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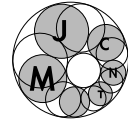


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# On the Rankin—Selberg problem in short intervals

Aleksandar Ivić (Belgrade)

**Abstract:** If

$$\Delta(x) := \sum_{n \leq x} c_n - Cx \quad (C > 0)$$

denotes the error term in the classical Rankin—Selberg problem, then we obtain a non-trivial upper bound for the mean square of  $\Delta(x+U) - \Delta(x)$  for a certain range of  $U = U(X)$ . In particular, under the Lindelöf hypothesis for  $\zeta(s)$ , it is shown that

$$\int_X^{2X} (\Delta(x+U) - \Delta(x))^2 dx \ll_\varepsilon X^{9/7+\varepsilon} U^{8/7},$$

while under the Lindelöf hypothesis for the Rankin—Selberg zeta-function the integral is bounded by  $X^{1+\varepsilon} U^{4/3}$ . An analogous result for the discrete second moment of  $\Delta(x+U) - \Delta(x)$  also holds.

**Keywords:** the Rankin—Selberg problem, short intervals, mean square bounds

**AMS Subject classification:** 11N37, 44A15, 26A12

## 1. Introduction and statement of results

The classical Rankin—Selberg problem consists of the estimation of the error term function

$$\Delta(x) := \sum_{n \leq x} c_n - Cx, \tag{1.1}$$

where the notation is as follows. Let  $\varphi(z)$  be a holomorphic cusp form of weight  $\kappa$  with respect to the full modular group  $SL(2, \mathbb{Z})$ , so that

$$\varphi\left(\frac{az+b}{cz+d}\right) = (cz+d)^\kappa \varphi(z) \quad (a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1)$$

when  $\Im z > 0$  and  $\lim_{\Im z \rightarrow \infty} \varphi(z) = 0$  (see e. g., R. A. Rankin [17] for basic notions).

We denote by  $a(n)$  the  $n$ -th Fourier coefficient of  $\varphi(z)$  and suppose that  $\varphi(z)$  is a normalized eigenfunction for the Hecke operators  $T(n)$ , that is,  $a(1) = 1$  and  $T(n)\varphi = a(n)\varphi$  for every  $n \in \mathbb{N}$ . The classical example is  $a(n) = \tau(n)$ , when  $\kappa = 12$ . This is the Ramanujan  $\tau$ -function defined by

$$\sum_{n=1}^{\infty} \tau(n)x^n = x\{(1-x)(1-x^2)(1-x^3)\cdots\}^{24} \quad (|x| < 1).$$

The constant  $C (> 0)$  in (1.1) may be written down explicitly (see e. g., [11]), and  $c_n$  is the convolution function defined by

$$c_n = n^{1-\kappa} \sum_{m^2|n} m^{2(\kappa-1)} \left| a\left(\frac{n}{m^2}\right) \right|^2. \quad (1.2)$$

This is a multiplicative arithmetic function, namely  $c_{mn} = c_m c_n$  when  $(m, n) = 1$ , since  $a(n)$  is multiplicative. The classical Rankin—Selberg bound of 1939 is

$$\Delta(x) = O(x^{3/5}), \quad (1.3)$$

hitherto unimproved. In fact, this bound is one of the longest standing unimproved bounds of Analytic number theory. In their works, done independently, R. A. Rankin [16] derives (1.3) from a general result of E. Landau [15], while A. Selberg [19] states the result with no proof. Note that, by the Möbius inversion formula, (1.2) is equivalent to

$$|a(n)|^2 n^{1-\kappa} = \sum_{d^2|n} \mu(d) c_{n/d^2}.$$

Therefore using (1.1), (1.3) and partial summation we obtain

$$\sum_{n \leq x} |a(n)|^2 = Dx^\kappa + O(x^{\kappa-2/5}) \quad (D > 0),$$

and conversely the above formula yields (1.1) with (1.3).

