

Moscow Journal

of
***Combinatorics
and
Number Theory***



Moscow Journal of Combinatorics and Number Theory. 2011. Vol. 1. Iss. 1. 80 p.

The journal was founded in 2010

The aim of this journal is to publish original, high-quality research articles from a broad range of interests within combinatorics, number theory and allied areas. One volume of four issues is published annually.

*Published by the Moscow Institute of Physics and Technology
with the support of Yandex and Microsoft.*

Website

<http://mjcnt.phystech.edu>

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URSS Publishers

56, Nakhimovsky Prospekt,
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Издательство «УРСС»

Нахимовский пр-т, 56
Москва,
Российская Федерация,
117335

Журнал зарегистрирован в Федеральной службе по надзору в сфере массовых коммуникаций, связи и охраны культурного наследия 3 сентября 2010 г. Свидетельство ПИ № ФС77-41900.

Формат 70×100/16. Печ. л. 5. Зак. № 4714.

Отпечатано в ООО «ЛЕНАНД».

117312, Москва, пр-т Шестидесятилетия Октября, 11А, стр. 11.


ISSN 2220-5438

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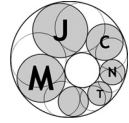
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Multicolored matchings in hypergraphs

Noga Alon (Tel-Aviv)

Abstract: For a collection of (not necessarily distinct) matchings $\mathcal{M} = (M_1, M_2, \dots, M_q)$ in a hypergraph, where each matching is of size t , a matching M of size t contained in the union $\bigcup_{i=1}^q M_i$ is called a *rainbow matching* if there is an injective mapping from M to \mathcal{M} assigning to each edge e of M a matching $M_i \in \mathcal{M}$ containing e .

Let $f(r, t)$ denote the maximal k for which there exists a collection of k matchings, each of size t , in some r -partite r -uniform hypergraph, such that there is no rainbow matching of size t .

Aharoni and Berger showed that $f(r, t) \geq 2^{r-1}(t-1)$, proved that the equality holds for $r = 2$ as well as for $t = 2$ and conjectured that the equality holds for all r, t . We show that in fact $f(r, t)$ is much bigger for most values of r and t , establish an upper bound and point out a relation between the problem of estimating $f(r, t)$ and several results in additive number theory, which provides new insights into some of those results.

Keywords: Rainbow matchings, Additive Latin transversals

AMS Subject classification: 05C65, 05D15

Received: 30.11.2010; **revised:** 30.11.2010

1. Introduction

A *matching* in a hypergraph is a collection of pairwise disjoint edges. For a collection of (not necessarily distinct) matchings $\mathcal{M} = (M_1, M_2, \dots, M_q)$ in a hypergraph, where each matching is of size t , a matching M of size t contained in the union $\bigcup_{i=1}^q M_i$ is called a *rainbow matching* if there is an injective mapping from M to \mathcal{M} assigning to each edge e of M a matching $M_i \in \mathcal{M}$ containing e .

Let $f(r, t)$ denote the maximal k for which there exists a collection of k matchings, each of size t , in some r -partite r -uniform hypergraph, such that there is no rainbow matching of size t .

Aharoni and Berger [1] showed that $f(r, t) \geq 2^{r-1}(t-1)$ for all $r, t > 1$, proved that the equality holds for $r = 2$ as well as for $t = 2$ and conjectured that the equality holds for all $r, t > 1$.

CONJECTURE 1 ([1]). For every integers $r, t > 1$, $f(r, t) = 2^{r-1}(t-1)$.

In this note we observe that this question is closely related to a well studied problem in additive number theory. Using this relation we show that the conjecture is false for every pair (r, t) with $t \geq 3$ odd and $r \geq 4$ as well as for $t = 4, 6, 8$ and all sufficiently large r , and for every even $t \geq 10$ and $r \geq 4$. In addition, we describe a probabilistic lower bound for $f(r, t)$ showing that for all sufficiently large t and all r we have $f(r, t) > 2.71828^{r-1}$, and prove a (much larger) upper bound: $f(r, t) \leq \frac{t^{rt}(t-1)}{t!}$. We conclude by pointing out that the known value of $f(2, t)$ provides a new graph theoretic proof of an old result of Erdős, Ginzburg and Ziv, and by discussing several extensions and open problems.

