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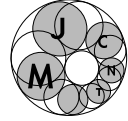
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On the sum of the digits of multiples

Antal Balog (Budapest), Cecile Dartyge (Nancy)

Abstract: Let $s_g(n)$ be the sum of digits function of n in base g . Almost all integers n have a multiple hn with $s_g(hn)$ significantly smaller than $s_g(n)$. We consider the opposite direction. For any fixed $K \geq 1$ we give a lower bound for the frequency of those integers n satisfying

$$s_g(hn) \geq \frac{1}{K} s_g(n)$$

for all integers $h \geq 1$.

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1. Introduction

Let $g \geq 2$ be an integer and $s_g(n)$ denote the sum of digits in base g of the positive integer n . The relationship between $s_g(n)$ and $s_g(hn)$ raises interesting questions. We introduce the function

$$F(n) := \min_{h \geq 1} s_g(hn).$$

Clearly $F(n) \leq s_g(n)$. In fact in [1] it is proved that if n is not a power of g then there exists $h \notin \{g^k : k \in \mathbb{N}\}$ such that $s_g(hn) = s_g(n)$. On the other hand, $F(n)$ is much smaller than $s_g(n)$ in many cases, for example, if n is a prime and g is a quadratic non-residue modulo n , then by Euler lemma $g^{(n-1)/2} + 1 \equiv 0 \pmod{n}$, that is $F(n) = 2$. It is somewhat harder to find examples for $F(n)$ big. A natural question is to study the following set:

$$S_g(x) := \{n \leq x : F(n) = s_g(n)\},$$

and more generally, for any given $K \geq 1$,

$$S_g(x, K) := \left\{ n \leq x : F(n) \geq \frac{1}{K} s_g(n) \right\}.$$

In [1] it is proved that these sets are small, more precisely:

$$\left| \left\{ n \leq x : F(n) \geq \frac{1}{K} s_g(n) \right\} \right| \ll \frac{x}{(\log x)^{1/2}}.$$

In 1980, Stolarsky [3] obtained the following lower bound (strictly speaking he only studied the case $g = 2$, but his method worked in general)

$$|S_g(x)| \geq \frac{1}{g} x^{1/2}.$$

Our main objective in this paper is to improve the exponent in the lower bound and to extend the result to the sets $S_g(x, K)$.

THEOREM 1. *For any fixed integers $g \geq 2$, $K \geq 1$ and $k \geq 0$ we have*

$$|S_g(x, K)| \geq \frac{1}{g^{K-1} \binom{g+k}{k+1}} x^\theta,$$

with

$$\theta = \theta_{g,K,k} = \frac{(K-1) \log g + \log \binom{g+k}{k+1}}{(k+K+1) \log g}. \quad (1)$$

In the classical case of $K = 1$ Stolarsky's result corresponds to the choice of $k = 0$, while our method always gives a better exponent. For example, when $g = 2$ we take $k = 1$ to get

$$|S_2(x)| \geq \frac{1}{3} x^\theta, \quad \theta = \frac{\log 3}{3 \log 2} = 0.5283 \dots \quad (2)$$

Other numerical examples are

g	10	100	1000	10^6	10^{12}	10^{24}
θ	0.5856...	0.6645...	0.7213...	0.8119...	0.8808...	0.9277...

Using the straightforward bounds

$$\frac{g^{k+1}}{(k+1)^k} \leq \binom{g+k}{k+1} \leq g^{k+1},$$

the choice $k \leq \log g < k+1$ implies

COROLLARY 1. *Let $g \geq 2$ and $K \geq 1$ be integers. We have*

$$|S_g(x, K)| \geq \frac{1}{g^{K+\log g}} x^{1-(1+\log(1+\log g))/(K+\log g)}.$$

Note that everything is explicit in Corollary 1, allowing us to choose K or g as functions of x . At the end of the paper we will indicate that the above choice of k provides the optimum of the present method, at least when g is big. The same calculation indicates that for $K \gg \log g$, the optimal choice is $k = 0$. In other words our iterative process does not improve upon the direct argument, which is

