Reprint from

ISSN 2220-5438

Moscow Journal of Combinatorics and Number Theory

URSS

МФТИ

Volume 7 • Issue 4

2017

Moscow Journal of Combinatorics and Number Theory Volume 7 • Issue 4 2017

URSS

Moscow Journal
of Combinatorics
and Number Theory
2017, vol. 7, iss. 4,
pp. 39–57, [pp. 307–325]

On-line and list on-line colorings of graphs and hypergraphs

Alina Khuzieva, Dmitry Shabanov, Polina Svyatokum (Moscow)

Abstract: The paper deals with on-line and list on-line colorings of graphs and hypergraphs. On-line coloring of a hypergraph is a game with two players, Lister and Painter, in which Lister picks a vertex one by one (or a set of vertices) and Painter should choose a color for the given vertex (or choose a subset to be colored). The problem is to find an extremal value of some characteristics which admits a winning strategy for Painter. We establish the asymptotic behavior of the list on-line chromatic number for the complete multipartite graphs and hypergraphs. We also prove some results for the related property B-type problems for on-line colorings.

Keywords: list colorings, on-line colorings, independent sets, property B problem

AMS Subject Classification: 05C15, 05C65

Received: 13.05.2017; **revised:** 19.09.2017

1. Introduction

The work is devoted to the on-line colorings of graphs and hypergraphs. Let us start with recalling some definitions.

1.1. Definitions

Let $H = (V, E)$ be a hypergraph (or a graph). A vertex subset $W \subset V$ is called *independent* in H if it does not contain completely any edge of H, i.e. for every $A \in E$, $A \setminus W \neq \varnothing$.

A *vertex coloring* f is a mapping from the vertex set V to some set of colors C. A coloring is called *proper* for H if there is no monochromatic edges in E under it. *The chromatic number* of H, $\chi(H)$, is the minimum r such that there is a proper coloring for H with r colors (H is r-colorable).

Another classical notion concerning colorings of graphs and hypergraphs is the list chromatic number. A hypergraph $H = (V, E)$ is said to be r-*choosable* (or list r-colorable) if for every *list assignment* $L = \{L(v) : \ v \in V\}$ such that $|L(v)| = r$ for any $v \in V$ (r-*uniform* list assignment), there exists a proper coloring from the lists, i.e. for every $v \in V$, we should use a color from $L(v)$. The *list chromatic number* of H, denoted by $\chi_l(H)$, is the minimum r such that H is r-choosable.

The concept of the list chromatic number was recently brought to the on-line setting, see [1]–[3]. Suppose $H = (V, E)$ is a hypergraph and $r \geqslant 2$ is an integer. Two players, Lister and Painter, play the following $Game_1(H, r)$ game. Let us set $X_0 = \varnothing$. In the round number i Lister presents a non-empty set of vertices $V_i \subset V \setminus (X_0 \cup \ldots \cup X_{i-1})$ and Painter chooses an independent subset $X_i \subset V_i$, i.e. the vertices of X_i are colored with color number i. After i rounds the vertices in $X_1 \cup \ldots \cup X_i$ are colored. If a vertex v belongs to exactly l sets $V_{j_1}, \ldots, V_{j_l}$, $1 \leqslant j_1 < \ldots < j_l \leqslant i$ then v is said to have l *permissible* colors after i rounds. The winning rule is the following.

- Lister wins if after some round there exists a non-colored vertex with r permissible colors.
- Otherwise Painter wins, i.e. after some round all the vertices are colored.

Hypergraph H is said to be r-*paintable* (or list on-line r-colorable) if Painter has a winning strategy in (H, r)-game. The minimum r such that H is r-*paintable* is called *the list on-line chromatic number* and denoted by $\chi_{ol}(H)$. It is easy to see that

$$\chi(H) \leqslant \chi_l(H) \leqslant \chi_{ol}(H).$$

1.2. Colorings of the complete multipartite graphs and hypergraphs

List colorings of graphs and hypergraphs were introduced independently by Vizing (see [4]) and by Erdős, Rubin and Taylor (see [5]). One of the first results concerning the list chromatic number states that it can be much larger than the usual chromatic number. In particular, the authors in [5] showed that the list chromatic number of the complete bipartite graph $K_{m,m}$ with m vertices in any part grows as binary logarithm of m:

$$\chi_l(K_{m,m}) = (1+o(1))\log_2 m \text{ as } m \to \infty. \tag{1}$$

Surprisingly the above asymptotic representation remains true for the list on-line chromatic number. In [6] Duraj, Gebowski and Kozik showed that

$$\chi_{ol}(K_{m,m}) = \log_2 m + O(1) \text{ as } m \to \infty. \tag{2}$$

This provides the first example of a graph for which the difference between the list on-line chromatic number and the list chromatic number can be arbitrarily large. Since $\chi_l(K_{m,m}) = \log_2 m - \Omega(\log_2 \log_2 m)$ (see [6]) we have

$$\chi_{ol}(K_{m,m}) - \chi_l(K_{m,m}) = \Omega(\log_2 \log_2 m).$$

The result (1) for $K_{m,m}$ was generalized in different ways. The first generalization considers the complete r-partite graph K_{m*r} with equal size of parts m. Krivelevich and Gazit established (see [7]) the asymptotic behavior of $\chi_l(K_{m*r})$ for fixed $r \geqslant 3$ and growing m:

$$\chi_l(K_{m*r}) = (1+o(1))\log_{\frac{r}{r-1}} m \text{ as } m \to \infty. \tag{3}$$

In [9] Shabanov showed that the same asymptotic representation holds when $\ln r = o(\ln m)$.

