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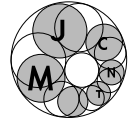
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A numerically explicit version of the Pólya–Vinogradov inequality

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Abstract: A new numerically explicit version of the Pólya–Vinogradov inequality is proved in the paper. The proof is based on new ideas of V. A. Bykovskii and improves on a recent result obtained by C. Pomerance.

Keywords: Dirichlet character, character sums, Pólya–Vinogradov inequality

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In the paper [13] C. Pomerance proved a new explicit version of the famous Pólya–Vinogradov inequality for character sums. In this paper we present an improvement of Pomerance’s result. Our approach is based on a recent construction due to V. Bykovskii [2], [3]. In [2] and [3] this construction was used to obtain new upper bounds for the discrepancy of good lattice points sets. The present paper is organized as follows. In Section 1 we give a brief survey of classic and recent results on the topic. In Section 2 we formulate our main result. In Section 3 we formulate two lemmas by Pomerance. In Section 4 we describe Bykovskii’s construction. In Section 5 we complete the proof of our main result.

1. Introduction

Let $\chi \pmod{q}$ be a primitive Dirichlet character. Put

$$S_\chi = \max_{0 \leq M < N \leq q} \left| \sum_{n=M}^N \chi(n) \right|, \quad T_\chi = \max_N \left| \sum_{a=0}^N \chi(a) \right|.$$

A character is defined to be even or odd if $\chi(-1) = 1$ or $\chi(-1) = -1$ respectively. In the case of even characters one has

$$S_\chi = 2T_\chi. \quad (1)$$

In 1918 Pólya [12] and Vinogradov [17] independently proved that for any non-principal Dirichlet character the inequality

$$S_\chi \leq c\sqrt{q} \log q \quad (2)$$

holds with an absolute constant c . In 2007 Granville and Soundararajan [6] proved that for every primitive Dirichlet character $\chi \pmod{q}$ of odd order g we have

$$T_\chi \ll \sqrt{q}(\log q)^{1-\delta_g/2+o(1)}, \quad \text{where } \delta_g = 1 - \frac{g}{\pi} \sin \frac{\pi}{g}, \quad q \rightarrow \infty.$$

This result has recently been improved by Goldmakher [5]. He obtained the following estimate:

$$T_\chi \ll \sqrt{q}(\log q)^{1-\delta_g+o(1)}.$$

Under the General Riemann Hypothesis, Montgomery and Vaughan [10, Theorem 3] proved that

$$S_\chi \ll \sqrt{q} \log \log q.$$

In fact, this is the best-possible result. Paley [11] proved that there exists an infinite class of quadratic characters $\chi_n \pmod{q_n}$ such that

$$S_{\chi_n} \gg \sqrt{q_n} \log \log q_n.$$

An important problem is to find the most precise form of the inequality (2). So far, two types of results have been obtained. Results of the first type don't include explicit bounds for the remainder terms, whereas results of the second type define all constants explicitly. As a rule, results of the first type are characterized by having more accurate constants in the main term.

1.1. Asymptotic results

Landau [9] proved that

$$S_\chi \leq \left(\frac{1}{\pi\sqrt{2}} + o(1) \right) \sqrt{q} \log q \quad \text{if } \chi(-1) = 1$$

and

$$S_\chi \leq \left(\frac{1}{2\pi} + o(1) \right) \sqrt{q} \log q \quad \text{if } \chi(-1) = -1.$$

Hildebrand [8] obtained that

$$T_\chi \leq \left(\frac{2}{3\pi^2} + o(1) \right) \sqrt{q} \log q \quad \text{if } \chi(-1) = 1$$

and

$$T_\chi \leq \left(\frac{1}{3\pi} + o(1) \right) \sqrt{q} \log q \quad \text{if } \chi(-1) = -1.$$

Later, Hildebrand [7] improved his result in the case of even characters. He proved the estimate

$$T_\chi \leq \left(\frac{c}{\pi\sqrt{3}} + o(1) \right) \sqrt{q} \log q \quad \text{for } \chi(-1) = 1,$$

where

$$c = \begin{cases} \frac{1}{4}, & \text{if } q \text{ is cubefree;} \\ \frac{1}{3}, & \text{otherwise.} \end{cases}$$

Granville and Soundararajan [6] obtained the following two inequalities:

$$T_\chi \leq \left(\frac{69}{70} \frac{c}{\pi\sqrt{3}} + o(1) \right) \sqrt{q} \log q \quad \text{if } \chi(-1) = 1$$

and

$$T_\chi \leq \left(\frac{c}{\pi} + o(1) \right) \sqrt{q} \log q \quad \text{if } \chi(-1) = -1.$$

At the moment of writing, this is the best known asymptotic estimate.