2. The lower bound

In this section we describe two methods that provide lower bounds for $f(r, t)$. The first one is based on a simple interpretation of the problem in terms of additive number theory, and the second one is probabilistic. Both methods can be used to construct counterexamples to Conjecture 1.

2.1. The first construction

Let $g(n, t)$ denote the least integer p , such that any sequence of at least p (not necessarily distinct) elements of the abelian group Z_t^n contains a sub-sequence of exactly t elements whose sum (in Z_t^n) is zero. Equivalently, this is the minimal number p , such that any set of at least p lattice points in Z^n contains a subset of exactly t points whose centroid is also a lattice point.

The problem of determining or estimating $g(n, t)$, suggested by Harborth in [17] received a considerable amount of attention. In particular, it is known that $2^n(t-1) + 1 \leq g(n, t) \leq (t-1)t^n + 1$ ([17]), $g(3, 3) = 19$ ([17], [8]), $g(4, 3) = 41$ ([20], [9], [8], [18]), $g(5, 3) = 91$ ([12], [11]), $g(n, 3) > 2.217^n$ for all sufficiently large n ([11], improving [15]), $g(n, t) \geq 1.125^{\lfloor n/3 \rfloor} (t-1)2^n + 1$ for every odd $t \geq 3$ and every n ([13]), $g(n, 3) \leq 2 \cdot \frac{3^n}{n}$ ([19]), $g(n, t) = o(n^t)$ for any fixed t , as n tends to infinity ([6]), and $g(n, t) \leq c(n)t$ ([6]).

THEOREM 1. *For all $r, t > 1$ we have $f(r, t) \geq g(r-1, t) - 1$.*

PROOF. By the definition of g , there is a sequence S of $|S| = g(r-1, t) - 1$ elements of Z_t^{r-1} containing no sub-sequence of t terms that sum to zero. Using this sequence, we define a collection of $|S|$ matchings, each of size t , in an r -uniform r -partite hypergraph on the vertex classes A_1, A_2, \dots, A_r , where each A_i is a copy of Z_t . Note that each matching will be a perfect matching.

For each element $s = (s_1, s_2, \dots, s_{r-1}) \in S$ let M_s be the matching whose i -th edge, for $0 \leq i < t$, is $(s_1 + i, s_2 + i, \dots, s_{r-1} + i, i)$, where the addition is in Z_t , and where for each j , $1 \leq j \leq r$, the j -th coordinate of the vector is interpreted as an element of A_j . This defines a family of $|S|$ perfect matchings in our hypergraph. A rainbow matching here corresponds to a choice of t distinct

elements $s^{(1)}, s^{(2)}, \dots, s^{(t)}$ of the sequence S , and an edge from each matching $M_{s^{(i)}}$ such that these t edges form a perfect matching. As these edges have to cover the last vertex class A_r , it follows that there is a permutation $\sigma \in S_t$, such that the rainbow matching consists of the edges $(s_1^{(i)} + \sigma(i), s_2^{(i)} + \sigma(i), \dots, s_{r-1}^{(i)} + \sigma(i), \sigma(i))$, $1 \leq i \leq t$.

This implies that for every j , $1 \leq j \leq r-1$, the t numbers $s_j^{(i)} + \sigma(i)$ form a permutation of Z_t , and hence in Z_t the two sums $\sum_{i=1}^t (s_j^{(i)} + \sigma(i))$ and $\sum_{i=1}^t \sigma(i)$ are equal. Thus $\sum_{i=1}^t s_j^{(i)} = 0$ for all $1 \leq j \leq r-1$, and the sum of the sub-sequence $s^{(1)}, \dots, s^{(t)}$ is zero in Z_t^{r-1} , contradicting the choice of the sequence S . It follows that there is no rainbow matching, which completes the proof. \square