The second generalization deals with the complete multi-partite uniform hypergraphs. Let $H_{m\times r}$ denote the complete r-partite r-uniform hypergraph with m vertices in every part. In [10] Haxell and Verstraëte proved that for fixed $r \geqslant 3$,

$$\chi_l(H_{m\times r}) = (1+o(1))\log_r m \text{ as } m \to \infty. \tag{4}$$

Recently the results (1), (3), (4) were extended by Shabanov and Shaikheeva (see [8]). Let $H(m,r,k)$ denote the complete r-partite k-uniform hypergraph with m vertices in every part, in which any edge takes exactly one vertex from some $k \leqslant r$ parts. Clearly, $H(m,r,2) = K_{m*r}$ and $H(m,r,r) = H_{m\times r}$. The authors in [8] showed that for fixed $2 \leqslant k \leqslant r$,

$$\chi_l(H(m,r,k)) = (1+o(1))\log_{\frac{r}{r-k+1}} m \text{ as } m \to \infty. \tag{5}$$

1.3. Main result

The main result of the current work provides the asymptotic behavior for the list on-line chromatic number of the complete r-partite k-uniform hypergraph $H(m, r, k)$. As in (2) the asymptotics of $\chi_{ol}(H(m, r, k))$ coincides with the asymptotics of the list chromatic number (5).

THEOREM 1. *For fixed* $2 \leqslant k \leqslant r$,

$$\chi_{ol}(H(m, r, k)) = (1 + o(1)) \log_{\frac{r}{r-k+1}} m \text{ as } m \to \infty. \tag{6}$$

The same asymptotic representation holds for any functions $r = r(m)$, $k = k(m)$, *such that* $\ln r = o(\ln m)$.

As immediate corollaries we obtain the analogues of (2) for (3) and (4): for fixed $r \geqslant 3$,

$$\chi_{ol}(K_{m*r}) = (1 + o(1)) \log_{\frac{r}{r-1}} m \text{ as } m \to \infty;$$

$$\chi_{ol}(H_{m\times r}) = (1 + o(1)) \log_r m \text{ as } m \to \infty.$$

The structure of the paper will be the following. In the next section we will discuss the connection of the list on-line colorings of multipartite hypergraphs with extremal property B-type problems. In Section 3 we will give the proofs of the obtained results.

2. Extremal property B-type problems

2.1. Connection with the property B problem

The close connection of the list colorings of complete multi-partite graphs with the classical property B problem was realized by Erdős, Rubin and Taylor in [5]. Recall that the property B problem is to find the value $m(n)$ equal to the minimum number of edges in an n-uniform non-2-colorable hypergraph. The obtained quantitative relation between $\chi_l(K_{m,m})$ and $m(n)$ is the following.

CLAIM 1. Suppose that $n, m \geqslant 2$ are integers.

1. If $2m < m(n)$ then $\chi_l(K_{m,m}) \leqslant n$.
2. If $m \geqslant m(n)$ then $\chi_l(K_{m,m}) > n$.

These inequalities together with the known bounds for $m(n)$ provide the asymptotics for $\chi_l(K_{m,m})$.

The same approach was used in [9] and [10] for investigating $\chi_l(K_{m*r})$ and $\chi_l(H_{m\times r})$. For K_{m*r}, the corresponding extremal value deals with panchromatic colorings. A vertex coloring of the hypergraph $H = (V, E)$ with r colors is said to be *panchromatic* if under this coloring every edge of E meets every of r colors. Let $p(n, r)$ denote the minimum possible number of edges in a n-uniform hypergraph that does not admit a panchromatic coloring with r colors. Kostochka showed [11] that $p(n, r)$ plays the same role for $\chi_l(K_{m*r})$ as $m(n)$ for $\chi_l(K_{m,m})$.

Haxell and Verstraëte considered another generalization of the property B problem to obtain the asymptotics for $\chi_l(H_{m\times r})$. They used the value $m(n, r)$, the minimum possible number of edges in an n-uniform non-r-colorable hypergraph.

Finally, Shabanov and Shaikheeva [8] introduced the property that lies "between" r-colorability and panchromatic r-colorability. Let us denote $[r] = \{1, \ldots, r\}$. A mapping $f : V \to \binom{[r]}{s}$ is called an *s-covering by r sets*, i.e. we assign s different colors to any vertex of H. Furthermore f is called an *s-covering by r independent sets* if for every $i = 1, \ldots, r$, a vertex subset

$$V_i = \{v \in V : i \in f(v)\}$$

is an independent set in H. It is easy to understand that

- a 1-covering by r independent sets is just a proper coloring with r colors;
- an $(r-1)$-covering f by r independent sets is equivalent to a panchromatic r-coloring (we can color a vertex with the remaining unassigned color).

The authors of [8] introduced the value $c(n, r, s)$, equal to the minimum possible number of edges in an n-uniform hypergraph that does not admit an s-covering by r independent sets. They also proved the following quantitative relation between $c(n, r, s)$ and $\chi_l(H(m, r, k))$.

Claim 2. Suppose that $n, m, r \geqslant 2$, $2 \leqslant k \leqslant r$ are integers.

1. If $rm < c(n, r, r-k+1)$ then $\chi_l(H(m, r, k)) \leqslant n$.
2. If $m \geqslant c(n, r, r-k+1)$ then $\chi_l(H(m, r, k)) > n$.

By using Claim 2 and the bounds for $c(n, r, s)$, one can easily obtain the asymptotics for the list chromatic number of $H(m, r, k)$.