1.2. Numerically explicit results

In this section we discuss numerically explicit versions of the Pólya–Vinogradov inequality. Qiu [14] proved that

$$S_\chi \leq \frac{4}{\pi^2} \sqrt{q} \log q + 0.38 \sqrt{q} + 0.608 \frac{1}{\sqrt{q}} + 0.116 \sqrt{q}.$$

Simalarides [15], [16] obtained the estimates

$$T_\chi \leq \frac{3}{4\pi} \sqrt{q} \log q + \left(2 - \frac{\log 2}{\pi} - \frac{\gamma}{2\pi} \right) \quad \text{if } \chi(-1) = 1$$

and

$$T_\chi \leq \frac{1}{\pi} \sqrt{q} \log q + \sqrt{q} + \frac{1}{2} \quad \text{if } \chi(-1) = -1.$$

Dobrowolski and Williams [4] proved that for any non-principal Dirichlet character $\chi \pmod{q}$ one has

$$S_\chi \leq \frac{1}{2 \log 2} \sqrt{q} \log q + 3 \sqrt{q}.$$

Bachman and Rachakonda [1] improved their result, proving that

$$S_\chi \leq \frac{1}{3 \log 3} \sqrt{q} \log q + 6.5 \sqrt{q}$$

for any non-principal Dirichlet character $\chi \pmod{q}$. In [13] Pomerance proved that

$$S_\chi \leq \frac{2}{\pi^2} \sqrt{q} \log q + \frac{4}{\pi^2} \sqrt{q} \log \log q + \frac{3}{2} \sqrt{q} \quad \text{if } \chi(-1) = 1 \quad (3)$$

and

$$S_\chi \leq \frac{1}{2\pi} \sqrt{q} \log q + \frac{1}{\pi} \sqrt{q} \log \log q + \sqrt{q} \quad \text{if } \chi(-1) = -1. \quad (4)$$

To be more precise, Pomerance showed that

$$S_\chi \leq \frac{2}{\pi^2} \sqrt{q} \log q + \frac{4}{\pi^2} \sqrt{q} \log \log q + \frac{4}{\pi^2} \sqrt{q} \left(\log 4 + \gamma + 1 + \frac{3}{n} + \frac{\log\left(\frac{4}{\pi}\right)}{\log q} + \frac{\pi^2}{12q^2 \log q} \right) + 1 \quad \text{if } \chi(-1) = 1, \quad (5)$$

and

$$\begin{aligned}
 S_\chi &\leq \frac{1}{2\pi}\sqrt{q}\log q + \frac{1}{\pi}\sqrt{q}\log\log q + \\
 &+ \frac{1}{\pi}\sqrt{q}\left(\log\left(\frac{4}{\pi}\right) + \gamma + \log 2 + 1 + \frac{3}{m} + \frac{\log\left(\frac{4}{\pi}\right)}{\log q} + \frac{\pi^2}{12q^2\log q}\right) + 1 \\
 &\text{if } \chi(-1) = -1, \tag{6}
 \end{aligned}$$

where $n = \lfloor 2\sqrt{q}\log q \rfloor$, $m = \lfloor (4/\pi)\sqrt{q}\log q \rfloor$ and q is big enough (see [13]). Until now, these bounds have been the best known numerically explicit versions of the Pólya–Vinogradov inequality.

2. The main result

We prove the following theorem, improving (3) and (4) in the second term.

THEOREM 1. *Let $\chi \pmod q$ be a primitive character. Then*

1. *If $\chi(-1) = 1$, we have*

$$S_\chi \leq \frac{2}{\pi^2}\sqrt{q}\log q + \frac{4}{\pi^2}\sqrt{q}(1 + \gamma + \log C_0) + \psi_1(q),$$