The above proposition, together with the known lower bounds for the function $g(n, t)$ imply that $f(4, 3) \geq 18$, $f(5, 3) \geq 40$, $f(6, 3) \geq 90$, and $f(r, 3) > 2.216^r$ for all sufficiently large r , showing that the assertion of Conjecture 1 fails for these values of the parameters. The known bounds also show that for every fixed odd t we have $f(r, t) \geq 1.125^{\lfloor (r-1)/3 \rfloor} (t-1)2^{r-1}$, which is strictly larger than $(t-1)2^{r-1}$ for all $r \geq 4$. It is easy to check that for every $t > 2$ and every r we have $f(r, t) \geq f(r, t-1)$, as one can simply take a large collection of matchings, each of size t , with no rainbow matching of size t , and add the same edge, disjoint from all the existing edges, to each of the matchings. This suffices to show that $f(r, t)$ exceeds $(t-1)2^{r-1}$ for all even values of $t \geq 10$ and $r \geq 4$ as well as for $t = 4, 6, 8$ and all large values of r .

The function $f(r, t)$ is likely to be much larger than $g(r-1, t) - 1$. Indeed, it is known, for example, that for every $t = 2^a$ which is a power of 2 we have $g(r-1, 2^a) = 2^{r-1}(2^a - 1) + 1$ (see [17]), while as mentioned above, $f(r, t)$ is larger by an exponential factor for all such $t \geq 4$ and large r .

2.2. The probabilistic construction

In this subsection we describe a simple probabilistic lower bound for $f(r, t)$, using the so-called alteration method (c.f., e.g., [7], chapter 3). For fixed large t and $r > b \log t$ for an appropriate absolute constant b this bound is better than the ones given in the previous subsection.

THEOREM 2. *For any real number $p \in (0, 1)$ we have $f(r, t) \geq p \cdot t^{r-1} - (t!)^{r-1} p^t$. Therefore, for every $\epsilon > 0$ and $t > t_0(\epsilon)$ we have $f(r, t) > (e - \epsilon)^{r-1}$, where $e = 2.718281828 \dots$ is the basis of the natural logarithm.*

PROOF. As before, all our matchings are perfect matchings in an r -partite r -uniform hypergraph on the classes of vertices A_1, A_2, \dots, A_r , where each A_r is a copy of Z_t . Every edge of this hypergraph is represented by a vector $s = (s_1, s_2, \dots, s_r) \in Z_t^r$, where the j^{th} coordinate is an element of A_j . For each vector $s = (s_1, s_2, \dots,$

$\dots, s_{r-1}, 0) \in Z_t^r$ with the last coordinate equal to 0, let M_s denote the matching consisting of the t edges $(s_1 + i, s_2 + i, \dots, s_{r-1} + i, i)$, ($0 \leq i < t$), where the addition is in Z_t . Let \mathcal{M} be a random collection of matchings obtained by picking each matching M_s for s as above, randomly and independently, with probability p , to be an element of \mathcal{M} . Let $X = X(\mathcal{M})$ be the random variable counting the number of matchings in \mathcal{M} , and let $Y = Y(\mathcal{M})$ be the random variable counting the number of rainbow matchings in the union of all the edges of \mathcal{M} . Clearly, the expectation of X is equal to pt^{r-1} .

We claim that the expectation of Y is at most $(t!)^{r-1}p^t$. Indeed, the total number of perfect matchings in the complete r -partite r -uniform hypergraph on the sets A_i is exactly $(t!)^{r-1}$. Some of these matchings cannot be rainbow matchings in the union of our randomly selected matchings, as they contain two edges that belong to the same matching M_s for some s . It is crucial to observe that for each matching that may become a rainbow matching, the probability that it lies in the union of the chosen matchings equals precisely p^t , since all its edges belong to different M_s and hence the choices are independent. This proves, by linearity of expectation, that the expected value of Y is at most $(t!)^{r-1}p^t$. Applying linearity of expectation once again we conclude that the expectation of the difference $X - Y$ is at least $pt^{r-1} - (t!)^{r-1}p^t$. Thus, \mathcal{M} can be chosen so that the inequality $X(\mathcal{M}) - Y(\mathcal{M}) \geq pt^{r-1} - (t!)^{r-1}p^t$ holds. Fix such an \mathcal{M} , and for each rainbow matching M it contains omit from the collection an arbitrary matching that contributes an edge to M . This gives a collection of at least $pt^{r-1} - (t!)^{r-1}p^t$ matchings with no rainbow one, as needed.