2.2. On-line analogues of the property B problem

The first on-line version of the property B problem was considered by Aslam and Dhagat in [12]. Suppose n, N are positive integers and there are two players, Lister and Painter. They play the following game $Game_2(N, n)$, which is parametrized by two numbers: the cardinality of edges n and the number of edges N. Values of these parameters are known to both players before the game. In each round, Lister reveals one vertex and declares in which edges it is contained. He cannot add vertices to edges which already contain n vertices. Painter must immediately assign any of two colors (0 or 1) to the presented vertex. When all the vertices have been revealed (i.e. all N edges contain n vertices each) Painter wins if there is no monochromatic edge in the constructed hypergraph. Otherwise Lister wins.

Let $m_{ol}(n)$ denote the minimum N such that Lister has a winning strategy in $Game_2(N, n)$. Clearly, $m_{ol}(n) \leqslant m(n)$ since for $N \geqslant m(n)$ Lister can just construct a non-2-colorable hypergraph. Aslam and Dhagat proved [12] that

$$m_{ol}(n) \geqslant 2^{n-1}. \tag{7}$$

Duraj, Gutowski and Kozik showed [6] that the above estimate is sharp up to a bounded factor:

$$m_{ol}(n) \leqslant 8 \cdot 2^n. \tag{8}$$

They also showed that $m_{ol}(n)$ plays the same role for $\chi_{ol}(K_{m,m})$ as $m(n)$ for $\chi_l(K_{m,m})$. This connection together with the bounds (7)-(8) implies the result (2).

In the current paper we consider the following extension of $Game_2(N, n)$. Suppose n, $s \leqslant r$ and N are positive integers. There are two players, Lister and Painter, who play the following game $Game_3(N, n, r, s)$, which is parametrized by four numbers:

- n is the cardinality of edges;
- N is the number of edges;
- r is the total number of colors;
- s is the number of colors that should be assigned to every vertex.

Again the values of these parameters are known to both players before the game. In each round, Lister reveals one vertex of a hypergraph and declares in which edges it is contained. He cannot add vertices to edges which already contain n vertices. Painter must immediately assign s colors from $[r] = \{1, \ldots, r\}$ to the presented vertex. When all the vertices have been revealed (i.e. all N edges contain exactly n vertices each) Painter wins if the obtained s-covering is a covering by r independent sets for the constructed n-uniform hypergraph. Otherwise Lister wins.

Let $c_{ol}(n, r, s)$ denote the minimum N such that Lister has a winning strategy in $Game_3$ (N, n, r, s). We obtain the following generalization of Claim 2.

LEMMA 1. *Suppose that $n, m, r \geqslant 2$, $2 \leqslant k \leqslant r$ are integers.*

1. *If $rm < c_{ol}(n, r, r-k+1)$ then $\chi_{ol}(H(m, r, k)) \leqslant n$.*
2. *If $m \geqslant c_{ol}(n, r, r-k+1)$ then $\chi_{ol}(H(m, r, k)) > n$.*

Lemma 1 is crucial in estimating the list on-line chromatic number of $H(m, r, k)$. However we will also need the bounds for the extremal value $c_{ol}(n, r, s)$, this question will be discussed in the next paragraph.

2.3. New results in extremal problems for on-line colorings

The following lemma gives a reasonable lower bound for $c_{ol}(n, r, s)$.

LEMMA 2. *For any $n \geqslant 2$, $r > s \geqslant 1$,*

$$c_{ol}(n, r, s) \geqslant \frac{r^{n-1}}{s^n}. \tag{9}$$

Note that for $r = 2$, $s = 1$ the bound (9) coincides with the bound (7) for $m_{ol}(n)$.

Recall that $c_{ol}(n, r, s)$ does not exceed its "off-line" version $c(n, r, s)$. It was shown in [8] by a probabilistic approach that for any $n > r > s \geqslant 1$,

$$c(n, r, s) \leqslant \frac{e}{2}\, n^2 \left(\frac{r}{s}\right)^n \ln \binom{r}{s} \left(1 + O\left(\frac{1}{n}\right) + O\left(\frac{s}{r}\right)\right). \tag{10}$$

So we can use the estimate (10) as an upper bound for $c_{ol}(n, r, s)$. However, as it was shown by Duraj, Gutowski and Kozik, much better results can be obtained for on-line colorings. We will give some of them in the most interesting cases: $s = 1$ and $s = r - 1$.

The value $c(n, r, 1)$ is well-known in the literature as $m(n, r)$, the minimum possible number of edges in an n-uniform non-r-colorable hypergraph. The problem of finding $m(n, r)$ was proposed by Erdős and Hajnal in the 60-s and since that time it had been intensively studied. The reader is referred to the survey [13] for the detailed history. Clearly, $m(n,2) = m(n)$ and it is known that

$$c\left(\frac{n}{\ln n}\right)^{1/2} 2^n \leqslant m(n) \leqslant \frac{e\ln 2}{4} n^2 2^n (1 + o(1)), \tag{11}$$

where $c > 0$ is an absolute constant. The upper bound is due to Erdős [14] and the lower is due to Radhakrishnan and Srinivasan [15]. Note that the above relations imply that $m(n)$ has a greater asymptotic order than its on-line analogue $m_{ol}(n)$ (see (7) and (8)). Similar estimates in the case of arbitrary number of colors r are the following (the lower bound is due to Cherkashin and Kozik [16]):

$$c\left(\frac{n}{\ln n}\right)^{(r-1)/r} r^{n-1} \leqslant m(n,r) \leqslant \frac{e}{2} n^2 r^n \ln r \left(1 + O\left(\frac{1}{n}\right)\right). \tag{12}$$

Since $m_{ol}(n,r) = c_{ol}(n,r,1)$ does not exceed $m(n,r)$ the upper bound from (12) holds for $m_{ol}(n,r)$. The following statement refines it significantly.