2. *If $\chi(-1) = -1$, we have*

$$S_\chi \leq \frac{1}{2\pi}\sqrt{q}\log q + \frac{1}{\pi}\sqrt{q}\left(1 + \gamma + \log \frac{2C_0}{\pi}\right) + \psi_2(q),$$

where γ is the Euler constant, $C_0 = 4\pi^{5/2} + 5$, and

$$\psi_1(q) = 1 + \frac{24}{\pi^2 C_0} + \frac{8}{\pi^2} \frac{\sqrt{q}}{\exp\left(\frac{4\pi\sqrt{q}}{C_0}\right) - 1}, \quad \psi_2(q) = 1 + \frac{3}{C_0} + \frac{2}{\pi} \frac{\sqrt{q}}{\exp\left(\frac{2\pi^2\sqrt{q}}{C_0}\right) - 1}.$$

Simple calculations show that for $q > \exp(C_0/4) \approx 1.38 \cdot 10^8$ the first and the second case of Theorem 1 improve on (5) and (6) respectively. The proof of this theorem is based on the ideas of Bykovskii [2] and Pomerance [13].

3. Pomerance's lemmas

In this section we formulate some results from Pomerance's paper [13].

LEMMA 1. *For all real numbers x and positive integers n we have*

$$\sum_{j=1}^n \frac{|\sin jx|}{j} < \frac{2}{\pi} \log n + \frac{2}{\pi} \left(\gamma + \log 2 + \frac{3}{n} \right).$$

PROOF. See [13, Lemma 3]. □

LEMMA 2. *For all real numbers α, β and positive integers n we have*

$$\sum_{m=1}^n \frac{1}{m} |\cos m\alpha - \cos m\beta| < \log n + \gamma + \log 2 + \frac{3}{n}.$$

PROOF. See [13, §4]. □

4. Bykovskii's method

In this section we describe the main construction from the paper [2], and then present our own modification of this construction. Let $\theta : [0, \infty) \rightarrow [0, 1]$ be the following function:

$$\theta(t) = \begin{cases} 0, & \text{if } 0 < t \leq \frac{1}{2}; \\ 2t - 1, & \text{if } \frac{1}{2} \leq t \leq 1; \\ 2 - t, & \text{if } 1 \leq t \leq 2; \\ 0, & \text{if } 2 \leq t. \end{cases}$$

For all $t \in (0, \infty)$ we have

$$\sum_{j=-\infty}^{\infty} \theta\left(\frac{t}{2^j}\right) = 1.$$

Let $\omega, \omega^* : (0, \infty) \rightarrow [0, 1]$ be defined as follows:

$$\omega(t) = \sum_{j=0}^{\infty} \theta\left(\frac{t}{2^j}\right) = \begin{cases} 0, & \text{if } 0 < t \leq \frac{1}{2}; \\ 2t - 1, & \text{if } \frac{1}{2} \leq t \leq 1; \\ 1, & \text{if } 1 \leq t, \end{cases}$$

$$\omega^*(t) = 1 - \omega(t) = \sum_{j=-\infty}^{-1} \theta\left(\frac{t}{2^j}\right) = \begin{cases} 1, & \text{if } 0 < t \leq \frac{1}{2}; \\ 2 - 2t, & \text{if } \frac{1}{2} \leq t \leq 1; \\ 0, & \text{if } 1 \leq t. \end{cases}$$

For any positive P and P' put

$$S(u; P, P') = \sum_{m=1}^{\infty} \frac{1}{m} \omega\left(\frac{m}{P}\right) \omega^*\left(\frac{m}{P'}\right) \sin 2\pi mu.$$

If $2P \leq P'$, we can write

$$S(u; P, P') = S(u; P) - S(u; P'),$$

where

$$S(u; P) = \sum_{m=1}^{\infty} \frac{1}{m} \omega\left(\frac{m}{P}\right) \sin 2\pi mu. \tag{7}$$

For any positive P define

$$G_P(u) = \sum_{n=-\infty}^{\infty} \frac{1}{1 + P^2(n + u)^2} = \frac{\pi}{P} \sum_{n=-\infty}^{\infty} \exp\left(-2\pi \frac{|n|}{P}\right) e(nu), \tag{8}$$

where $e(u) = \exp(2\pi iu)$. Bykovskii proved (see [2, Lemma 1]) that

$$|S(u; P)| \leq 14G_P(u).$$

We are going to improve on this result.

LEMMA 3. For any $u \in [0; 1]$ and $P > 1$ the inequality

$$|S(u; P)| \leq \frac{C_0}{2\pi^2} G_P(u)$$

is satisfied.

PROOF. Take $A \in [0; 1]$. The optimal value of the parameter A will be computed later. We are going to consider the following two cases separately.