$$|S_g(x, K)| \geq \frac{1}{g^K} x^{1-1/(K+1)}.$$

A little twist on the iteration, however, improves this to

THEOREM 2. *Let $g \geq 2$ and $K \geq 1$ be integers. We have*

$$|S_g(x, K)| \geq \frac{1}{6(2K-1)((K-1)!)^2 g^{2K}} x^{1-1/(2K)}.$$

We do not make any effort to get the best multiplicative constants in these lower bounds. On the other hand, Theorem 1 reflects our best exponent for $K = 1$, while Theorem 2 reflects our best exponent for $K \gg \log g$. The question raises naturally if it is possible to combine the proofs of the two theorems. There is a straightforward method resulting the new exponent

$$\theta'_{g,K,k} = \frac{(2K-3) \log g + \log \binom{g+k}{k+1}}{(k+2K-1) \log g}, \quad (3)$$

which is superior to (1) for $3 \leq K \ll \log g / \log \log g$.

There is also a more sophisticated method, based on a recent result of Mauduit, Pomerance and Sárközy [2] on the distribution of the values of $s_g(n)$. However, the improvement upon (3) is very small.

The function $s_g(n)$ is the ultimate example of a “q–additive” function. Our arguments may be useful in other situations, and some of our statements may extend to a wider subclass of q–additive functions.

2. Stolarsky’s construction revisited

The sum of digit function, s_g is a sub-additive function, that is, for all $m, n \in \mathbb{N}$

$$s_g(m + n) \leq s_g(m) + s_g(n). \quad (4)$$

Formula (4) is easily obtained if m is of type $m = ug^k$ with $0 \leq u \leq g - 1$. The general case for m is then obtained by iteration. It is also easy to see that there is equality in (4) if and only if “there is no carry in adding m and n ”. Equivalently, we have for $n > m$

$$s_g(n - m) \geq s_g(n) - s_g(m), \quad (5)$$

and there is equality in (5) if and only if no digits of m are bigger than the corresponding digits of n . This situation will be denoted by $n \supseteq m$.

The idea of Stolarsky was to consider the binary sum of digits of the multiples of

$$1 + 2^\ell + \dots + 2^{\ell(r-1)} =: n$$

for some integer $\ell \geq 1$. He proved that $s_2(hn) \geq s_2(n) \forall h \in \mathbb{N}$.

A natural extension of this idea to general basis is to consider the multiples of the integers

$$N_r := (g - 1)(1 + g + \dots + g^r) = g^{r+1} - 1,$$

with $r \geq 1$.

LEMMA 1. *For all $h \geq 1$ we have*

$$s_g(hN_r) \geq s_g(N_r). \quad (6)$$

This lemma follows from the next by induction.

LEMMA 2. *Writing $h = u + vg^{r+1}$, where $0 < u < g^{r+1}$ and $0 \leq v$ we have*

$$s_g(hN_r) \geq s_g((v + 1)N_r). \quad (7)$$

For $1 \leq h \leq g^{r+1}$ we have

$$s_g(hN_r) = s_g(N_r). \quad (8)$$

PROOF. Let $h > 1$. The obvious identity $(h-1)(N_r+1) = (h-1)g^{r+1}$ implies

$$hN_r = N_r - (h-1) + (h-1)g^{r+1} = N_r - (u-1) + (h-1-v)g^{r+1}.$$

Note that $0 < N_r - (u-1) < g^{r+1}$ and that the least significant $r+1$ digits of $(h-1-v)g^{r+1}$ are zero. Thus we have:

$$s_g(hN_r) = s_g(N_r - (u-1)) + s_g(h-1-v). \quad (9)$$

If $v = 0$ then $h-1 = u-1$ and we have by (5) (since $N_r \geq u-1$):

$$s_g(hN_r) = s_g(N_r) - s_g(u-1) + s_g(u-1) = s_g(N_r).$$

This proves the second statement (the case $h = g^{r+1}$ being trivial). If $v \geq 1$ then