Proposition 1. *For any r and n,*

$$m_{ol}(n,r) \leqslant n(r-1)^2 \cdot r^n. \tag{13}$$

For fixed r and growing n, the bound (13) is much better than the bound (12) for $m(n,r)$. If $r \sim \ln n$ then (13) is only $(\ln n)^4$ times greater than the lower bound in (12), so we can expect that $m(n,r)$ and $m_{ol}(n,r)$ do not have the same asymptotic order. However, for $r = 2$, the bound (13) is not good, a much stronger result (8) is known.

In the opposite situation when n is fixed and r is large, the bound (13) also is not the best possible since it is known that even $m(n,r)$ has the order $O_n(r^n)$. In fact, Alon [17] showed that $m(n,r)$ has the order r^n for large r and small n. His bounds were refined by Akolzin and Shabanov [18] as follows: if $r > n$ then

$$c_1 \cdot \frac{n}{\ln n} \cdot r^n \leqslant m(n,r) \leqslant c_2 \cdot n^3 \ln n \cdot r^n, \tag{14}$$

where c_1 and c_2 are some positive absolute constants. We show that the value $m_{ol}(n,r)$ also have the order r^n when r is large and n is fixed. The upper bound clearly follows from (14), but the lower bound r^{n-1} obtained in Lemma 2 is not enough. The next statement provides an improved bound.

Proposition 2. *Suppose $r > n$ and let us denote $a = \lfloor \frac{n-1}{n} r \rfloor$ and $b = r - a = \lceil \frac{r}{n} \rceil$. Then*

$$m_{ol}(n, r) \geqslant (n-1)ba^{n-1} + a^{n-1} = \Omega(r^n). \tag{15}$$

Finally, we discuss on-line panchromatic colorings, i.e. the problem of estimating $c_{ol}(n, r, s)$ when $s = r - 1$. Let $p_{ol}(n, r) = c_{ol}(n, r, r-1)$. "Off-line" version of the problem, the value $p(n, r)$, first appeared it the paper of Kostochka [11] and since that time has been studied in several papers. For instance, it was shown in [9] and [19] that

$$c_1 \frac{1}{r} \left(\frac{n}{r^2 \ln n} \right)^{1/2} \left(\frac{r}{r-1} \right)^n \leqslant p(n, r) \leqslant c_2 n^2 \left(\frac{r}{r-1} \right)^n \ln r, \tag{16}$$

where $c_1, c_2 > 0$ are some absolute constants. Cherkashin improved [20] the upper bound in (16) by a factor $1/r$ and gave a better lower bound for r large enough in comparison with n.

The lower bound $r^{-1} \left(\frac{r}{r-1} \right)^n$ for $p_{ol}(n, r)$ has been obtained in Lemma 2. The upper bound from (16) can be refined in the on-line case as follows.

Proposition 3. *Suppose $n > r$. Then*

$$p_{ol}(n, r) \leqslant 3r(r-1)^2 n \left(\frac{r}{r-1} \right)^{n+1}. \tag{17}$$

For fixed r and large n, the bound (17) is even closer to the lower bound in (16) than to the upper one.

In the next sections we proceed to the proofs of the above new results.

3. Proof of Theorem 1

We start with establishing auxiliary lemmas: Lemma 1 and Lemma 2.

3.1. Proof of Lemma 1

We follow the ideas from [6] and [8].

1) We have to show that $\chi_{ol}(H(m, r, k)) \leqslant n$, i.e. we have to prove that Painter has a winning strategy in $Game_1(H(m, r, k), n)$.

Let $W = W_1 \sqcup \ldots \sqcup W_r$ denote the vertex set of $H(m, r, k)$, where $W_1, \ldots, W_r$ are the parts of the graph. Our strategy for $Game_1(H(m, r, k), n)$ will use the winning strategy for $Game_3(rm, n, r, r - k + 1)$.

- Suppose $X_1, \ldots, X_{i-1}$ have been already chosen. In round i Lister chooses the set of vertices $V_i \subset W \setminus (X_1 \sqcup \ldots \sqcup X_{i-1})$.
- We assume that we are playing $Game_3(rm, n, r, r-k+1)$ and here Lister has chosen the edges with numbers V_i to contain vertex i.
- Since $rm < c_{ol}(n, r, r - k + 1)$ then Painter has a winning strategy in $Game_3(rm, n, r, r - k + 1)$. Let $\{\widetilde{j}_1, \ldots, \widetilde{j}_{r-k+1}\}$ be the choice of colors for the vertex i according to this strategy.
- Let $\{j_1, \ldots, j_{k-1}\} = [r]/\{\widetilde{j}_1, \ldots, \widetilde{j}_{r-k+1}\}$ be a complementary set of colors.
- Painter's choice of an independent set X_i will be the following:

$$X_i = V_i \cap \left(W_{j_1} \sqcup \ldots \sqcup W_{j_{k-1}}\right).$$

Since X_i is contained in a union of some $k-1$ parts of $H(m, r, k)$ then it will be independent in $H(m, r, k)$ by the construction of the hypergraph.

Suppose $w \in W_j$ is a vertex of $H(m, r, k)$. Everytime w is chosen by Lister as an element of V_i in $Game_1(H(m, r, k), n)$, i becomes a vertex of an edge w in $Game_3(rm, n, r, r - k + 1)$. The winning strategy in $Game_3(rm, n, r, r - k + 1)$ provides that after choosing n times the edge w there will be a vertex $i \in w$ such that color j will not be assigned to i (otherwise the obtained covering will not be a covering by independent sets). For such i, the independent set X_i will contain all the vertices in $V_i \cap W_j$, i.e. $w \in X_i$. Thus, every vertex of $H(m, r, k)$ is colored before it receives n permissable colors. The existence of the winning strategy for Painter is proved.