Although it seems very difficult at present to improve the bound in (1.3), recently there have been some results on the Rankin—Selberg problem (see the author's works [5–8]), in particular on mean square estimates. Namely, let as usual  $\mu(\sigma)$  denote the Lindelöf function

$$\mu(\sigma) := \limsup_{t \rightarrow \infty} \frac{\log |\zeta(\sigma + it)|}{\log t} \quad (\sigma \in \mathbb{R}). \quad (1.4)$$

Then we have (see [6, 7]; the exponent of  $\beta$  was misprinted as  $\frac{2}{5 - 2\mu(\frac{1}{2})}$ )

$$\int_0^X \Delta^2(x) dx \ll_{\varepsilon} X^{1+2\beta+\varepsilon}, \quad \beta = \frac{2}{5 - 4\mu(\frac{1}{2})}. \quad (1.5)$$

Here and later  $\varepsilon$  denotes positive constants which may be arbitrarily small, but are not necessarily the same at each occurrence, while  $\ll_{\varepsilon}$  means that the  $\ll$ -constant depends on  $\varepsilon$ . Note that with the sharpest known result (see M. N. Huxley [2])  $\mu(\frac{1}{2}) \leq 32/205$  we obtain  $\beta = 410/897 = 0.4457079\dots$ . The limit of (1.5) is the value  $\beta = 2/5$  if the Lindelöf hypothesis for  $\zeta(s)$  (that  $\mu(\frac{1}{2}) = 0$ ) is true.

In this work we are interested in mean square bounds for  $\Delta(x+U) - \Delta(x)$  in the range  $1 \ll U \leq X$ , especially when  $U$  is «short», namely when  $U = o(x)$  ( $x \rightarrow \infty$ ). First of all note that, since  $c_n \ll_{\varepsilon} n^{\varepsilon}$ , by (1.1) we have

$$\Delta(x+U) - \Delta(x) = \sum_{x < n \leq x+U} c_n - CU \ll_{\varepsilon} \sum_{x < n \leq x+U} n^{\varepsilon} - CU \ll_{\varepsilon} Ux^{\varepsilon}. \quad (1.6)$$

Although this bound may be considered as «trivial», there does not exist an analytic proof of it yet. Hence using (1.5) and (1.6) we have

$$\int_X^{2X} \left( \Delta(x+U) - \Delta(x) \right)^2 dx \ll_{\varepsilon} \min \left( X^{1+2\beta+\varepsilon}, X^{1+\varepsilon} U^2 \right) \quad (1 \ll U \leq X). \quad (1.7)$$

One can call then (1.7) the «trivial bound» for the mean square of  $\Delta(x+U) - \Delta(x)$ , and we seek a non-trivial bound, namely a bound which is (at least in certain ranges of  $U = U(X)$ ) sharper than (1.7).

Recently there has been work on the analogue of this problem for some related divisor problems. Let  $\Delta_k(x)$  denote the error term in the asymptotic formula for the summatory function of  $d_k(n)$ , generated by  $\zeta^k(s)$  ( $k \in \mathbb{N}$ ). Then in particular

$$\Delta_2(x) = \sum_{n \leq x} d(n) - x(\log x + 2\gamma - 1) \quad \left( d_2(n) \equiv d(n) = \sum_{\delta|n} 1 \right)$$

is the error term in the classical Dirichlet divisor problem and  $\gamma = -\Gamma'(1) = 0.5772157\dots$  is Euler's constant. The author [9] proved that, for

$$1 \ll U = U(X) \leq \frac{1}{2}\sqrt{X}, \quad c_3 = 8\pi^{-2}$$

and computable constants  $c_j$ , we have

$$\int_X^{2X} (\Delta_2(x+U) - \Delta_2(x))^2 dx = XU \sum_{j=0}^3 c_j \log^j \left( \frac{\sqrt{X}}{U} \right) + O_\varepsilon(X^{1/2+\varepsilon}U^2) + O_\varepsilon(X^{1+\varepsilon}U^{1/2}). \quad (1.8)$$

Thus for

$$X^\varepsilon \leq U = U(X) \leq X^{1/2-\varepsilon}$$

it is seen that (1.8) is a true asymptotic formula.