We can now choose p optimally in order to maximize the bound obtained. This is given by $p = \left(\frac{1}{t[(t-1)!]^{r-1}}\right)^{1/(t-1)}$ (but in fact even choosing $p = (t!)^{-(r-1)/t}$ gives the same asymptotic result.) Taking this value of p and using Stirling's formula we conclude that as t tends to infinity the bound obtained is at least $e^{(1+o(1))(r-1)} - 1$. As for $r \leq \frac{1}{2} \log t$, say, the lower bound $f(r, t) \geq 2^{r-1}(t-1)$, proved in [1], exceeds e^{r-1} , the desired estimate follows for all sufficiently large t and all r . This completes the proof. \square

3. The upper bound

In this section we prove an upper bound for $f(r, t)$. The proof is probabilistic and applies to matchings in general, not necessarily r -partite r -uniform hypergraphs. Let $F(r, t)$ denote the maximal k for which there exists a collection of k matchings, each of size t , in some r -uniform hypergraph, such that there is no rainbow matching of size t . Obviously, $F(r, t) \geq f(r, t)$ for all r and t , and it is not difficult to see that

$F(r, t) \leq \left(\frac{r^r}{r!}\right)^t f(r, t) \leq e^{rt} f(r, t)$. Indeed, given a collection \mathcal{M} of matchings, each of size t , in an arbitrary r -uniform hypergraph $H = (V, E)$, take a random partition $V = V_1 \cup V_2 \cup \dots \cup V_r$ of V into r pairwise disjoint sets, and let \mathcal{M}' consist

of all matchings in \mathcal{M} in which every edge intersects each V_i exactly once. Let H' be the hypergraph consisting of all the edges in all the matchings in \mathcal{M}' . Then H' is r -partite and the expected number of matchings in \mathcal{M}' is exactly $\left(\frac{r!}{r^r}\right)^t |\mathcal{M}|$. If there is no rainbow matching in $\cup_{M \in \mathcal{M}} M$, then there is no rainbow matching in $\cup_{M' \in \mathcal{M}'} M'$, which implies that $f(r, t) \leq \left(\frac{r!}{r^r}\right)^t F(r, t)$.

THEOREM 3. *For all r and t we have*

$$f(r, t) \leq F(r, t) \leq \frac{t^{rt}(t-1)}{t!}.$$

PROOF. Let \mathcal{M} be a collection of matchings in an r -uniform hypergraph, where each matching $M \in \mathcal{M}$ is of size t . Let $H = (V, E)$ be the hypergraph consisting of all the edges in all the matchings $M \in \mathcal{M}$, and let $c : V \rightarrow [t] = \{1, 2, \dots, t\}$ be a random function, assigning to each vertex $v \in V$, randomly and independently, a uniformly chosen color $c(v) \in [t]$. Call a matching $M \in \mathcal{M}$ *multicolored* if for every $i \in [t]$ it contains exactly one edge with all r vertices of color i . Note that for each $M \in \mathcal{M}$ the probability that M is multicolored is exactly $\frac{t!}{t^{rt}}$, as there are $t!$ ways to distribute the colors among the edges, and once this is done, the probability that each vertex saturated by the matching gets the color assigned to its edge is $\frac{1}{t}$.

