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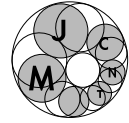
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# Almost all factorial subclasses of quasi-line graphs with respect to one forbidden subgraph

Victor Zamaraev (Nizhny Novgorod)

**Abstract:** For a graph property  $X$ , let  $X_n$  be the set of graphs with the vertex set  $\{1, \dots, n\}$  that satisfy the property  $X$ . A property  $X$  is called factorial if  $X$  is hereditary (i. e. closed under taking induced subgraphs) and  $n^{c_1 n} \leq |X_n| \leq n^{c_2 n}$  for some positive constants  $c_1$  and  $c_2$ . A graph  $G$  is a *quasi-line* if for every vertex  $v$ , the set of neighbors of  $v$  can be expressed as a union of two cliques. In the present paper we identify almost all factorial subclasses of quasi-line graphs defined by one forbidden induced subgraph. We use these new results to prove that the class  $\text{Free}(K_{1,3}, W_4)$  is factorial, which improves on a result of Lozin, Mayhill and Zamaraev [8].

**Keywords:** Hereditary class of graphs; Factorial class; Quasi-line graphs

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## 1. Introduction

A *graph property* is an infinite class of graphs closed under isomorphism. Given a property  $X$ , we write  $X_n$  for the set of graphs in  $X$  which have the vertex set  $\{1, 2, \dots, n\}$ . Following [3], we define the *speed* of the property  $X$  as  $|X_n|$ . A property is called *hereditary* if it is closed under taking induced subgraphs.

It is well-known that a graph property  $X$  is hereditary if and only if  $X$  can be described in terms of forbidden induced subgraphs. More formally, for a set of

graphs  $\mathcal{M}$  let us denote by  $\text{Free}(\mathcal{M})$  the class of graphs containing no induced subgraphs isomorphic to a graph in the set  $\mathcal{M}$ . Then  $X$  is a hereditary class if and only if  $X = \text{Free}(\mathcal{M})$  for some set  $\mathcal{M}$ . We call  $\mathcal{M}$  the set of forbidden induced subgraphs for the class  $X$  and say that graphs in  $X$  are  $\mathcal{M}$ -free. For the sake of simplicity, we are going to omit curly brackets in expressions of the form  $\text{Free}(\mathcal{M})$  if we are dealing with a specific set of graphs, e. g. the expression  $\text{Free}(\{K_{1,3}, W_4\})$  will be written as simply  $\text{Free}(K_{1,3}, W_4)$ .

In [9] Scheinerman and Zito have shown that for a hereditary property  $X$  the growth of  $|X_n|$  is far from arbitrary. They have defined four layers, which are comprised of classes with certain specific growth rates of the function  $\log_2 |X_n|$ :

- the *constant* layer contains classes  $X$  with  $\log_2 |X_n| = O(1)$ ;
- the *polynomial* layer contains classes  $X$  with  $\log_2 |X_n| = \Theta(\log_2 n)$ ;
- the *exponential* layer contains classes  $X$  with  $\log_2 |X_n| = \Theta(n)$ ;
- the *factorial* layer contains classes  $X$  with  $\log_2 |X_n| = \Theta(n \log_2 n)$ .

Independently, a similar result has been obtained by Alekseev in [2]. Moreover, Alekseev found all minimal (by inclusion) elements in each layer and provided the first three layers with complete structural characterizations. In [3] Balogh, Bollobás and Weinreich, also independently, gave their characterization of the first three layers as well as characterized the lower part of the factorial layer. They described properties with the speed  $n^{(1-1/k+o(1))n}$  for each natural  $k \geq 2$  and showed in [4] that for any property  $X$  with a speed above that range we have  $|X_n| \geq \mathcal{B}_n \approx (n/\log n)^n$ , where  $\mathcal{B}_n$  is  $n$ -th Bell number. The factorial layer is the minimal one for which no complete structural characterization is known. Graph properties beyond the factorial layer are called *superfactorial*. More formally, a graph property  $X$  is superfactorial if for all positive constants  $c$  and  $n_0$  there exists  $n \geq n_0$  such that  $|X_n| > n^{cn}$ .

The factorial layer contains many classes of theoretical or practical importance, such as line graphs, interval graphs, permutation graphs, threshold graphs, forests, planar graphs, classes of graphs of bounded vertex degrees, etc. On the other hand, except for the definition, very little can be said about the factorial layer in general. This motivated some authors to study factorial properties on a systematic basis [7, 8, 10].