A result analogous to (1.8) holds if  $\Delta_2(x+U) - \Delta_2(x)$  is replaced by the function  $E(x+U) - E(x)$ , with different constants  $c_j$ , where

$$E(T) := \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt - T \left( \log \frac{T}{2\pi} + 2\gamma - 1 \right)$$

is the error term in the mean square formula for  $|\zeta(\frac{1}{2} + it)|$ . For an extensive account on  $E(T)$  see e. g., F. V. Atkinson's classical work [1], and the author's monographs [4], [3].

In the general case, when  $k > 2$ , the above problem becomes more difficult. In [10] we obtained mean square estimates for  $\Delta_k(x+U) - \Delta_k(x)$ . To formulate the results, first we define  $\sigma(k)$  as a number satisfying  $\frac{1}{2} \leq \sigma(k) < 1$ , for which

$$\int_0^T |\zeta(\sigma(k) + it)|^{2k} dt \ll_{\varepsilon} T^{1+\varepsilon}$$

holds for a fixed integer  $k \geq 2$ . From zeta-function theory (see [4], and in particular Section 7.9 of E. C. Titchmarsh [22]) it is known that such a number exists for any given  $k \in \mathbb{N}$ , but it is not uniquely defined, as one has

$$\int_0^T |\zeta(\sigma + it)|^{2k} dt \ll_{\varepsilon} T^{1+\varepsilon} \quad (\sigma(k) \leq \sigma < 1).$$

From Chapter 8 of [4] it follows that one has

$$\sigma(2) = \frac{1}{2}, \quad \sigma(3) = \frac{7}{12}, \quad \sigma(4) = \frac{5}{8},$$

etc., but it is not easy to write down (the best known value of)  $\sigma(k)$  explicitly as a function of  $k$ . Note that the Lindelöf hypothesis that  $\mu(\frac{1}{2}) = 0$  is equivalent to the fact that  $\sigma(k) = \frac{1}{2}$  ( $\forall k \in \mathbb{N}$ ). Then the result of [10] states: Let  $k \geq 3$  be a fixed integer. If  $\sigma(k) = \frac{1}{2}$ , then

$$\int_X^{2X} (\Delta_k(x+U) - \Delta_k(x))^2 dx \ll_{\varepsilon} X^{1+\varepsilon} U^{4/3} \tag{1.9}$$

$$(X^{\varepsilon} \leq U = U(X) \leq X^{1-\varepsilon}).$$

If  $\frac{1}{2} < \sigma(k) < 1$ , and  $\theta(k)$  is any constant satisfying  $2\sigma(k) - 1 < \theta(k) < 1$ , then there exists  $\varepsilon_1 = \varepsilon_1(k) > 0$  such that

$$\int_X^{2X} (\Delta_k(x+U) - \Delta_k(x))^2 dx \ll_{\varepsilon_1} X^{1-\varepsilon_1} U^2$$

$$(X^{\theta(k)} \leq U = U(X) \leq X^{1-\varepsilon}).$$

It is clear that, if the Lindelöf hypothesis is true for  $\zeta(s)$ , then (1.9) holds for all natural numbers  $k \geq 2$ .