Due to the linearity of expectation, the expected number of multicolored matchings in \mathcal{M} is $|\mathcal{M}| \cdot \frac{t!}{t^{rt}}$. If $|\mathcal{M}| > \frac{t^{rt}(t-1)}{t!}$, then this expectation exceeds $t-1$, and hence there exists a coloring in which there are at least t multicolored matchings $M_1, M_2, \dots, M_t \in \mathcal{M}$. Fix such a coloring, and let $e_i \in M_i$ be the edge of M_i in which all the vertices are of color i ($1 \leq i \leq t$). Then the matching $\{e_1, e_2, \dots, e_t\}$ is a rainbow matching. This shows that any collection of more than $\frac{t^{rt}(t-1)}{t!}$ matchings contains a rainbow matching, implying that $F(r, t) \leq \frac{t^{rt}(t-1)}{t!}$, as needed. □

It is worth noting that the upper bound in the above Theorem can be slightly improved by coloring the vertices randomly with some $t' > t$ colors, calling a matching multicolored if there is a set T of t colors, such that for each $i \in T$ there is an edge of the matching with all the vertices of color i . It is easy to check that here, too, a collection of t multicolored matchings must contain a rainbow matching, and one can choose the optimal value of t' to (slightly) improve the upper bound.

4. Concluding remarks and open problems

- The authors of [1] defined a function $f(r, s, t)$ for r and $t \geq s$ as the maximal k for which there exists a collection of k matchings, each of size t ,

in some r -partite r -uniform hypergraph, such that there is no matching of size t in which there are at least s edges that belong to distinct matchings. Thus, $f(r, s, t)$ is exactly the function $f(r, t)$ considered in the previous sections. They showed that $f(r, s, t) \geq 2^{r-1}(s-1)$, and that the equality holds for $r=2$ and for $s=t=2$. The upper bound proved in Section 3 can be modified to yield improved upper bounds for this function when $s < t$ (although in general these are still far from the lower bound). Indeed, given a collection \mathcal{M} of matchings, each of size t , in an r -uniform hypergraph $H = (V, E)$, consider a random coloring of V by s colors, where to each vertex, randomly and independently, the color i is assigned with probability

p_i , so that $\sum_{i=1}^s p_i = 1$, $p_i \geq 0$. Let q_1, q_2, \dots, q_s be positive numbers whose

sum equals t . Call a matching $M \in \mathcal{M}$ q -multicolored if it contains exactly q_i monochromatic edges of color $i \in [s] = \{1, 2, \dots, s\}$ for every $i \in [s]$. Note that a collection S of s q -multicolored matchings always contains a matching of size t using at least one edge of each matching in S . By choosing the numbers q_i and the probabilities p_i optimally and by computing the expected number of q -multicolored matchings we get an upper bound for $f(r, s, t)$ (and in fact for $F(r, s, t)$ which is defined in the obvious way.) Here, too, one can use more than s colors and more than one vector (q_1, \dots, q_s) to improve the estimate in some cases. As an example, for $t > s = 2$ and any r one can take $p_1 = 1/t, p_2 = (t-1)/t, q_1 = 1, q_2 = t-1$ and conclude that

$$f(r, 2, t) \leq F(r, 2, t) \leq \left(\frac{t^t}{(t-1)^{t-1}} \right)^r.$$

- The connection between the function $f(r, t)$ and problems in additive number theory described in Section 2 leads to some new insights about problems in additive combinatorics using the known results about $f(r, t)$. In particular, one can get a new proof of an old theorem of Erdős, Ginzburg and Ziv [14], that asserts that any sequence of $2t-1$ elements of Z_t contains a subsequence of exactly t terms whose sum in Z_t is zero. (In the notation of Section 2, this is the known fact that $g(1, t) = 2t-1$.) Several proofs of this result are known, see [5] for five of them. A common feature of all those proofs is that they first establish the result for all prime values of t and then use the fact that the validity of the result for t_1 and t_2 implies its validity for the product $t_1 t_2$. Here is a new short proof, based on the result of [1] (following [10]), that $f(2, t) = 2t-2$ for all t . Note that the proof in [1], [10] is graph theoretic, based on an alternating path argument, and works directly for all (prime or non-prime) t .