In [7] Lozin, Mayhill and Zamaraev revealed a variety of factorial properties defined by two forbidden induced subgraphs. They also pointed out some minimal classes of graphs defined by two forbidden induced subgraphs for which the membership in the factorial layer is an open question. One of such classes is  $\text{Free}(K_{1,3}, C_4)$ ,

which was for some time the only open case in the collection of classes defined by two forbidden induced subgraphs with at most four vertices. Recently, it was proved in [8] that  $\text{Free}(K_{1,3}, C_4)$  is factorial, with the main step of the proof showing that the class of  $C_4$ -free quasi-line graphs is factorial.

In the present paper we study subclasses of quasi-line graphs defined by one forbidden induced subgraph and identify almost all factorial cases. Using these new results, we also show that the class  $\text{Free}(K_{1,3}, W_4)$  is factorial, improving on a result of [8].

## 2. Preliminaries

### 2.1. Notation and Terminology

All graphs in this paper are assumed to be finite, undirected, without loops or multiple edges. By  $V(G)$  and  $E(G)$  we denote respectively the vertex set and the edge set of a graph  $G$ . By  $N(x)$  we denote the neighborhood of a vertex  $x$  in  $G$ . The subgraph of  $G$  induced by a set of vertices  $U \subseteq V(G)$  is denoted by  $G[U]$ . By  $G \setminus M$ , where  $M \subseteq V(G)$ , we denote the graph  $G[V(G) \setminus M]$ . A *union* of two graphs  $G_1(V_1, E_1)$  and  $G_2(V_2, E_2)$  is the graph  $U(V_1 \cup V_2, E_1 \cup E_2)$ . By  $G + H$  we denote a disjoint union (in the sense that  $G$  and  $H$  have disjoint vertex sets) of graphs  $G$  and  $H$ . The complement of a graph  $G$  is denoted by  $\overline{G}$ .

As usual, by  $C_n$ ,  $P_n$ , and  $K_{n,m}$  we denote a chordless cycle, a chordless path and a complete bipartite graph with parts of size  $n$  and  $m$ . The graph  $K_{1,m}$  is usually referred to as a star. By  $\Phi_{p,q}$  we are going to denote the graph obtained from two stars  $K_{1,p}$  and  $K_{1,q}$  by connecting their central vertices with an edge and subdividing this edge once. Also,  $T_{1,2,3}$  is the graph obtained from  $K_{1,3}$  by subdividing one of its edges once and another edge twice. By  $W_n$  we denote an  $n$ -wheel graph, which is the graph with  $n + 1$  vertices formed by connecting a single vertex to all vertices of  $C_n$ .

By  $\mathcal{B}$  and  $\overline{\mathcal{B}}$  we denote respectively the class of bipartite graphs and the class of co-bipartite graphs (complements of bipartite graphs). It is known [1] that the classes  $\mathcal{B}$  and  $\overline{\mathcal{B}}$  are superfactorial and that their speeds can be both written as  $2^{n^2/4+o(n^2)}$ . A graph  $G$  is a *quasi-line* if for every vertex  $v$  the set of neighbors of  $v$  induces a co-bipartite graph. Denote by  $\mathcal{Q}$  the class of quasi-line graphs. It is easy to see that  $\overline{\mathcal{B}} \subset \mathcal{Q}$  and hence  $\mathcal{Q}$  is also superfactorial.

In order to proceed, we are going to need the characterization of quasi-line graphs which has recently been obtained by Chudnovsky and Seymour [5].

## 2.2. Quasi-line graphs

A *circular interval graph*  $G = (V, E)$  is defined by the following construction. Take a circle  $C$ , and let  $V(G)$  be a finite set of points of  $C$ . Take a subset of intervals  $I$  of  $C$  (an *interval* is a proper subset of  $C$  homeomorphic to  $[0, 1]$ ) and say that  $u, v \in V(G)$  are adjacent in  $G$  if  $\{u, v\}$  is a subset of one of the intervals. A special case of a circular interval graph is the *linear interval graph* which is defined in the same way, only taking  $C$  to be a line instead of a circle.