2) We have to show that $\chi_{ol}(H(m, r, k)) > n$, i.e. Lister has a winning strategy in $Game_1$ $(H(m, r, k), n)$. Again our strategy will follow the winning strategy for $Game_3(m, n, r, r - k + 1)$.

Recall that $W = W_1 \sqcup \ldots \sqcup W_r$ denotes the vertex set of $H(m, r, k)$. Every W_j has exactly m vertices, so let us denote $W_j = \{w_{1,j}, \ldots, w_{m,j}\}$.

- Suppose $V_1, X_1, \ldots, V_{i-1}, X_{i-1}$ have been already chosen. In round i we have to choose the set of vertices $V_i \subset W \setminus (X_1 \sqcup \ldots \sqcup X_{i-1})$.
- Once again we assume that we are playing $Game_3(m, n, r, r-k+1)$ and there is a winning strategy for Lister since $m \geqslant c_{ol}(n, r, r - k + 1)$.

- Now let $a_1, \dots, a_q \in \{1, \dots, m\}$ denote the set of edges which this strategy assigns to vertex i.
- Lister's choice of a set V_i is the following:

$$V_i = \bigcup_{j=1}^{r} \bigcup_{y=1}^{q} \{w_{a_y,j}\} \setminus (X_1 \sqcup \ldots \sqcup X_{i-1}) .$$

Roughly speaking, Lister chooses a set of rows in matrix $\|w_{l,j}\|$, $l = 1, \dots, m$, $j = 1, \dots, r$, and forms V_i as a set of all available elements in chosen rows.

Suppose Painter chooses X_i as an independent set in V_i. In fact, Painter chooses the vertices from some $k-1$ parts $W_{j_1}, \dots, W_{j_{k-1}}$, i.e. he chooses $k-1$ columns in matrix $\|w_{l,j}\|$ and forms X_i as an intersection of V_i with these columns. Such an answer can be interpreted as Painter's choice of colors $[r] \setminus \{j_1, \dots, j_{k-1}\}$ for covering vertex i in $Game_3(m, n, r, r-k+1)$.

Let us understand that Lister always wins by this strategy in $Game_1(H(m, r, k), n)$. The winning strategy in $Game_3(m, n, r, r-k+1)$ provides that after some round there will be a color j which will be assigned to any of n vertices of some edge $q \in \{1, \dots, m\}$. This corresponds to the following situation in $Game_1(H(m, r, k), n)$:

1. column j has never been chosen as a part of an independent X_i, when Lister chooses row q,
2. vertex $w_{q,j}$ has not been colored,
3. vertex $w_{q,j}$ has been chosen n times as an element of V_i, i.e. it has n permissable colors.

Hence Lister always wins by using the described strategy. Lemma 1 is proved.

3.2. Proof of Lemma 2

The proof follows the ideas from [12]. We have to prove that for $N < \frac{r^{n-1}}{s^n}$, Painter has a winning strategy in $Game_3(N, n, r, s)$. Let us describe it.

Suppose that the first l vertices $v_1, \dots, v_l$ have already been colored with s colors each. For every color $j \in \{1, \dots, r\}$, let $V_j(l)$ denote the set of revealed vertices colored with j. Every edge A can be considered as a function of l, where $A(l)$ denote an edge subset revealed after round l. If $A(l) \subset V_j(l)$ then A is said to be currently monochromatic of color j. We assume that an empty edge is monochromatic of every color. In this case we define the weight of A in color j as follows:

$$w_j(A,l) = \left(\frac{r}{s}\right)^{|A(l)|}.$$

Painter's strategy will be the following. Suppose that Lister states that a vertex v_{l+1} is assigned to edges $A_1, \dots, A_q$. Then Painter calculates r numbers $b_j(v_{l+1})$, $j = 1, \dots, r$, where

$$b_j(v_{l+1}) = \sum_{u: A_u(l) \subset V_j(l)} w_j(A_u, l) = \sum_{A: A(l) \subset V_j(l), v_{l+1} \in A} w_j(A, l).$$

Suppose that $b_{j_1}(v_{l+1}), \dots, b_{j_s}(v_{l+1})$ are the smallest s numbers among them. Then Painter assigns colors $j_1, \dots, j_s$ to vertex v_{l+1}.

Let us prove that this is a winning strategy. After every round l we can define the total weight of currently monochromatic edges:

$$w(l) = \sum_{j=1}^{r} \sum_{A: A(l) \subset V_j(l)} w_j(A, l).$$

We will show that $w(l) \geqslant w(l+1)$, i.e. the total weight decreases. Indeed, let $j_1, \dots, j_s$ be the colors assigned to vertex v_{l+1} in round $l+1$. Then our strategy implies that

$$w(l+1) = w(l) - \sum_{j=1}^{r} \sum_{A: A(l) \subset V_j(l), v_{l+1} \in A} w_j(A, l) + \sum_{u=1}^{s} \sum_{A: A(l) \subset V_{j_u}(l), v_{l+1} \in A} w_{j_u}(A, l+1) =$$

$$= w(l) - \sum_{j=1}^{r} b_j(v_{l+1}) + \frac{r}{s} \sum_{u=1}^{s} b_{j_u}(v_{l+1}) \leqslant w(l)$$

since $b_{j_1}(v_{l+1}), \dots, b_{j_s}(v_{l+1})$ are the smallest s numbers among $b_j(v_{l+1})$, $j = 1, \dots, r$.

Let us finish the proof. Suppose our strategy fails and at the end of the game there is an edge A, which is monochromatic in color j. Then the total weight of the monochromatic edges at the end of the game is at least $(r/s)^n$. But at the beginning the total weight is equal to rN which is smaller than $(r/s)^n$, a contradiction. Lemma 2 is proved.