Given a sequence $a_1, a_2, \dots, a_{2t-1}$ of elements of Z_t , define a family $\mathcal{M} = \{M_1, \dots, M_{2t-1}\}$ of $2t-1$ perfect matchings in a bipartite graph on the color classes A_1, A_2 , where each A_i is a copy of Z_t . The matching M_i is defined by a_i as in Section 2, that is, $M_i = \{(a_i + j, j) \in A_1 \times A_2 : j \in Z_t\}$, where addition is in Z_t . Since $f(2, t) = 2t-2$, there is a rainbow matching, implying that there is a set $I \subset \{1, 2, \dots, 2t-1\}$, $|I| = t$, and a bijection $\sigma : I \rightarrow Z_t$, such

that the edges $(a_i + \sigma(i), \sigma(i))$, $i \in I$, form a perfect matching. In particular, the elements $a_i + \sigma(i)$, $i \in I$, form a permutation of Z_t , implying that we have $\sum_{i \in I} (a_i + \sigma(i)) = \sum_{j \in Z_t} j$ in Z_t , whence $\sum_{i \in I} a_i = 0$ in Z_t , as needed.

Note that the argument works for any abelian group of order t . It may seem that the above proof gives a stronger result than the Erdős—Ginzburg—Ziv Theorem, as it does not only give a subsequence of t terms whose sum is zero, but in fact it provides a subsequence to which one can add a permutation of the elements of Z_t and get a permutation. However, by an old result of M. Hall [17], these two assertions are equivalent in any abelian group, that is, a sequence of t elements in an abelian group of order t can be expressed as the pointwise difference of two permutations if and only if the sum of its elements is zero. The proof in [17] is also based on an alternating path argument. Note that for prime t the assertion of Hall's Theorem can be easily deduced from a special case of Theorem 1.2 in [4].

- In [1] it is proved that $f(r, 2) = 2^{r-1}$, using a special case of the main result of [2]. It is interesting to note that this is also equivalent to the assertion of Corollary 1.2 in [3]. This corollary asserts that the largest n for which the complement of a perfect matching of n edges can be covered by r subgraphs, each being a vertex disjoint union of complete graphs, is 2^{r-1} . To see the equivalence, let the edges of the missing matching be $\{a_i, b_i\}$, and let the subgraphs be H_1, \dots, H_r , where H_i is the disjoint union of cliques $C_{i,1}, \dots, C_{i,q_i}$. Put $V_i = \{C_{i,1}, \dots, C_{i,q_i}\}$ and consider the sets V_i as the vertex classes of an r -partite r -uniform hypergraph. For each edge $\{a_i, b_i\}$ as above, let $M_i = \{e_i, f_i\}$ be a matching of size 2 in this hypergraph, where e_i consists of all the vertices C_{i,j_i} with $a_i \in C_{i,j_i}$ and f_i consists of all the vertices C_{i,j_i} with $b_i \in C_{i,j_i}$. There is no rainbow matching here, since for every $i \neq j$ the intersection of e_i and f_j is not empty (since the edge $a_i b_j$ must belong to some subgraph H_i). This gives a correspondence between the families of matchings of size 2 without a rainbow matching, and the coverings of the complements of graph-matchings by subgraphs that are disjoint unions of cliques, showing that indeed the fact that $f(r, 2) = 2^{r-1}$ is equivalent to the covering result stated above. The proofs in [2], [3] apply linear algebra tools based on some simple properties of the exterior algebra.
- The problem of determining $f(r, t)$ or obtaining better estimates for it remains open. In particular, it seems interesting to determine the asymptotic behaviour of $f(r, 3)$ (which is exponential in r) more accurately, and to find out whether or not for any fixed r there is a constant $c(r)$, such that $f(r, t) \leq c(r)t$ for all t .

Acknowledgements. Research supported in part by an ERC Advanced grant, by a USA—Israeli BSF grant, by the Oswald Veblen Fund and by the Bell Companies Fellowship.

This paper was initiated during a visit at the Department of Combinatorics and Optimization of the University of Waterloo. The stimulating research atmosphere at the department and, in particular, fruitful discussions with Penny Haxell and Nick Wormald are gratefully appreciated.

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