$$h-1-v = u + vg^{r+1} - 1 - v = (v+1)N_r - (N_r - (u-1)) \quad (10)$$

and by (5) we find

$$s_g(h-1-v) \geq s_g((v+1)N_r) - s_g(N_r - (u-1)).$$

Inserting this into (9) gives

$$s_g(hN_r) \geq s_g((v+1)N_r).$$

This proves Lemma 2. □

PROOF OF LEMMA 1. The second statement of Lemma 2 proves (6) for $h \leq g^{r+1}$. Since $h = u + vg^{r+1} > v+1$ we can use induction based on the first statement of Lemma 2 when $u \neq 0$, or based on $s_g(hN_r) = s_g(vN_r)$ when $u = 0$. □

Let $\mathcal{H} := \{h \geq 1 : s_g(hN_r) = s_g(N_r)\}$. The importance of \mathcal{H} is reflected in the fact that for any $h \in \mathcal{H}$ and for any $m \geq 1$ we have by Lemma 1

$$s_g(mhN_r) \geq s_g(N_r) = s_g(hN_r),$$

that is $hN_r \in S_g(x)$, whenever $hN_r \leq x$. We proved in Lemma 2 that $h \in \mathcal{H}$ if $1 \leq h \leq g^{r+1}$. This observation covers Stolarsky's argument.

3. How to go further?

For simpler exposition we introduce a variant of the relation \triangleright . We write $n \triangleright m \pmod{g^{r+1}}$ to express that the least significant $r+1$ digits of n are not smaller than the corresponding digits of m . That is if the g -basis expansion of n and m are

$$n = n_0 + n_1g + n_2g^2 + \dots, \quad m = m_0 + m_1g + m_2g^2 + \dots,$$

then

$$n_j \geq m_j, \quad \text{for } j = 0, \dots, r.$$

LEMMA 3. *Let $h = u + vg^{r+1}$, where $1 \leq u < g^{r+1}$ and $0 \leq v$. If $v+1 \in \mathcal{H}$ and $u-1 \triangleright v \pmod{g^{r+1}}$, then $h \in \mathcal{H}$.*

PROOF. By (9) we have

$$s_g(hN_r) = s_g(N_r - (u-1)) + s_g(h-1-v).$$

By (10) we have

$$s_g(h-1-v) = s_g(u-1 + vN_r) = s_g((v+1)N_r - (N_r - (u-1))).$$

If

$$(v+1)N_r \triangleright N_r - (u-1), \tag{11}$$

then $s_g((v+1)N_r - (N_r - (u-1))) = s_g((v+1)N_r) - s_g(N_r - (u-1))$. In this case the condition $v+1 \in \mathcal{H}$ implies that $h \in \mathcal{H}$. Now it remains to check that (11) is satisfied. It is enough to compare the $r+1$ least significant digits as the right hand side itself is $< g^{r+1}$. Note the identity $(v+1)N_r = vg^{r+1} + N_r - v$. Since N_r is constructed to have maximal digits (11) is indeed equivalent to

$$N_r - v \triangleright N_r - (u-1) \pmod{g^{r+1}},$$

and this is equivalent to the condition $u-1 \triangleright v \pmod{g^{r+1}}$. This ends the proof of Lemma 3. \square

LEMMA 4. Let $h \in \mathbb{N}$, $h = h_0 + h_1 g^{r+1} + \dots + h_k g^{k(r+1)}$ with $0 \leq h_0, \dots, h_k < g^{r+1}$. If $h_0 \geq 1$ and if $h_0 - 1 \triangleright h_1 \triangleright h_2 \triangleright \dots \triangleright h_k$ then $h \in \mathcal{H}$.

PROOF. First note that the conditions imply

$$0 \leq h_k \leq \dots \leq h_1 \leq h_0 - 1 < g^{r+1} - 1.$$

We prove this lemma by induction on k . The case $k = 0$ follows immediately from Lemma 2. Similarly for $k = 1$, $h = h_0 + h_1 g^{r+1}$, we have $h_1 + 1 \in \mathcal{H}$ by Lemma 2, and we can apply Lemma 3.