1. Let $P\|u\| \leq A$.

We can write

$$|\sin 2\pi m u| \leq 2\pi \|m u\| \tag{9}$$

and

$$\left| \sum_{m=m_1}^{m_2} \sin 2\pi m u \right| \leq \frac{1}{2\|u\|}. \tag{10}$$

The proof of (10) can be found in [2, Lemma 1]. By using partial summation and applying (10) we obtain

$$\left| \sum_{m>B/\|u\|} \frac{\sin 2\pi m u}{m} \right| \leq \frac{1}{2\|u\|} \frac{\|u\|}{B} \leq \frac{1}{2B}. \tag{11}$$

Let $B \in [A; 1]$. The optimal value of the parameter B will be calculated later. We can see that

$$P \leq \frac{A}{\|u\|} \leq \frac{B}{\|u\|}.$$

Therefore, from (7), (10), and (11) we have

$$|S(u; P)| \leq \sum_{m \leq B/\|u\|} \frac{|\sin 2\pi m u|}{m} + \left| \sum_{m > B/\|u\|} \frac{\sin 2\pi m u}{m} \right| \leq \sum_{m \leq B/\|u\|} \frac{2\pi \|m u\|}{m} + \frac{1}{2B}.$$

If $B \leq 1/2$, then we have

$$|S(u; P)| \leq 2\pi B + \frac{1}{2B},$$

and otherwise

$$\begin{aligned}
 |S(u; P)| &\leq 2\pi \frac{1}{2} + 2\pi \sum_{1/2\|u\| < m \leq B/\|u\|} \frac{1 - mu}{m} + \frac{1}{2B} \leq \\
 &\leq 2\pi \left(\frac{1}{2} + \log 2B - \left(B - \frac{1}{2} \right) \right) + \frac{1}{2B} = 2\pi \left(1 - B + \log 2B \right) + \frac{1}{2B}.
 \end{aligned}$$

By the definition of the function $G_P(u)$ we have

$$G_P(u) \geq \frac{1}{1 + P^2\|u\|^2} \geq \frac{1}{1 + A^2}.$$

Thus, we can write

$$|S(u; P)| \leq C_1(1 + A^2)G_P(u),$$

where

$$C_1 = \begin{cases} 2\pi B + \frac{1}{2B}, & \text{if } A \leq B \leq \frac{1}{2}; \\ 2\pi(1 - B + \log 2B) + \frac{1}{2B}, & \text{if } \max\left(\frac{1}{2}, A\right) \leq B \leq 1. \end{cases}$$

2. Let $P\|u\| > A$.

The proof is going to be similar to the proof of the second case of Lemma 2 in [2], and therefore we are going to use the relevant notation from [2]. Lemma 1 of [2] states that for any $P' > 2P$ we have

$$|S(u; P, P')| = \frac{1}{2} \left| \sum_{n=-\infty}^{\infty} (\eta(n - \|u\|; P, P') - \eta(n + \|u\|; P, P')) \right|,$$

where

$$\eta(w; P, P') = \int_{P/2}^{P'} \omega\left(\frac{x}{P}\right) \omega^*\left(\frac{x}{P'}\right) e(-wx) \frac{dx}{x}.$$

Applying the ideas of this lemma, we obtain

$$|\eta(w; P, P')| \leq \frac{1}{4\pi^2 w^2} \left(\frac{10}{P^2} + \frac{16}{P'^2} \right).$$

Therefore, we can write

$$|S(u; P, P')| \leq \sum_{n=-\infty}^{\infty} \frac{1}{4\pi^2(n+u)^2} \left(\frac{10}{P^2} + \frac{16}{P'^2} \right),$$

which leads to

$$|S(u; P)| \leq \frac{5}{2\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{P^2(n+u)^2}.$$

For any integer n we have

$$P^2(n+u)^2 > \frac{A^2}{1+A^2} (1+P^2(n+u)^2).$$

This leads to the following estimate:

$$|S(u; P)| \leq \frac{5}{2\pi^2} \left(1 + \frac{1}{A^2} \right) \sum_{n=-\infty}^{\infty} \frac{1}{1+P^2(n+u)^2} < \frac{5}{2\pi^2} \left(1 + \frac{1}{A^2} \right) G_P(u).$$