3.3. Completion of the proof

Let us deduce the asymptotics for the list on-line chromatic number of the hypergraph $H(m, r, k)$. If we denote $n = \chi_l(H(m, r, k))$ then Lemma 1 implies that

$$c_{ol}(n-1, r, r-k+1) \leqslant m \text{ and } c_{ol}(n, r, r-k+1) > mr.$$

By using bounds (9) and (10) for $c_{ol}(n, r, r-k+1)$ we obtain that

$$(n-2)\ln\frac{r}{r-k+1} - \ln(r-k+1) \leqslant \ln m; \tag{18}$$

$$\ln m + \ln r < n\ln\frac{r}{r-k+1} + 2\ln n + \ln\ln\binom{r}{r-k+1} + O(1). \tag{19}$$

We assume that the function $r = r(m)$ satisfies the condition $\ln r = o(\ln m)$ when $m \to \infty$. Hence the inequality (18) implies that

$$\limsup_{m\to\infty}\frac{n\ln\frac{r}{r-k+1}}{\ln m} \leqslant 1 + \lim_{m\to\infty}\frac{2\ln r}{\ln m} = 1. \tag{20}$$

Moreover, it follows from (18) that $\ln n = O(\ln\ln m) = o(\ln m)$. Thus from (19) we get

$$\liminf_{m\to\infty}\frac{n\ln\frac{r}{r-k+1}}{\ln m} \geqslant 1 - \lim_{m\to\infty}\frac{O(\ln r + \ln n)}{\ln m} = 1. \tag{21}$$

Finally, from (20) and (21) we obtain the asymptotics for the list on-line chromatic number of $H(m, r, k)$:

$$\lim_{m\to\infty}\frac{\chi_{ol}(H(m,r,k))\ln\frac{r}{r-k+1}}{\ln m} = \lim_{m\to\infty}\frac{\chi_{ol}(H(m,r,k))}{\log_{\frac{r}{r-k+1}} m} = 1.$$

Theorem 1 is established.

4. Other proofs

4.1. Proof of Proposition 1

The proof follows the ideas from [6]. We have to show that for $N = n(r-1)^2 r^n$, Lister has a winning strategy in $Game_3(N, n, r, 1)$. The strategy will be the following.

Suppose that the first l vertices $v_1, \dots, v_l$ have already been colored. For every color $j \in \{1, \dots, r\}$, let $V_j(l)$ denote the set of revealed vertices colored with j. We divide the number of edges into r parts $E_1, \dots, E_r$, with $(r-1)^2 nr^{n-1}$ edges in every part. Every edge A again can be considered as a function of l, where $A(l)$ denote an edge subset revealed after round l. If $A(l) \subset V_j(l)$ then A is said to be currently monochromatic of color j. We assume that an empty edge $A(0)$ is monochromatic of color j if $A \in E_j$. If $|A(l)| = i$, $i = 0, \dots, n-1$, then edge A is said to be at level i after the round l. A monochromatic (j, i)-block is a set of r^{n-i-1} currently monochromatic edges of color j, which are currently at level i.

Lister's strategy can be described as follows.

- For every color $j = 1, \dots, r$, he chooses the largest $i = i(j)$ such that there exists a (j, i)-block B_j.
- He chooses the union $B_1 \sqcup \dots \sqcup B_r$ as a set of edges that will contain the next vertex v_{l+1}.

Clearly, Lister wins if after some round there is a monochromatic edge at level n. The total number of blocks at the beginning is equal to $r \cdot (r-1)^2 nr^{n-1}/r^{n-1} = r(r-1)^2 n$. For any Painter's choice of color for v_{l+1}, the total number of blocks remains the same. Indeed, for chosen color the number of blocks of this color will increase by $r-1$ (plus r blocks on the next level minus 1 chosen block on the current level) but the number of blocks of any other color will decrease by 1.

The game continues until there is no monochromatic blocks in some color (after that Painter can always choose this color for all the remaining vertices) or Lister wins. Suppose the first situation appears. It implies that after some round there is no monochromatic blocks, say, of color 1. In every other color there can be

1. at most $r-1$ monochromatic blocks on any level from 1 to $n-2$ (we always use the block on the largest level);
2. at most r monochromatic blocks on level $n-1$;
3. at most $(r-1)^2 n - 1$ blocks on level 0.

Thus the total number of blocks will be at most

$$\begin{aligned}(r-1)\left((r-1)(n-2) + r + (r-1)^2 n - 1\right) &= (r-1)^2\left(n - 2 + 1 + (r-1)n\right) \\ &= (r-1)^2 (rn-1)\end{aligned}$$

which is less than $r(r-1)^2 n$, a contradiction. We have shown that the number of blocks should be constant. Thus Lister always wins.

4.2. Proof of Proposition 2

We follow the proof of Alon from [17]. Suppose $N < (n-1)ba^{n-1} + a^{n-1}$, we have to show that Painter has a winning strategy in $Game_3(N, n, r, 1)$. The first part of the strategy will be the same as in Lemma 2.

Suppose that the first l vertices $v_1, \dots, v_l$ have already been colored. For every color $j \in \{1, \dots, r\}$, let $V_j(l)$ denote the set of revealed vertices colored with j. Every edge A can be considered as a function of l, where $A(l)$ denote an edge subset revealed after round l. If $A(l) \subset V_j(l)$ then A is said to be currently monochromatic of color j. We assume that an empty edge is monochromatic in every color. In this case we define the weight of A in color j as follows:

$$w_j(A, l) = a^{|A(l)|}.$$

Painter's strategy will be the following. Suppose that Lister states that a vertex v_{l+1} is assigned to edges $A_1, \dots, A_q$. Then Painter calculates a numbers $d_j(v_{l+1})$, $j = 1, \dots, a$, where

$$d_j(v_{l+1}) = \sum_{u: A_u(l) \subset V_j(l)} w_j(A_u, l) = \sum_{A: A(l) \subset V_j(l), v_{l+1} \in A} w_j(A, l).$$

Suppose that $d_q(v_{l+1})$ is the smallest number among $d_1(v_{l+1}), \dots, d_a(v_{l+1})$.