Suppose that the lemma is true for $k - 1$ for some $k \geq 2$ and let $u = h_0$ and $v = h_1 + h_2 g^{r+1} + \dots + h_k g^{(k-1)(r+1)}$; so that $h = u + v g^{r+1}$. By Lemma 3, $h \in \mathcal{H}$ whenever $u - 1 \triangleright v \pmod{g^{r+1}}$ and $v + 1 \in \mathcal{H}$. The first condition is clearly satisfied since $h_0 - 1 \triangleright h_1$ and $v \equiv h_1 \pmod{g^{r+1}}$. By our induction hypothesis the second condition also holds. \square

LEMMA 5. Let $\mathcal{E}_{k,r}$ denote the set of the $k + 1$ tuples (h_0, h_1, \dots, h_k) satisfying $0 \leq h_0, \dots, h_k < g^{r+1}$, $h_0 \geq 1$ and $h_0 - 1 \triangleright h_1 \triangleright \dots \triangleright h_k$. We have

$$|\mathcal{E}_{k,r}| = \binom{g+k}{k+1}^{r+1} - \binom{g+k-1}{k}^{r+1}. \quad (12)$$

PROOF. Note that $|\mathcal{E}_{0,r}| = g^{r+1} - 1$. Let $k \geq 1$. For all $(h_0, h_1, \dots, h_k) \in \mathcal{E}_{k,r}$ we write

$$h_0 - 1 = \sum_{\ell=0}^r \varepsilon_0^{(\ell)} g^\ell, \quad h_j = \sum_{\ell=0}^r \varepsilon_j^{(\ell)} g^\ell, \quad j = 1, \dots, k,$$

with $0 \leq \varepsilon_j^{(\ell)} \leq g - 1$.

The $k + 1$ -tuple (h_0, \dots, h_k) is in $\mathcal{E}_{k,r}$ if and only if the following conditions hold

$$0 \leq h_0 - 1 < g^{r+1} - 1, \quad (13)$$

and for all $0 \leq \ell \leq r$

$$g - 1 \geq \varepsilon_0^{(\ell)} \geq \varepsilon_1^{(\ell)} \geq \varepsilon_2^{(\ell)} \geq \dots \geq \varepsilon_k^{(\ell)} \geq 0. \quad (14)$$

For any given $0 \leq \ell \leq r$, the number of $(\varepsilon_0^{(\ell)}, \varepsilon_1^{(\ell)}, \dots, \varepsilon_k^{(\ell)})$ satisfying (14) is $\binom{g+k}{k+1}$, the number of choices of $k + 1$ not necessarily different elements out

of $0, \dots, g-1$, since each choice has exactly one decreasing order. Thus the number of all sets of $0 \leq \varepsilon_j^{(\ell)} \leq g-1$ satisfying (14) is $\binom{g+k}{k+1}^{r+1}$. However, sets with $\varepsilon_0^{(0)} = \dots = \varepsilon_0^{(r)} = g-1$, that is with $h_0 - 1 = g^{r+1} - 1 (= N_r)$, do not lead to a valid choice of h_0 , as they violate (13). Much the same way, the number of $(\varepsilon_0^{(\ell)}, \varepsilon_1^{(\ell)}, \dots, \varepsilon_k^{(\ell)})$ satisfying

$$g-1 = \varepsilon_0^{(\ell)} \geq \varepsilon_1^{(\ell)} \geq \varepsilon_2^{(\ell)} \geq \dots \geq \varepsilon_k^{(\ell)} \geq 0, \quad \ell = 0, \dots, r$$

is $\binom{g+k-1}{k}^{r+1}$. This ends the proof of Lemma 5. \square

4. The sets $S_g(x, K)$

Now we consider the sets $\mathcal{H}_K := \{h \geq 1 : s_g(hN_r) \leq Ks_g(N_r)\}$. Clearly $\mathcal{H}_1 = \mathcal{H}$. The importance of these sets is coded again in the fact, if $h \in \mathcal{H}_K$ then (by Lemma 1), for all $m \geq 1$ we have

$$s_g(mhN_r) \geq s_g(N_r) \geq \frac{1}{K}s_g(hN_r),$$

that is $hN_r \in S_g(x, K)$ whenever $hN_r \leq x$.