It is easy to prove that

$$f_1(A) = C_1(1+A^2)$$

is an increasing function if $A \in [0; 1]$. Therefore, we should use the value of A that satisfies the equation

$$f_1(A) = \frac{5}{2\pi^2} \left(1 + \frac{1}{A^2} \right).$$

If $A \leq \frac{1}{2\sqrt{\pi}}$, then

$$\min_{A \leq B \leq 1/2} \left(2\pi B + \frac{1}{2B} \right) = 2\sqrt{\pi},$$

and therefore

$$f_1(A) = 2\sqrt{\pi}(1+A^2), \quad 0 \leq A \leq \frac{1}{2\sqrt{\pi}}.$$

Solving the system of equations

$$\begin{cases} 2\sqrt{\pi}(1+A^2) = \frac{5}{2\pi^2} \left(1 + \frac{1}{A^2} \right) \\ 0 \leq A \leq \frac{1}{2\sqrt{\pi}} \end{cases} \quad (12)$$

yields

$$A = \frac{\sqrt{5}}{2\pi^{5/4}}.$$

Substituting this value of A in the estimates that have been obtained earlier leads to

$$|S(u; P)| \leq f_1 \left(\frac{\sqrt{5}}{2\pi^{5/4}} \right) G_P(u) = \left(2\sqrt{\pi} + \frac{5}{2\pi^2} \right) G_P(u),$$

concluding the proof. □

LEMMA 4. For any $q \in \mathbb{N}$ and $P > 1$ we have

$$\sum_{a=1}^q G_P \left(\frac{a}{q} \right) = \pi \frac{q}{P} + 2\pi \frac{q}{P} \frac{1}{\exp \left(2\pi \frac{q}{P} \right) - 1}.$$

PROOF. Let us take

$$\delta_q(a) = \frac{1}{q} \sum_{x=1}^q e \left(\frac{ax}{q} \right) = \begin{cases} 1, & \text{if } a \equiv 0 \pmod{q}; \\ 0, & \text{else,} \end{cases} \tag{13}$$

and then apply (8) to write

$$\begin{aligned} \sum_{a=1}^q G_P \left(\frac{a}{q} \right) &= \frac{\pi}{P} \sum_{a=1}^q \sum_{n=-\infty}^{\infty} \exp \left(-2\pi \frac{|n|}{P} \right) e \left(n \frac{a}{q} \right) = \\ &= \frac{\pi}{P} \sum_{n=-\infty}^{\infty} \exp \left(-2\pi \frac{|n|}{P} \right) q \delta_q(n) = \\ &= \pi \frac{q}{P} \left(1 + 2 \sum_{n=1}^{\infty} \exp \left(-2\pi \frac{n}{P} \right) \delta_q(n) \right) = \\ &= \pi \frac{q}{P} \left(1 + 2 \sum_{n=1}^{\infty} \exp \left(-2\pi \frac{qn}{P} \right) \right) = \pi \frac{q}{P} \left(1 + 2 \frac{\exp \left(-2\pi \frac{q}{P} \right)}{1 - \exp \left(-2\pi \frac{q}{P} \right)} \right), \end{aligned}$$

proving the lemma. □

5. Proof of Theorem 1

Let $0 \leq a < b \leq 1$, and let $\lambda(x; a, b)$ be a function defined on $[0; 1)$ as

$$\lambda(x; a, b) = \begin{cases} \frac{1}{2}, & \text{if } x = a; \\ 1, & \text{if } a < x < b; \\ \frac{1}{2}, & \text{if } x = b; \\ 0, & \text{otherwise.} \end{cases}$$

Its Fourier expansion can be written as

$$\lambda(x; a, b) = b - a + \frac{1}{\pi}S(x - a) - \frac{1}{\pi}S(x - b),$$

where

$$S(x) = \sum_{m=1}^{\infty} \frac{\sin 2\pi mx}{m}.$$

For all $0 \leq M < N \leq q$ we can write

$$\begin{aligned} \sum_{a=M}^N \chi(a) &= \sum_{a=1}^q \chi(a) \lambda\left(\frac{a}{q}; \frac{M}{q}, \frac{N}{q}\right) + \frac{\chi(M) + \chi(N)}{2} = \\ &= \frac{1}{\pi} \sum_{a=1}^q \chi(a) \left(S\left(\frac{a-M}{q}\right) - S\left(\frac{a-N}{q}\right) \right) + \frac{\chi(M) + \chi(N)}{2}. \end{aligned} \quad (14)$$