Recently the author and J. Wu [12] obtained a new upper bound for

$$\sum_{h \leq H} \Delta_k(N, h)$$

for  $1 \leq H \leq N$ ,  $k \in \mathbb{N}$ ,  $k \geq 3$ , where  $\Delta_k(N, h)$  is the (expected) error term in the asymptotic formula for

$$\sum_{N < n \leq 2N} d_k(n) d_k(n+h).$$

Now we state our results on the mean square of  $\Delta(x+U) - \Delta(x)$  as the following

**THEOREM 1.** *If  $\mu = \mu(\frac{1}{2})$  is defined by (1.4) then, for  $1 \leq U = U(X) \leq X$ ,*

$$\int_X^{2X} \left( \Delta(x+U) - \Delta(x) \right)^2 dx \ll_{\varepsilon} X^{(9+12\mu)/(7+4\mu)+\varepsilon} U^{8/(7+4\mu)}. \quad (1.10)$$

*If*

$$Z\left(\frac{1}{2} + it\right) \ll_{\varepsilon} (|t| + 1)^{\varepsilon} \quad (1.11)$$

*holds, which is the Lindelöf hypothesis for the Rankin–Selberg zeta-function, then the above integral is bounded by  $X^{1+\varepsilon} U^{4/3}$ .*

**COROLLARY 1.** *The bound in (1.10) improves (1.7) for*

$$X^{(1+4\mu)/(3+4\mu)} \leq U \leq X^{(16\mu^2-8\mu+9)/(20-16\mu)}.$$

**COROLLARY 2.** *If the Lindelöf hypothesis for  $\zeta(s)$  that  $\mu = \mu(\frac{1}{2}) = 0$  is true, then (1.10) reduces to*

$$\int_X^{2X} \left( \Delta(x+U) - \Delta(x) \right)^2 dx \ll_{\varepsilon} X^{9/7+\varepsilon} U^{8/7}, \quad (1.12)$$

*and (1.12) improves (1.7) for*

$$X^{1/3} \leq U \leq X^{9/20}.$$

There also exists a discrete analogue of Theorem 1. This is

**THEOREM 2.** *If  $\mu = \mu(\frac{1}{2})$  is defined by (1.4) then, for  $1 \leq U = U(X) \leq X$ ,*

$$\sum_{X < n \leq 2X} (\Delta(n+U) - \Delta(n))^2 \ll_{\varepsilon} X^{(9+12\mu)/(7+4\mu)+\varepsilon} U^{8/(7+4\mu)}. \quad (1.13)$$

*If (1.11) holds, then the above sum is bounded by  $X^{1+\varepsilon} U^{4/3}$ .*

It seems hard to ascertain what should be the true order of magnitude of the function  $\Delta(x+U) - \Delta(x)$ . From (1.8) it seems plausible that

$$\Delta_2(x+U) - \Delta_2(x) \ll_{\varepsilon} x^{\varepsilon} \sqrt{U} \quad (x^{\varepsilon} \leq U = U(x) \leq x^{1/2-\varepsilon}), \quad (1.14)$$

which is a very strong conjecture made by M. Jutila [13], but it is not clear whether there is sufficient analogy between  $\Delta(x+U) - \Delta(x)$  and  $\Delta_2(x+U) - \Delta_2(x)$  to make any predictions about the order of  $\Delta(x+U) - \Delta(x)$  from (1.14).

## 2. Proof of the Theorems

There are two natural tools to study  $\Delta(x)$ . The first is the explicit, truncated formula for  $\Delta(x)$ , of the Voronoï type, namely

$$\Delta(x) = \frac{x^{3/8}}{2\pi} \sum_{k \leq K} c_k k^{-5/8} \sin \left( 8\pi(kx)^{1/4} + \frac{3\pi}{4} \right) + O_{\varepsilon} \left( x^{3/4+\varepsilon} K^{-1/4} \right),$$

where the parameter  $K$  satisfies  $1 \ll K \ll x$ . The proof of this result can be found in [11]. However, the error term is much too large for our present purpose. Therefore we resort to the use of another natural tool in the study of  $\Delta(x)$ . This is the Rankin—Selberg zeta-function

$$Z(s) := \sum_{n=1}^{\infty} c_n n^{-s}, \quad (2.1)$$

defined initially for  $s = \sigma + it$ ,  $\sigma > 1$ , and for other values of  $s$  by analytic continuation. It has a simple pole at  $s = 1$  with residue equal to  $C$  (cf. (1.1)), and is otherwise regular. For every  $s \in \mathbb{C}$  it satisfies the functional equation

$$\Gamma(s + \kappa - 1)\Gamma(s)Z(s) = (2\pi)^{4s-2}\Gamma(\kappa - s)\Gamma(1 - s)Z(1 - s). \quad (2.2)$$