LEMMA 6. *Let $K \geq 2$ be an integer and $h = u + vg^{r+1}$, where $0 \leq u < g^{r+1}$. If $v \in \mathcal{H}_{K-1}$ then $h \in \mathcal{H}_K$.*

PROOF. Here again we start out from the formula (9). There is nothing to prove for $u = 0$. For $u \geq 1$ we have

$$s_g(hN_r) = s_g(N_r - (u-1)) + s_g(h-1-v) = s_g(N_r - (u-1)) + s_g(u-1 + vN_r).$$

Next we use the sub-additivity of s_g and the fact that N_r has maximal digits, that is $N_r \supseteq u-1$

$$s_g(hN_r) \leq s_g(N_r - (u-1)) + s_g(u-1) + s_g(vN_r) = s_g(N_r) + s_g(vN_r).$$

Thus if $v \in \mathcal{H}_{K-1}$, then for all

$$0 \leq u < g^{r+1}, \quad h = u + vg^{r+1} \in \mathcal{H}_K,$$

and we are done. \square

PROOF OF THEOREM 1. In the previous section, actually in Lemma 4, we proved that $h = h_0 + h_1 g^{r+1} + \dots + h_k g^{k(r+1)}$ with $0 \leq h_0, \dots, h_k < g^{r+1}$ is in \mathcal{H} whenever $h_0 \geq 1$ and $(h_0, \dots, h_k) \in \mathcal{E}_{k,r}$. The same is true if $h_0 = 0$, $h_1 \geq 1$ and $(h_1, \dots, h_k) \in \mathcal{E}_{k-1,r}$, and so on. All of these h satisfy $h < g^{(k+1)(r+1)}$. Adding also $h = g^{(k+1)(r+1)}$ we get

$$|\{h \in \mathcal{H} : h \leq g^{(k+1)(r+1)}\}| \geq |\mathcal{E}_{k,r}| + \dots + |\mathcal{E}_{0,r}| + 1 = \binom{g+k}{k+1}^{r+1}.$$

Thus we have by Lemma 6

$$|\{h \in \mathcal{H}_2 : h \leq g^{(k+2)(r+1)}\}| \geq g^{r+1} \binom{g+k}{k+1}^{r+1}.$$

Iterating this observation we obtain

$$|\{h \in \mathcal{H}_K : h \leq g^{(k+K)(r+1)}\}| \geq g^{(K-1)(r+1)} \binom{g+k}{k+1}^{r+1}.$$

Finally we collect all these pieces. If $g^{(k+K+1)(r+1)} \leq x$, $h \in \mathcal{H}_K$ and $h \leq g^{(k+K)(r+1)}$, then $hN_r \in S_g(x, K)$, that is

$$|S_g(x, K)| \geq g^{(K-1)(r+1)} \binom{g+k}{k+1}^{r+1}. \quad (15)$$

For fixed g, K, k and (sufficiently large) x we choose the integer $r \geq 1$ by

$$g^{(k+K+1)(r+1)} \leq x < g^{(k+K+1)(r+2)},$$

that is

$$r+2 > \frac{\log x}{(k+K+1) \log g}.$$

The lower bound in (15) can be modified as

$$|S_g(x, K)| \geq \frac{1}{g^{K-1} \binom{g+k}{k+1}} \left[g^{(K-1)} \binom{g+k}{k+1} \right]^{r+2} > \frac{1}{g^{K-1} \binom{g+k}{k+1}} x^\theta,$$

with

$$\theta = \frac{(K-1) \log g + \log \binom{g+k}{k+1}}{(k+K+1) \log g},$$

as in (1). This completes the proof. \square

PROOF OF THEOREM 2. We can suppose $K \geq 2$, as for $K = 1$ this follows by taking $k = 0$ in Theorem 1. We suppose that h is of the following type

$$h = u + v_1 g^{r+1} + \dots + v_{K-1} g^{(K-1)(r+1)} + w_1 g^{K(r+1)} + \dots + w_{K-1} g^{(2K-2)(r+1)},$$

where

$$1 \leq u \leq v_1 \leq \dots \leq v_{K-1} < g^{r+1}, \quad 0 \leq w_{K-1} \leq \dots \leq w_1 \leq v_{K-1} - 1. \quad (16)$$