The Gauss sum $\tau(\chi)$ is defined as

$$\tau(\chi) = \sum_{a=1}^q \chi(a) e\left(\frac{a}{q}\right).$$

For a primitive Dirichlet character $\chi \pmod{q}$ we have

$$|\tau(\chi)| = \sqrt{q}. \quad (15)$$

It has been proved by Pomerance (see equation (10) in [13]) that

$$\bar{\chi}(m)\tau(\chi) = \begin{cases} \sum_{a=1}^q \chi(a) \cos \frac{2\pi am}{q}, & \text{if } \chi \text{ is even;} \\ i \sum_{a=1}^q \chi(a) \sin \frac{2\pi am}{q}, & \text{if } \chi \text{ is odd.} \end{cases} \tag{16}$$

5.1. The case of even characters

It follows from (14) that

$$\sum_{a=1}^N \chi(a) = \frac{1}{\pi} \sum_{a=1}^q \chi(a) \left(S\left(\frac{a}{q}\right) - S\left(\frac{a-N}{q}\right) \right) + \frac{\chi(N)}{2}.$$

Let $P > 0$. By the definition of the functions $\omega(t)$ and $\omega^*(t)$, we have

$$\begin{aligned} S(u) &= \sum_{m \leq P} \frac{\sin 2\pi mu}{m} \omega^*\left(\frac{m}{P}\right) + \sum_{m=1}^{\infty} \frac{\sin 2\pi mu}{m} \omega\left(\frac{m}{P}\right) = \\ &= \sum_{m \leq P} \frac{\sin 2\pi mu}{m} \omega^*\left(\frac{m}{P}\right) + S(u; P). \end{aligned} \tag{17}$$

Hence, we can write

$$\begin{aligned} \sum_{a=1}^N \chi(a) &= \frac{1}{\pi} \sum_{a=1}^q \chi(a) \sum_{m \leq P} \frac{\sin 2\pi m \frac{a}{q} - \sin 2\pi m \frac{a-N}{q}}{m} \omega^*\left(\frac{m}{P}\right) + \\ &+ \frac{1}{\pi} \sum_{a=1}^q \chi(a) \left(S\left(\frac{a}{q}; P\right) - S\left(\frac{a-N}{q}; P\right) \right) + \frac{\chi(N)}{2}. \end{aligned} \tag{18}$$

From (16) we also have

$$\begin{aligned} \frac{1}{\pi} \sum_{a=1}^q \chi(a) \sum_{m \leq P} \frac{\sin 2\pi m \frac{a}{q} - \sin 2\pi m \frac{a-N}{q}}{m} \omega^*\left(\frac{m}{P}\right) &= \\ &= \frac{1}{\pi} \sum_{m \leq P} \frac{\bar{\chi}(m)\tau(\chi)}{m} \sin 2\pi m \frac{N}{q} \omega^*\left(\frac{m}{P}\right), \end{aligned}$$

and (15) implies that

$$\left| \frac{1}{\pi} \sum_{a=1}^q \chi(a) \sum_{m \leq P} \frac{\sin 2\pi m \frac{a}{q} - \sin 2\pi m \frac{a-N}{q}}{m} \omega^* \left(\frac{m}{P} \right) \right| \leq \frac{\sqrt{q}}{\pi} \sum_{m \leq P} \frac{|\sin 2\pi m \frac{N}{q}|}{m}. \quad (19)$$

From (19) and Lemma 3 it follows that

$$\left| \sum_{a=1}^N \chi(a) \right| \leq \frac{\sqrt{q}}{\pi} \sum_{m \leq P} \frac{|\sin 2\pi m \frac{N}{q}|}{m} + \frac{2}{\pi} \sum_{a=1}^q \frac{C_0}{2\pi^2} G_P \left(\frac{a}{q} \right) + \frac{1}{2}. \quad (20)$$

By Lemma 1 and Lemma 4 we have

$$\begin{aligned} \left| \sum_{a=1}^N \chi(a) \right| &\leq \frac{2}{\pi^2} \sqrt{q} \left(\log P + \gamma + \log 2 + \frac{3}{P} \right) + \\ &+ 2 \frac{C_0}{2\pi^2} \frac{q}{P} + 2 \frac{C_0}{2\pi^2} \frac{q}{P} \frac{2}{\exp \left(2\pi \frac{q}{P} \right) - 1} + \frac{1}{2}. \end{aligned} \quad (21)$$