A vertex  $v \in V(G)$  is *simplicial* if the set of neighbors of  $v$  is a clique. A *strip* is a triple  $(G, a, b)$ , where  $G$  is graph and  $a$  and  $b$  are two designated simplicial vertices of  $G$  called the *ends* of the strip. A strip  $(G, v_1, v_n)$ , where  $n > 1$ , is called a *linear interval strip* if  $G$  is a linear interval graph with the vertices given in order as  $v_1, \dots, v_n$ . A *glue* of two vertex-disjoint strips  $(G, a, b)$  and  $(G', a', b')$  is the graph resulting from the union of  $G \setminus \{a, b\}$  and  $G' \setminus \{a', b'\}$  together with the adjunction of all possible edges between the neighbors of  $a(b)$  and the neighbors of  $a'(b')$ .

Let  $G_0$  be a disjoint union of cliques with an even vertex set  $V(G_0) = \{a_1, b_1, \dots, a_k, b_k\}$ , where  $a_i, b_i$  are nonadjacent,  $i = 1, \dots, k$ . For  $i = 1, \dots, k$  let  $(G'_i, a'_i, b'_i)$  be  $k$  vertex-disjoint strips that do not intersect the vertex set of  $G_0$ . For  $i = 1, \dots, k$  let  $G_i$  be the graph obtained by gluing  $(G_{i-1}, a_i, b_i)$  together with  $(G'_i, a'_i, b'_i)$ . The resulting graph  $G_n$  is called the *composition* of the strips  $(G'_i, a'_i, b'_i)$ ,  $1 \leq i \leq k$ , with the disjoint union of cliques  $G_0$ .

In [5] Chudnovsky and Seymour have generalized the notion of a circular interval graph. A graph  $G(V, E)$  is a *fuzzy circular interval* if the following conditions hold:

1. There is a map  $\phi$  from  $V(G)$  to a circle  $C$  (not necessarily injective).
2. There is a set of intervals  $I$  of  $C$ , none including another, such that no point of  $C$  is an endpoint of more than one interval, and also
  - a) if any two vertices  $u$  and  $v$  are adjacent, then  $\phi(u)$  and  $\phi(v)$  belong to a common interval;
  - b) if  $\phi(u)$  and  $\phi(v)$  belong to the same interval, which is not the interval with endpoints  $\phi(u)$  and  $\phi(v)$ , then the vertices  $u$  and  $v$  are adjacent.

In other words, for a fuzzy circular interval graph its *fuzzy representation*  $(\phi, I)$  completely describes adjacencies, excepting the adjacencies for vertices  $u$  and  $v$  such that  $I$  contains an interval with the endpoints  $\phi(u)$  and  $\phi(v)$ . For such vertices the adjacency is fuzzy. Note that if we require  $\phi$  to be injective, the definition of a fuzzy circular interval graph is equivalent to the definition of a circular interval graph.

*Fuzzy linear interval graphs* and *fuzzy linear interval strips* are defined similarly, with a provision that if  $a$  and  $b$  are the endpoints of a strip, then  $\phi(a)$  and  $\phi(b)$  are different from  $\phi(v)$  for all other vertices  $v$  of  $G$ . Chudnovsky and Seymour proved the following structural result [5].

**THEOREM 1.** *A connected quasi-line graph  $G$  is either a fuzzy circular interval graph or a composition of fuzzy linear interval strips with a disjoint union of cliques.*

In [8] Lozin, Mayhill, and Zamaraev proved that the class  $\text{Free}(K_{1,3}, C_4)$  is factorial. In fact, they reduced the original problem to the class of  $C_4$ -free quasi-line graphs and showed that this class is factorial. One of the steps of this proof is the following theorem.

**THEOREM 2.** *A connected quasi-line graph  $G$  without an induced  $C_4$  subgraph is either a circular interval graph or a composition of linear interval strips with a disjoint union of cliques.*

Theorem 2 says that if we forbid an induced  $C_4$  in quasi-line graphs, we then eliminate fuzziness in their representation. A result from [8] motivates us to investigate the following question:

*how to identify graphs  $H$  such that the class  
of  $H$ -free quasi-line graphs is factorial?* (1)

A result from [7] partially answers a similar question for the class of bipartite graphs, and hence for the class of co-bipartite graphs (since  $|X_n| = |\overline{X}_n|$ ):

**THEOREM 3.** *If a graph  $G$  is not isomorphic to any induced subgraph of  $T_{1,2,3}$ ,  $\Phi_{p,q} + O_1$ , or  $P_7$ , then the class  $\text{Free}(G) \cap \mathcal{B}$  is superfactorial. If  $G = T_{1,2,3}$  or  $G = \Phi_{p,q} + O_1$  for any natural  $p, q$ , then  $\text{Free}(G) \cap \mathcal{B}$  is factorial.*

In the next section we prove the same result for the class of quasi-line graphs. Before proceeding to the next section, we need to define certain notions and formulate several auxiliary statements.