We start from the identity

$$\begin{aligned} hN_r &= (N_r - (u - 1)) + (N_r - (v_1 - u))g^{r+1} + \dots + \\ &+ (N_r - (v_{K-1} - v_{K-2}))g^{(K-1)(r+1)} + (v_{K-1} - w_1 - 1)g^{K(r+1)} + \\ &+ (w_1 - w_2)g^{(K+1)(r+1)} + \dots + w_{K-1}g^{(2K-1)(r+1)}. \end{aligned}$$

In the above representation all coefficients are non negative and $< g^{r+1}$, moreover $N_r \geq u - 1, \dots, N_r \geq v_{K-1} - v_{K-2}$, and we conclude

$$\begin{aligned} s_g(hN_r) &= Ks_g(N_r) - s_g(u - 1) - s_g(v_1 - u) - \dots - s_g(v_{K-1} - v_{K-2}) + \\ &+ s_g(v_{K-1} - w_1 - 1) + s_g(w_1 - w_2) + \dots + s_g(w_{K-1}). \end{aligned}$$

Thus $h \in \mathcal{H}_K$ whenever

$$\begin{aligned} s_g(v_{K-1} - w_1 - 1) + s_g(w_1 - w_2) + \dots + s_g(w_{K-1}) &\leq \\ &\leq s_g(u - 1) + s_g(v_1 - u) + \dots + s_g(v_{K-1} - v_{K-2}). \end{aligned} \quad (17)$$

There is a one-to-one correspondence between systems of integers satisfying (16) and systems of integers satisfying

$$0 \leq X_1, \dots, X_K, Y_1, \dots, Y_K, \quad X_1 + \dots + X_K = Y_1 + \dots + Y_K < N_r. \quad (18)$$

The correspondence is given by

$$\begin{aligned} u &= 1 + Y_1 & w_1 &= X_2 + \dots + X_K \\ v_1 &= 1 + Y_1 + Y_2 & & \vdots \\ & \vdots & w_{K-2} &= X_{K-1} + X_K \\ v_{K-1} &= 1 + Y_1 + \dots + Y_K, & w_{K-1} &= X_K \end{aligned}$$

and the condition (17) is equivalent to

$$s_g(X_1) + \dots + s_g(X_K) \leq s_g(Y_1) + \dots + s_g(Y_K). \quad (19)$$

It is an easy classical problem to compute M , the number of solutions to (18). If a solution of (18) does not satisfy (19), then changing the role of (X_1, \dots, X_K) and (Y_1, \dots, Y_K) we get one that satisfies (19). In other words, at least half of the solutions of (18) satisfy (19) as well. To compute M we write

$$r(n) = |\{n = X_1 + \dots + X_K, \quad 0 \leq X_j\}| = \binom{n + K - 1}{K - 1}.$$

Finally we have (with some numerical work)

$$M = \sum_{n=0}^{N_r-1} (r(n))^2 = \sum_{n=0}^{N_r-1} \binom{n + K - 1}{K - 1}^2 \geq \frac{g^{(2K-1)(r+1)}}{3(2K-1)((K-1)!)^2}.$$

We constructed at least $M/2$ elements h of \mathcal{H}_K , satisfying $h < g^{(2K-1)(r+1)}$. With these h we have $hN_r \in S_g(x, K)$ whenever $hN_r \leq x$ which is the case if $g^{2K(r+1)} \leq x < g^{2K(r+2)}$. Collecting all pieces we arrive at

$$|S_g(x, K)| \geq \frac{g^{(2K-1)(r+1)}}{6(2K-1)((K-1)!)^2} \geq \frac{1}{6(2K-1)((K-1)!)^2 g^{2K}} x^{1-1/2K}. \quad \square$$

