope if and only if the corresponding Delone tiling is a dicing.

4. The second conjecture of Voronoi for zonotopes

LEMMA 1. *Let $Z = Z(V)$ be a d-dimensional zonotope. Any facet of Z is generated by some $(d-1)$-dimensional subset U of V, and conversely, any $(d-1)$-dimensional subset of V generates a facet $Z(U)$ of the zonotope $Z(V)$.*

PROOF. Let π be a hyperplane of some facet of Z. The zonotope $Z(V) = Z(\mathbf{v}_1, \dots, \mathbf{v}_n)$ is a projection of a cube $C^n \subset \mathbb{R}^n$ into $\mathbb{R}^d$ along an $(n-d)$-dimensional subspace ψ. Consider the hyperplane $\pi \times \psi$ in $\mathbb{R}^n$. This hyperplane is a supporting plane for the cube C^n and hence it determines a face F of C^n. The face F is a cube of some dimension and this face is generated by the edges of C^n that are projected onto the vectors of V parallel to π. Hence F is projected onto a parallel copy of the zonotope $Z(U)$ for some $(d-1)$-dimensional subset U of V. The converse can be proven in an analogous way. ☐

THEOREM 4. *The conjecture 1 is true for space filling zonotopes.*

PROOF. Assume that Z is a d-dimensional space filling zonotope, i. e. a zonotope which is also a parallelohedron. Then the first conjecture of Voronoi is true for Z [2, Theorem 1.1], i. e. there exists an affine transformation $\mathcal{A}$ such that the zonotope $\mathcal{A}Z$ is a Dirichlet—Voronoi polytope of some lattice Λ with respect to the usual Euclidean metric as the quadratic form. The Delone tiling for Λ with respect to the Euclidean metric is a dicing $\mathfrak{D} = \mathcal{D}(\pm\mathbf{d}_1, \dots, \pm\mathbf{d}_n)$, so by Theorem 3 the zone vectors of the zonotope $\mathcal{A}Z$ can be written as $\omega_i\mathbf{d}_i$.

Any vector that connects the centers O_1 and O_2 of copies of $\mathcal{A}Z$ with a joint facet F is perpendicular to this facet because any point of F is equidistant from O_1 and O_2. By Lemma 1 there are $d-1$ vectors in the set $\{\omega_i\mathbf{d}_i\}_{i=1}^n$ that are parallel to F, so by Theorem 1 there is a vector in $\mathcal{E}$ perpendicular to F. This means that $\overrightarrow{O_1O_2}$ lies in $\mathcal{E}$. The converse is also true: if we take any vector $\mathbf{x}$ from $\mathcal{E}$, then $d-1$ linearly independent vectors from $\mathcal{D}$ that are perpendicular to $\mathbf{x}$ will determine a facet of Z. Therefore, $\mathcal{E}$ and $\mathcal{F}(Z)$ coincide.

Consider a unimodular representation of the dicing $\mathfrak{D}$. In this representation the set $\mathcal{E}$ contains the standard basis of $\mathbb{R}^d$, and all the other vectors of $\mathcal{E}$ have integer coordinates. Hence in a unimodular representation the set $\mathcal{F}(Z)$ contains the standard basis of $\mathbb{R}^d$ and all the other vectors of this set have integer coordinates in this basis. So, in a unimodular representation the lattice $\Lambda(Z)$ coincides with $\mathbb{Z}^d$ and $\mathcal{F}(Z)$ contains its basis. The latter property is invariant under linear transformations and under changing the basis in $\mathbb{R}^d$, which proves the theorem. □

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