### 2.3. Auxiliary Notions and Statements

Let  $G$  be a graph. A set of graphs  $H_1, \dots, H_k$  such that their union is  $G$  will be called a *covering* of  $G$ . The following lemma is a powerful tool which often helps determine whether a graph property is a member of the factorial layer [7].

LEMMA 1. *Let  $X$  be a class of graphs and  $m$  be a constant. If every graph  $G \in X$  can be covered by graphs belonging a class  $Y$  with  $\log_2 |Y_n| = O(n \log n)$  in such a way that every vertex of  $G$  is covered by at most  $m$  graphs, then  $\log_2 |X_n| = O(n \log n)$ .*

Using the definition of a glue, it is easy to prove the following lemma.

LEMMA 2. *Let  $(G_1, a_1, b_1)$  and  $(G_2, a_2, b_2)$  be two strips, and let  $G$  be their glue. If  $G_1 \setminus \{a_1, b_1\}$  or  $G_2 \setminus \{a_2, b_2\}$  contains  $H$  as an induced subgraph, then  $G$  also contains  $H$  as an induced subgraph.*

COROLLARY 1. *If  $G$  is a composition of strips  $(G_i, a_i, b_i)$ ,  $i = 1, \dots, k$ , with a disjoint union of cliques, and  $G$  does not contain  $H$  as an induced subgraph, then none of  $G_i \setminus \{a_i, b_i\}$ ,  $i = 1, \dots, k$ , contain  $H$  as an induced subgraph.*

Let  $G$  be a fuzzy circular interval graph with a representation  $(\phi, I)$ . If  $[p, q]$  is an interval of  $I$  such that  $\phi^{-1}(p)$  and  $\phi^{-1}(q)$  are both nonempty, then we call the pair  $(\phi^{-1}(p), \phi^{-1}(q))$  a *fuzzy pair*, where  $\phi^{-1}(p)$  denotes the set  $\{v \in V(G) | \phi(v) = p\}$ . Note that  $G[\phi^{-1}(p)]$  and  $G[\phi^{-1}(q)]$  are cliques, and hence the subgraph induced by  $\phi^{-1}(p) \cup \phi^{-1}(q)$  is co-bipartite. The following lemma was proved in [6].

LEMMA 3. *Let  $G$  be a fuzzy circular interval graph with a representation  $(\phi, I)$ . If  $G[\phi^{-1}(p) \cup \phi^{-1}(q)]$  is a  $C_4$ -free subgraph for every fuzzy pair  $(\phi^{-1}(p), \phi^{-1}(q))$ , then  $G$  is a circular interval graph.*

### 3. The main results

THEOREM 4. *The class  $\text{Free}(H) \cap \mathcal{Q}$  is at most factorial if and only if  $\text{Free}(H) \cap \overline{\mathcal{B}}$  is.*

PROOF. The 'only if' part immediately follows from the fact that  $\overline{\mathcal{B}} \subset \mathcal{Q}$ . In order to prove the converse suppose that  $\text{Free}(H) \cap \overline{\mathcal{B}}$  is at most factorial and estimate the number of  $n$ -vertex  $H$ -free quasi-line graphs. Note that Lemma 1 allows us to be restricted to connected graphs only, and hence by Theorem 1 we only need to estimate the number of  $n$ -vertex  $H$ -free fuzzy circular interval graphs and the number of  $n$ -vertex  $H$ -free compositions of fuzzy linear interval strips with a disjoint union of cliques.