1. If $d_q(v_{l+1}) < a^{n-1}$ then Painter colors v_{l+1} with color q.
2. If $d_q(v_{l+1}) \geqslant a^{n-1}$ then Painter colors v_{l+1} with any of the colors from $\{a+1, \dots, r\}$ which have not been used $(n-1)$ times.
3. If $d_q(v_{l+1}) \geqslant a^{n-1}$ and every color from $\{a+1, \dots, r\}$ have been used $(n-1)$ times then Painter colors v_{l+1} with color q.

If Painter follows the second alternative then vertex v_{l+1} is said to be *special.* Let $s(l)$ denote the number of special vertices after round l.

Let us prove that this is really a winning strategy. After every round l we can define the total weight function as follows:

$$w(l) = \sum_{j=1}^{a} \sum_{A: A(l) \subset V_j(l)} w_j(A, l) + s(l)a^n.$$

We will show that $w(l) \geqslant w(l+1)$, i.e. the total weight decreases. Indeed, let q be a color assigned to vertex v_{l+1} in round $l+1$. If v_{l+1} is not a special vertex then our strategy implies that

$$w(l+1) = w(l) - \sum_{j=1}^{a} \sum_{A:A(l)\subset V_j(l), v_{l+1}\in A} w_j(A,l) + \sum_{A:A(l)\subset V_q(l), v_{l+1}\in A} w_q(A,l+1) =$$

$$= w(l) - \sum_{j=1}^{a} d_j(v_{l+1}) + ad_q(v_{l+1}) \leqslant w(l)$$

since $d_q(v_{l+1})$ is the smallest number among $d_j(v_{l+1}), j = 1, \dots, a$. If v_{l+1} is a special vertex then

$$w(l+1) = w(l) - \sum_{j=1}^{a} \sum_{A:A(l)\subset V_j(l), v_{l+1}\in A} w_j(A,l) + a^n =$$

$$= w(l) - \sum_{j=1}^{a} d_j(v_{l+1}) + a^n \leqslant w(l)$$

since every $d_j(v_{l+1}), j = 1, \dots, a$, is at least a^{n-1}.

Now let us finish the proof. Suppose our strategy fails and at the end of the game there is an edge A, which is monochromatic of color j. Clearly, $j \in \{1, \dots, a\}$, because every color from $\{a+1, \dots, r\}$ can be used only $n-1$ times. Moreover, since the last vertex of A was assigned a color from $\{1, \dots, a\}$ there is already $(n-1)b$ special vertices. So the total weight at the end of the game is at least $a^n + (n-1)ba^n$. But at the beginning the total weight is equal to aN which is smaller than $a^n + (n-1)ba^n$, a contradiction. Hence Painter always wins. Proposition 2 is proved.

4.3. Proof of Proposition 3

The proof follows the general approach of the proof of Proposition 1. We have to show that for $N \geqslant 3r(r-1)^2 n \left(\frac{r}{r-1}\right)^{n+1}$, Lister has a winning strategy in $Game_3(N, n, r, r-1)$.

Let us divide the number of edges into r parts $E_1, \dots, E_r$ with exactly $3n(r-1) \cdot a_n$ edges in every part, where the value a_n is defined as follows:

$$a_0 = 1, \ a_m = \left\lceil \frac{r}{r-1} a_{m-1} \right\rceil, \ m = 1, \dots, n.$$

Clearly, $a_n \leqslant \frac{r}{r-1} a_{n-1} + 1$. Thus $a_n \leqslant \sum_{i=0}^{n} \left(\frac{r}{r-1}\right)^i \leqslant (r-1)\left(\frac{r}{r-1}\right)^{n+1}$. Since $N \geqslant 3rn(r-1) \cdot a_n$ the division can be made. We omit all the remaining edges.

Suppose that the first l vertices $v_1, \dots, v_l$ have already been colored. For every color $j \in \{1, \dots, r\}$, let $V_j(l)$ denote the set of revealed vertices which are not colored with j. Every edge A is considered as a function of l, where $A(l)$ denote an edge subset revealed after round l. If $A(l) \subset V_j(l)$ and $A \in E_j$ then we say that A is not colored with j, empty edge also satisfies this property. If $|A(l)| = i$, $i = 0, \dots, n-1$, then edge A is said to be at level i after round l. Finally, a (j, i)-block is a set of a_{n-i} edges not colored with j at level i.

Lister's strategy can be described as follows.

- For every color $j = 1, \dots, r$, he chooses the largest $i = i(j)$ such that there exists a (j, i)-block B_j.
- He chooses the union $B_1 \sqcup \dots \sqcup B_r$ as a set of edges that will contain the next vertex v_{l+1}.

Clearly, Lister wins if after some round there is an edge at level n, because such an edge will not meet some of the colors. The total number of blocks at the beginning is equal to $3rn(r-1)$. For any Painter's choice of color for v_{l+1}, the total number of blocks cannot decrease. Indeed, for chosen color the number of blocks not colored with this color will decrease by 1 but since $a_m \geqslant (1 + 1/(r-1))a_{m-1}$ the number of blocks not colored with any other color j will increase by at least $1/(r-1)$ (minus one block on the current level $i(j)$, plus $r/(r-1)$ blocks on the next one). Thus, the total number of blocks does not decrease.