The Rankin—Selberg zeta-function  $Z(s)$  belongs to the Selberg class  $\mathcal{S}$  of Dirichlet series of degree four. For the definition and properties of  $\mathcal{S}$  see e. g., the seminal paper [20] of A. Selberg and the review paper of Kaczorowski—Perelli [14].

One also has the decomposition

$$Z(s) := \sum_{n=1}^{\infty} c_n n^{-s} = \zeta(s) \sum_{n=1}^{\infty} b_n n^{-s} = \zeta(s)B(s), \quad (2.3)$$

say, where  $B(s)$  belongs to the class  $\mathcal{S}$  of of degree three, and moreover the function  $B(s)$  is holomorphic for  $\Re s > 0$ . This follows from G. Shimura's work [21] (see also A. Sankaranarayanan [18]). The coefficients  $b_n$  in (2.3) are multiplicative and satisfy  $b_n \ll_{\varepsilon} n^{\varepsilon}$  (see [18]). Actually the coefficients  $b_n$  are bounded by a log-power in mean square, but this stronger property is not needed here.

If we suppose that

$$\int_X^{2X} \left| B\left(\frac{1}{2} + it\right) \right|^2 dt \ll_{\varepsilon} X^{\theta+\varepsilon} \quad (\theta \geq 1), \quad (2.4)$$

and use the elementary fact (see Chapters 7 and 8 of [4] for the results on the moments of  $|\zeta(\frac{1}{2} + it)|$ ) that

$$\int_X^{2X} \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt \ll X \log X, \quad (2.5)$$

then from (2.3)–(2.5) and the Cauchy—Schwarz inequality for integrals we obtain

$$\int_X^{2X} \left| Z\left(\frac{1}{2} + it\right) \right| dt \ll_{\varepsilon} X^{(\theta+1)/2+\varepsilon}. \quad (2.6)$$

As  $B(s)$  belongs to the Selberg class of degree three, then  $B(\frac{1}{2} + it)$  in (2.5) can be written as a sum of two Dirichlet polynomials (e. g., by the reflection principle discussed in [4], Chapter 4), each of length  $\ll X^{3/2}$ , plus a manageable error term. Thus by the mean value theorem for Dirichlet polynomials (op. cit.) we

have  $\theta \leq 3/2$ , and any improvement on the value of  $\theta$  would give an improvement of (1.3), as shown by the author in [6], [7].

To prove (1.10), we start from (2.1) and Perron's inversion formula (see e. g., the Appendix of [4]) to obtain

$$\sum_{n \leq x} c_n = \frac{1}{2\pi i} \int_{1+\varepsilon-i\tau}^{1+\varepsilon+i\tau} \frac{x^s}{s} Z(s) ds + O_\varepsilon(X^{1+\varepsilon}T^{-1}), \quad (2.7)$$

where  $X \leq x \leq 2X$ ,  $1 \ll \tau \ll X$  and  $T \leq \tau \leq 2T$  will be suitably chosen a little later. We replace the segment of integration by the contour joining the points

$$1 + \varepsilon - i\tau, \quad \frac{1}{2} - i\tau, \quad \frac{1}{2} + i\tau, \quad 1 + \varepsilon + i\tau.$$