OUTLINE OF THE PROOF OF (3). We will only present the main idea, since the improvement is not significant. A natural way to combine the two previous proofs is to consider integers h of type $h = h' + h'' g^{(2K-3)(r+1)}$, where $h' < g^{(2K-3)(r+1)}$ plays the role of h in the proof of Theorem 2 (with $K - 1$ in place of K), and $h'' < g^{(k+1)(r+1)}$ plays the role of h in the proof of Theorem 1. By the subadditivity of $s_g(n)$ we have

$$s_g(hN_r) \leq s_g(h'N_r) + s_g(h'' g^{(k+1)(r+1)} N_r) \leq K s_g(N_r).$$

The previous calculations show that there are $\gg g^{(2K-3)(r+1)} \binom{g+k}{k+1}^{r+1}$ such h , all of them satisfy $hN_r \leq g^{(k+2K-1)(r+1)}$. This proves (3). \square

5. Some computations on the θ

In this last section we justify the choice of k in the cases mentioned in the introduction. For fixed g and K we denote $\theta_k = \theta_{g,K,k}$, see (1). We will be brief with some details of computations. Since

$$\theta_k - \theta_{k+1} = \frac{\theta_{k+1} \log g - \log \frac{g+k+1}{k+2}}{(k+K+1) \log g},$$

we have for all $k \geq 0$ that

$$\theta_k \geq \theta_{k+1} \quad \text{if and only if} \quad \theta_{k+1} \log g \geq \log \frac{g+k+1}{k+2}. \quad (20)$$

In other words, if the lower bound for θ_{k+1} in (20) is satisfied for all $k \geq k_0$ then θ_k is decreasing for $k \geq k_0$. The maximal $\theta_{g,K,k}$ in Theorem 1 must be attained with $k \leq k_0$. Indeed, we prove

LEMMA 7. *If $K \gg \log g$ then $\theta_k \geq \theta_{k+1}$ for all $k \geq 0$. For all $K \geq 1$ we have*

$$\theta_k \geq \theta_{k+1} \quad \text{if} \quad k \gg \log g.$$

PROOF. Since $(g+t)/(t+1)$ is a decreasing function of the variable t on the positive real numbers, we have

$$\begin{aligned} (k+K+2) \theta_{k+1} \log g &= (K-1) \log g + \sum_{j=0}^{k+1} \log \frac{g+j}{j+1} \geq \\ &\geq K \log g + \log \frac{g+k+1}{k+2} + \int_1^{k+1} \log \frac{g+t}{t+1} dt. \end{aligned}$$

Computing the integral we get from (20) that $\theta_k \geq \theta_{k+1}$ follows if

$$\begin{aligned} (K-1) \left(\log(k+2) - \log \frac{g+k+1}{g} \right) + (g-1) \log \frac{g+k+1}{g+1} + 2 \log 2 &\geq \\ &\geq 2 \log(g+1) - \log g, \end{aligned}$$

and this follows if

$$(K-1) \log \frac{4}{3} + (g-1) \log \frac{g+k+1}{g+1} \geq \log(g+1), \quad (21)$$

by the inequalities

$$\log(k+2) - \log \frac{g+k+1}{g} \geq \log \frac{4}{3} \quad \text{for all } g \geq 2, \quad k \geq 0,$$

and

$$2 \log(g+1) - \log g - 2 \log 2 \leq \log(g+1) \quad \text{for all } g \geq 2.$$

Finally one can see that for $K \gg \log g$ the first term, for $k \gg \log g$ the second term in the left hand side of (21) majorize the right hand side. This proves the lemma. \square

The same calculation is valid for $\theta'_{g,K,k}$ as well. One has to write $2K-2$ in place of K , as is seen from comparing (1) and (3).

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