The game continues until there is no blocks not colored with some color or Lister wins. In fact, in the first case Painter does not necessarily win, but we will show that even such situation is impossible. Suppose it appears and there is no blocks not colored with some color q. Due to the strategy for every $j \neq q$, the number of blocks not colored with j

- is at most 3 on every level form 1 to $n-1$ (since we always choose the largest level and add at most 2 new blocks to the next level);
- is at most $3n(r-1) - 1$ on level 0.

Hence the total number of the remaining blocks is at most

$$(r-1)\,(3(n-1) + 3n(r-1) - 1) = (r-1)\,(3nr-4)$$

which is less than $3nr(r-1)$, a contradiction, since we have shown that the total number of blocks cannot decrease. Thus Lister always wins.

Acknowledgements

This work was supported by Russian Foundation of Fundamental Research (grant № 15–01–03530-a) and by the grant of the President of Russian Federation MD–5650.2016.1

Bibliography

1. ***U. Schauz,*** *Mr. Paint and Mrs. Correct*, Electronic Journal of Combinatorics, **16**:1 (2009), Research Paper № R77.
2. ***U. Schauz,*** *A paintability version of the combinatorial nullstellensatz, and list colorings of k-partite k-uniform hypergraphs*, Electronic Journal of Combinatorics, **17** (2010), Research Paper № R176.
3. ***X. Zhu,*** *On-line list colouring of graphs*, Electronic Journal of Combinatorics, **16**:1 (2009), Research Paper № R127.
4. ***V. G. Vizing,*** *Coloring the vertices of a graph in prescribed colors*, Metody diskretnogo analiza v teorii kodov i skhem, Sobolev Institute of Mathematics, Novosibirsk, **29** (1976), 3–10 (in Russian).
5. ***P. Erdős, A. L. Rubin and H. Taylor,*** *Choosability in graphs*, Proc. West Coast Conference on Combinatorics, Graph Theory and Computing, 1979, Congr. Numer, **26** (1980), 125–157.
6. ***L. Duraj, G. Gutowski and J. Kozik,*** *Chip games and paintability*, Electronic journal of combinatorics, **23**:3 (2016), Paper № P3.3.
7. ***N. Gazit and M. Krivelevich,*** *On the asymptotic value of the choice number of complete multi-partite graphs*, Journal of Graph Theory, **52** (2006), 123–134.
8. ***D. A. Shabanov and T. M. Shaikheeva,*** *On the list chromatic number of the complete multi-partite hypergraphs and multiple coverings by independent sets*, preprint, https://mipt.ru/education/chairs/dm/laboratoriya-prodvinutoy-kombinatoriki-i-setevykh-prilozheniy/preprinty
9. ***D. A. Shabanov,*** *On a generalization of Rubin's theorem*, Journal of Graph Theory, **67**:3 (2011), 226–234.
10. ***P. Haxell and J. Verstraëte,*** *List coloring hypergraphs*, Electronic Journal of Combinatorics, **17** (2010), Research paper № 129.
11. ***A. V. Kostochka,*** *On a theorem by Erdős, Rubin and Taylor on choosability of complete bipartite graphs*, Electronic Journal of Combinatorics, **9** (2002).
12. ***J. Aslam and A. Dhagat,*** *On-line algorithms for 2-coloring hypergraphs via chip games*, Theoretical Computer Science, **112**:2 (1993), 355–369.
13. ***A. M. Raigorodskii and D. A. Shabanov,*** *The Erdős–Hajnal problem, its generalizations and related problems*, Russian Mathematical Surveys, **66**:5 (2011), 933–1002.

14. ***P. Erdős,*** *On a combinatorial problem*, *II*, Acta Mathematica of the Academy of Sciences, Hungary, **15**:3–4 (1964), 445–447.
15. ***J. Radhakrishnan and A. Srinivasan,*** *Improved bounds and algorithms for hypergraph two-coloring*, Random Structures and Algorithms, **16**:1 (2000), 4–32.
16. ***D. Cherkashin and J. Kozik***, *A note on random greedy coloring of uniform hypergraphs*, Random Structures and Algorithm, **47**:3 (2015), 407–413.
17. ***N. Alon,*** *Hypergraphs with high chromatic number*, Graphs and Combinatorics, **1**:1 (1985), 387–389.
18. ***I. A. Akolzin and D. A. Shabanov,*** *Colorings of hypergraphs with large number of colors*, Discrete Mathematics, **339**:12 (2016), 3020–3031.
19. ***A. P. Rozovskaya and D. A. Shabanov,*** *Extremal problems for panchromatic colourings of uniform hypergraphs*, Discrete Mathematics and Applications, **22**:2 (2012), 185–206.
20. ***D. D. Cherkashin,*** *A note on panchromatic colorings*, arXiv:1705.03797v1.

Alina E. Khuzieva

Moscow Institute
of Physics and Technology,
Department
of Discrete Mathematics,
141700, Institutskiy per. 9,
Dolgoprudny,
Moscow Region, Russia
smaleva@phystech.edu

Dmitry A. Shabanov

Moscow Institute
of Physics and Technology,
Laboratory
of Advanced Combinatorics
and Network Applications,
141700, Institutskiy per. 9,
Dolgoprudny,
Moscow Region, Russia;
Lomonosov Moscow
State University,
Faculty of Mechanics
and Mathematics,
Department
of Probability Theory,
119991, Leninskie gory 1,
Moscow, Russia;
National Research University
Higher School of Economics
(HSE),
Faculty of Computer Science,
101000, Myasnitskaya Str. 20,
Moscow, Russia
dmitry.shabanov@phystech.edu

Polina O. Svyatokum

National Research University
Higher School of Economics
(HSE),
Faculty of Computer Science,
101000, Myasnitskaya Str. 20,
Moscow, Russia
posvyatokum@edu.hse.ru