We encounter the simple pole of  $Z(s)$  at  $s = 1$  and the residue will furnish  $Cx$ , the main term in (1.1). Hence by the residue theorem (2.7) gives, once with  $x$  and once with  $x + U$ ,

$$\begin{aligned} \Delta(x + U) - \Delta(x) &= \frac{1}{2\pi i} \int_{\frac{1}{2}-i\tau}^{\frac{1}{2}+i\tau} \frac{(x + U)^s - x^s}{s} Z(s) ds + \\ &+ O_\varepsilon(X^{1+\varepsilon}T^{-1}) + O(R(x, \tau)), \end{aligned} \quad (2.8)$$

where we set

$$R(x, \tau) := \frac{1}{\tau} \int_{\frac{1}{2}}^{1+\varepsilon} x^\sigma |Z(\sigma + i\tau)| d\sigma.$$

From (2.6) (with  $\theta = 3/2$ ) and the convexity of mean values (see e. g., [4], Lemma 8.3) we have

$$\int_T^{2T} |Z(\sigma + it)| dt \ll_\varepsilon T^{(3-\sigma)/2+\varepsilon} \quad \left( \frac{1}{2} \leq \sigma \leq 1 \right), \quad (2.9)$$

and the integral in (2.9) is  $\ll_{\varepsilon} T^{1+\varepsilon}$  for  $\sigma \geq 1$ . It follows that

$$\begin{aligned} \int_T^{2T} R(x, \tau) d\tau &\ll_{\varepsilon} \frac{1}{T} \int_{\frac{1}{2}}^{1+\varepsilon} x^{\sigma} \left( \int_T^{2T} |Z(\sigma + i\tau)| d\tau \right) d\sigma \\ &\ll_{\varepsilon} \frac{1}{T} \max_{\frac{1}{2} \leq \sigma \leq 1+\varepsilon} \left( \frac{x}{\sqrt{T}} \right)^{\sigma} T^{3/2+\varepsilon} \ll_{\varepsilon} XT^{\varepsilon}, \end{aligned}$$

since  $T \ll X$ . Note that this holds uniformly in  $X \leq x \leq 2X$ . Therefore there exists  $T_0 \in [T, 2T]$  for which

$$R(x, T_0) \ll_{\varepsilon} X^{1+\varepsilon} T^{-1} \quad (X \leq x \leq 2X)$$

holds uniformly in  $x$ . It is  $\tau = T_0$  that is chosen in (2.7) and  $T$  is the basic parameter to be determined. Then using

$$\frac{(x+U)^s - x^s}{s} = \int_0^U (x+v)^{s-1} dv$$

we obtain from (2.8), since  $T \leq T_0 \leq 2T$ ,

$$\begin{aligned} \Delta(x+U) - \Delta(x) &= \\ &= \frac{1}{2\pi i} \int_{\frac{1}{2}-i\tau}^{\frac{1}{2}+i\tau} \left( \int_0^U (x+v)^{s-1} dv \right) Z(s) ds + O_{\varepsilon}(X^{1+\varepsilon} T^{-1}). \end{aligned} \quad (2.10)$$

On squaring (2.10) and integrating, we obtain

$$\begin{aligned} &\int_X^{2X} \left( \Delta(x+U) - \Delta(x) \right)^2 dx \\ &\ll_{\varepsilon} \int_X^{2X} \left| \int_{-\tau}^{\tau} \int_0^U (x+v)^{\frac{1}{2}-1+it} Z\left(\frac{1}{2}+it\right) dv dt \right|^2 dx + X^{3+\varepsilon} T^{-2}. \end{aligned} \quad (2.11)$$

Let now  $\psi(x) (\geq 0)$  be a smooth function supported in  $[X/2, 5X/2]$ , such that

$$\psi(x) = 1 \quad \text{when} \quad X \leq x \leq 2X$$

and

$$\psi^{(r)}(x) \ll_r X^{-r} \quad (r = 0, 1, 2, \dots).$$

By using the Cauchy—Schwarz inequality for integrals it is seen that the integral on the right-hand side of (2.11) does not exceed