PROPOSITION 1. *The class of  $H$ -free fuzzy circular interval graphs is at most factorial.*

Let  $G$  be a graph from the class under consideration with a representation  $(\phi, I)$ , where  $I = \{[a_1, b_1], \dots, [a_k, b_k]\}$  and the pairs  $a_1, b_1, \dots, a_k, b_k$  are distinct. For

each interval  $[a_i, b_i] \in I$  remove from  $G$  all edges  $(a, b)$  such that  $a \in \phi^{-1}(a_i)$ ,  $b \in \phi^{-1}(b_i)$  and denote the resulting graph by  $G_0$ . Obviously for  $i = 1, \dots, k$  we have that  $G_0[\phi^{-1}(a_i) \cup \phi^{-1}(b_i)]$  is  $C_4$ -free and hence from Lemma 3 we can deduce that  $G_0$  is a circular interval graph. By  $G_i$ , where  $i = 1, \dots, k$ , denote graphs  $G[\phi^{-1}(a_i) \cup \phi^{-1}(b_i)]$ . It is clear that  $G_i \in \text{Free}(H) \cap \overline{\mathcal{B}}$ ,  $i = 1, \dots, k$ , and that the graph  $G$  is a union of  $G_0, G_1, \dots, G_k$ . As the class of circular interval graphs is factorial [8],  $\text{Free}(H) \cap \overline{\mathcal{B}}$  is at most factorial (by assumption), and every vertex of  $G$  is covered by at most 2 graphs. By Lemma 1, the class of  $H$ -free fuzzy circular interval graphs is at most factorial.

PROPOSITION 2. *The number of fuzzy linear interval strips  $(G, a, b)$ , where  $|V(G)| = n$  and  $G \setminus \{a, b\}$  is  $H$ -free, is at most  $n^{cn}$ , where  $c$  is a constant independent of  $n$ .*

$G$  is a union of three graphs:  $G \setminus \{a, b\}$ ,  $G[a \cup N(a)]$  and  $G[b \cup N(b)]$ . The first of them belongs to the class of  $H$ -free fuzzy linear interval graphs, which is at most factorial since it is a subclass of the class of  $H$ -free fuzzy circular interval graphs. The remaining two graphs belong to the class of complete graphs. Using Lemma 1, it is easy to conclude the proof.

PROPOSITION 3. *The number of  $n$ -vertex  $H$ -free compositions of fuzzy linear interval strips with disjoint unions of cliques is at most  $n^{c'n}$ , where  $c'$  is a constant independent of  $n$ .*

Let  $G$  be an  $n$ -vertex  $H$ -free composition of strips  $(G_i, a_i, b_i)$ ,  $1 \leq i \leq k$ , with a disjoint union of cliques. Without loss of generality we may assume that each gluing adds at least one vertex to the resulting composition, and hence for every composing strip  $(G_i, a_i, b_i)$  we have  $|V(G_i)| \geq 3$ .

Any  $n$ -vertex composition of  $k$  strips is completely determined by a  $2k$ -vertex labeled graph  $G_0$ , which is a disjoint union of cliques, given together with a partition of its vertices into pairs  $\{a_1, b_1, \dots, a_k, b_k\}$ , and an ordered set of  $k$  labeled strips  $(G_i, a_i, b_i)$  ( $i = 1, \dots, k$ ). Note that  $k \leq n$ , since each gluing adds at least one vertex to the resulting graph. Denote  $|V(G_i)|$  by  $n_i = t_i + 2$ , where  $t_i$  is the number of vertices the  $i$ -th strip contributes to the resulting graph. Therefore,  $\sum_{i=1}^k t_i = n$ ,

and hence  $\sum_{i=1}^k n_i = n + 2k \leq 3n$ .

Suppose that we have been given a fixed ordered partition  $S_1 \cup \dots \cup S_k$  of  $V = \{1, \dots, n\}$  such that  $V(G_i) = S_i \cup \{n + 1, n + 2\}$ . The labels  $n + 1$  and  $n + 2$

are used for the endpoint vertices in each strip which are deleted in the process of gluing. Since  $G$  is  $H$ -free, by Corollary 1 all graphs  $G_i \setminus \{a_i, b_i\}$  ( $i = 1, \dots, k$ ) are also  $H$ -free, and Claim 2 implies that the number of different strips  $(G_i, a_i, b_i)$  is at most  $n_i^{cn_i}$ . Therefore, the number of different compositions with the above partition does not exceed

$$(2k)^{4k} \prod_{i=1}^k n_i^{cn_i} \leq (2n)^{4n} n^{\sum_{i=1}^k cn_i} \leq n^{11cn},$$

where  $(2k)^{4k}$  is an upper bound on the number of different labeled  $G_0$  graphs with their vertices partitioned into pairs. The number of different ordered partitions of  $V$  into  $k$  sets does not exceed  $k^n$ , and therefore the total number of labeled  $n$ -vertex  $H$ -free compositions of fuzzy linear interval strips with disjoint unions of cliques is bounded by

$$\sum_{k=1}^n n^{11cn} k^n \leq n^{(11c+1)n+1} < n^{c'n}$$

for some constant  $c'$ .