Let us take $P = \frac{C_0}{2} \sqrt{q}$, obtaining

$$\begin{aligned} \left| \sum_{a=1}^N \chi(a) \right| &\leq \frac{1}{\pi^2} \sqrt{q} \log q + \frac{2}{\pi^2} \sqrt{q} \left(1 + \gamma + \log C_0 \right) + \\ &+ \frac{12}{\pi^2 C_0} + \frac{4}{\pi^2} \frac{\sqrt{q}}{\exp \left(\frac{4\pi\sqrt{q}}{C_0} \right) - 1} + \frac{1}{2}. \end{aligned} \quad (22)$$

Now applying (1) proves Theorem 1 in the case of even characters.

5.2. The case of odd characters

From (14) and (17) we have that

$$\begin{aligned} \sum_{a=M}^N \chi(a) &= \frac{1}{\pi} \sum_{a=1}^q \chi(a) \sum_{m \leq P} \frac{\sin 2\pi m \frac{a-M}{q} - \sin 2\pi m \frac{a-N}{q}}{m} \omega^* \left(\frac{m}{P} \right) + \\ &+ \frac{1}{\pi} \sum_{a=1}^q \chi(a) \left(S \left(\frac{a-M}{q}; P \right) - S \left(\frac{a-N}{q}; P \right) \right) + \frac{\chi(M) + \chi(N)}{2}. \end{aligned} \quad (23)$$

Applying (16) yields

$$\begin{aligned} & \frac{1}{\pi} \sum_{a=1}^q \chi(a) \sum_{m \leq P} \frac{\sin 2\pi m \frac{a-M}{q} - \sin 2\pi m \frac{a-N}{q}}{m} \omega^* \left(\frac{m}{P} \right) = \\ & = -\frac{i}{\pi} \sum_{m \leq P} \frac{\bar{\chi}(m) \tau(\chi)}{m} \left(\cos 2\pi m \frac{M}{q} - \cos 2\pi m \frac{N}{q} \right) \omega^* \left(\frac{m}{P} \right), \end{aligned}$$

and taking into account (15) and Lemma 2 allows us to write

$$\begin{aligned} & \left| \frac{1}{\pi} \sum_{a=1}^q \chi(a) \sum_{m \leq P} \frac{\sin 2\pi m \frac{a-M}{q} - \sin 2\pi m \frac{a-N}{q}}{m} \omega^* \left(\frac{m}{P} \right) \right| \leq \\ & \leq \frac{\sqrt{q}}{\pi} \sum_{m \leq P} \frac{1}{m} \left| \cos 2\pi m \frac{M}{q} - \cos 2\pi m \frac{N}{q} \right| \leq \\ & \leq \frac{\sqrt{q}}{\pi} \left(\log P + \gamma + \log 2 + \frac{3}{P} \right). \end{aligned} \tag{24}$$

Thus, from (23), (24), and Lemma 3 we have

$$\left| \sum_{a=M}^N \chi(a) \right| \leq \frac{\sqrt{q}}{\pi} \left(\log P + \gamma + \log 2 + \frac{3}{P} \right) + \frac{2}{\pi} \sum_{a=1}^q \frac{C_0}{2\pi^2} G_P \left(\frac{a}{q} \right) + 1.$$

Applying Lemma 4 yields

$$\begin{aligned} & \left| \sum_{a=M}^N \chi(a) \right| \leq \frac{\sqrt{q}}{\pi} \left(\log P + \gamma + \log 2 + \frac{3}{P} \right) + \\ & + 2 \frac{C_0}{2\pi^2} \frac{q}{P} + 2 \frac{C_0}{2\pi^2} \frac{q}{P} \frac{2}{\exp \left(2\pi \frac{q}{P} \right) - 1} + 1. \end{aligned}$$

Now let us take $P = \frac{C_0}{\pi} \sqrt{q}$, obtaining

$$\begin{aligned} & \left| \sum_{a=M}^N \chi(a) \right| \leq \frac{1}{2\pi} \sqrt{q} \log q + \frac{1}{\pi} \sqrt{q} \left(1 + \gamma + \log \frac{2C_0}{\pi} \right) + \\ & + \frac{3}{C_0} + \frac{2}{\pi} \frac{\sqrt{q}}{\exp \left(\frac{2\pi^2 \sqrt{q}}{C_0} \right) - 1} + 1. \end{aligned}$$

This concludes the proof of Theorem 1. \square

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