$$\begin{aligned} & U \int_{X/2}^{5X/2} \psi(x) \int_0^U \left| \int_{-\tau}^{\tau} (x+v)^{-\frac{1}{2}+it} Z\left(\frac{1}{2}+it\right) dt \right|^2 dv dx = \\ & = U \int_0^U \int_{-\tau}^{\tau} \int_{-\tau}^{\tau} Z\left(\frac{1}{2}+it\right) Z\left(\frac{1}{2}-iy\right) J dy dt dv, \end{aligned}$$

say, where

$$J = J(X; v, t, y) := \int_{X/2}^{5X/2} \psi(x)(x+v)^{-1}(x+v)^{i(t-y)} dx.$$

Integrating by parts we obtain, since  $\psi(X/2) = \psi(5X/2) = 0$ ,

$$J = \frac{-1}{i(t-y)+1} \int_{X/2}^{5X/2} (x+v)^{i(t-y)} \left( \psi'(x) - \frac{1}{x+v} \psi(x) \right) dx.$$

By repeating this process it is seen that each time our integrand will be decreased by the factor of order

$$\ll \frac{X}{|t-y|+1} \cdot \frac{1}{X} \ll_{\varepsilon} X^{-\varepsilon}$$

for  $|t-y| \geq X^{\varepsilon}$ . Thus if we fix any  $A > 0$ , the contribution of  $|t-y| \geq X^{\varepsilon}$  will be  $\ll X^{-A}$  if we integrate by parts  $r = r(\varepsilon, A)$  times. For  $|t-y| \leq X^{\varepsilon}$  we estimate the corresponding contribution to  $J$  trivially as  $O(1)$  to obtain that the integral on the

right-hand side of (2.11) is

$$\begin{aligned} &\ll_{\varepsilon} U^2 \int_{-\tau}^{\tau} \int_{-\tau, |t-y| \leq X^{\varepsilon}}^{\tau} \left| Z\left(\frac{1}{2} + it\right) Z\left(\frac{1}{2} + iy\right) \right| dy dt + 1 \\ &\ll_{\varepsilon} U^2 \int_{-\tau}^{\tau} \left| Z\left(\frac{1}{2} + it\right) \right|^2 \left( \int_{t-X^{\varepsilon}}^{t+X^{\varepsilon}} dy \right) dt + 1 \\ &\ll_{\varepsilon} U^2 X^{\varepsilon} T^{\frac{3}{2} + 2\mu(\frac{1}{2})}. \end{aligned}$$

Here we used the elementary inequality

$$|ab| \leq \frac{1}{2} (|a|^2 + |b|^2),$$

and the bound (cf. (2.4) with  $\theta = 3/2$ )

$$\begin{aligned} \int_T^{2T} \left| Z\left(\frac{1}{2} + it\right) \right|^2 dt &= \int_T^{2T} \left| B\left(\frac{1}{2} + it\right) \right|^2 \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 dt \\ &\ll_{\varepsilon} T^{2\mu(\frac{1}{2}) + \varepsilon} \int_T^{2T} \left| B\left(\frac{1}{2} + it\right) \right|^2 dt \ll_{\varepsilon} T^{\frac{3}{2} + 2\mu(\frac{1}{2}) + \varepsilon}. \end{aligned}$$

Therefore it is seen that the left-hand side of (2.11) is

$$\ll_{\varepsilon} X^{\varepsilon} (U^2 T^{\frac{3}{2} + 2\mu(\frac{1}{2})} + X^3 T^{-2}). \quad (2.12)$$

With the choice

$$T = X^{3/(\frac{7}{2} + 2\mu(\frac{1}{2}))} U^{-2/(\frac{7}{2} + 2\mu(\frac{1}{2}))}$$

the terms in (2.12) are equalized. The condition  $1 \ll T \ll X$  is trivial, and (2.12) yields (1.10). Note that in proving (1.9) we could use power moments of  $|\zeta(\frac{1}{2} + it)|$  for which there is certainly more information than for the moments of  $|Z(t)|$ . This reflects the quality of the bounds in (1.9) and (1.10).