From Theorem 1, Claim 1, and Claim 3 it follows that the class of  $H$ -free quasi-line graphs is at most factorial. □

Using Theorem 3, Theorem 4, and the fact that  $\overline{B} \subset \mathcal{Q}$ , we can now state a theorem similar to Theorem 3:

**THEOREM 5.** *If a graph  $H$  is not isomorphic to any induced subgraph of  $\overline{T_{1,2,3}}$ ,  $\overline{\Phi_{p,q} + O_1}$ , or  $\overline{P_7}$ , then the class  $\text{Free}(H) \cap \mathcal{Q}$  is superfactorial. If  $H = \overline{T_{1,2,3}}$  or  $H = \overline{\Phi_{p,q} + O_1}$  for some natural  $p$  and  $q$ , then  $\text{Free}(H) \cap \mathcal{Q}$  is factorial.*

It is easy to show that for every co-bipartite graph  $H$  with at least three vertices the class  $\text{Free}(H) \cap \mathcal{Q}$  contains at least one of the two classes,  $\text{Free}(K_3, K_{1,2})$  and  $\overline{\text{Free}(K_3, K_{1,2})}$ , which are factorial [2]. Hence, for every co-bipartite graph  $H$  with at least three vertices the class  $\text{Free}(H) \cap \mathcal{Q}$  is at least factorial. This allows us to answer the question (1) for any graph distinct from  $\overline{P_7}$ .

**THEOREM 6.** *Let  $H$  be a graph with at least three vertices such that  $H$  is not isomorphic to  $\overline{P_7}$ . Then  $\text{Free}(H) \cap \mathcal{Q}$  is factorial if and only if  $H$  is isomorphic to some induced subgraph of  $\overline{T_{1,2,3}}$  or  $\overline{\Phi_{p,q} + O_1}$  for some natural  $p$  and  $q$ .*

Theorem 6 allows us to generalize a result from [8] which states that the class  $\text{Free}(K_{1,3}, C_4)$  is factorial.

**THEOREM 7.** *The class  $\text{Free}(K_{1,3}, W_4)$  is factorial.*

**PROOF.** Using exactly the same argument as for the class  $\text{Free}(K_{1,3}, C_4)$  [8] (see Theorem 7 and Lemma 3), we can reduce the original question to the class of  $W_4$ -free quasi-line graphs. In other words, if the class of  $W_4$ -free quasi-line graphs is at most factorial, then  $\text{Free}(K_{1,3}, W_4)$  is factorial.  $W_4$  is an induced subgraph of  $\overline{\Phi_{1,1}} + O_1$ , and thus by Theorem 6 the class of  $W_4$ -free quasi-line graphs is factorial.  $\square$

## 4. Open problems

The question whether the class  $\text{Free}(\overline{P_7}) \cap \mathcal{Q}$  is a member of the factorial layer is still open. By Theorem 4, in order to answer this question we only need to consider the smaller class of  $\overline{P_7}$ -free co-bipartite graphs (or the class of  $P_7$ -free bipartite graphs).

Studying generalizations of quasi-line graphs could be another interesting research direction. Let  $k$  be a fixed natural number. Denote by  $\mathcal{L}_k$  the class of such graphs that the set of neighbors of every vertex can be expressed as a union of  $k$  cliques. We believe that a statement similar to Theorem 5 is also true for the classes  $\mathcal{L}_k$ ,  $k \in \mathbb{N}$

**CONJECTURE 1.** Let  $k \in \mathbb{N}$ . If a graph  $H$  is not isomorphic to any induced subgraph of  $\overline{T_{1,2,3}}$ ,  $\overline{\Phi_{p,q}} + O_1$ , or  $\overline{P_7}$ , then the class  $\text{Free}(H) \cap \mathcal{L}_k$  is superfactorial. If  $H = \overline{T_{1,2,3}}$  or  $H = \overline{\Phi_{p,q}} + O_1$  for some natural  $p$  and  $q$ , then  $\text{Free}(H) \cap \mathcal{L}_k$  is factorial.

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