Finally note that if (1.11) holds, which is the Lindelöf hypothesis for the Rankin—Selberg zeta-function, then obviously

$$\int_X^{2X} \left| Z\left(\frac{1}{2} + it\right) \right|^2 dt \ll_{\varepsilon} X^{1+\varepsilon}. \quad (2.13)$$

This would replace (2.12) by

$$\ll_{\varepsilon} X^{\varepsilon} (U^2 T + X^3 T^{-2}).$$

The choice  $T = XU^{-2/3}$  yields then

$$\int_X^{2X} (\Delta(x+U) - \Delta(x))^2 dx \ll_{\varepsilon} X^{1+\varepsilon} U^{4/3}, \quad (2.14)$$

which is non-trivial in the whole range  $1 \ll U \leq X$ . Clearly for the proof (2.13) suffices instead of the stronger (1.11). The bound (2.14) is the analogue of (1.9). This completes the proof of Theorem 1.

To prove (1.13) of Theorem 2 we employ the method developed in [9]. We can assume that  $U$  and  $X$  are natural numbers, for otherwise we shall make an admissible error by using trivial estimation. Using (1.1) it is seen that integral in (1.10) is equal to

$$\begin{aligned} & \sum_{X \leq m \leq 2X-1} \int_m^{m+1-0} \left( \sum_{x < n \leq x+U} c_n - CU \right)^2 dx = \\ &= \sum_{X \leq m \leq 2X-1} \int_m^{m+1-0} \left( \sum_{m < n \leq m+U} c_n - CU \right)^2 dx = \\ &= \sum_{X \leq m \leq 2X-1} \int_m^{m+1-0} (\Delta(m+U) - \Delta(m))^2 dx = \\ &= \sum_{X \leq m \leq 2X-1} (\Delta(m+U) - \Delta(m))^2 = \\ &= \sum_{X \leq n \leq 2X} (\Delta(n+U) - \Delta(n))^2 + O_{\varepsilon}(X^{\varepsilon} U^2). \end{aligned}$$

Here in the last step we used (1.6). Since the error term above is absorbed in the expression on the right-hand side of (1.13), the proof of Theorem 2 is finished.

## Bibliography

1. **F. V. Atkinson**, *The mean value of the Riemann zeta-function*, Acta Math. **81** (1949), 353–376.
2. **M. N. Huxley**, *Exponential sums and the Riemann zeta-function V*, Proc. London Math. Soc. (4) **90** (2005), 1–41.
3. **A. Ivić**, *The mean values of the Riemann zeta-function*, LNS **82** (1991), Tata Institute of Fundamental Research, Bombay (distr. by Springer Verlag, Berlin etc.).
4. **A. Ivić**, *The Riemann Zeta-Function*, John Wiley & Sons, New York (1985); 2nd ed. Dover, Mineola (2003).
5. **A. Ivić**, *Estimates of convolutions of certain number-theoretic error terms*, Intern. J. Math. and Math. Sciences. No. **1–4** (2004), 1–23.
6. **A. Ivić**, *Convolutions and mean square estimates of certain number-theoretic error terms*, Publ. Inst. Math., Nouv. Sér. **80 (94)** (2006), 141–156.
7. **A. Ivić**, *On some mean square estimates in the Rankin–Selberg problem*, Applicable Analysis and Discrete Mathematics **1** (2007), 1–11.
8. **A. Ivić**, *On the fourth moment in the Rankin–Selberg problem*, Archiv Math. **90** (2008), 412–419.
9. **A. Ivić**, *On the divisor function and the Riemann zeta-function in short intervals* The Ramanujan Journal **19**, No. 2 (2009), 207–224.
10. **A. Ivić**, *On the mean square of the divisor function in short intervals*, Journal de Théorie des Nombres de Bordeaux **21** (2009), 195–205.
11. **A. Ivić**, **K. Matsumoto**, **Y. Tanigawa**, *On Riesz mean of the coefficients of the Rankin–Selberg series*, Math. Proc. Camb. Phil. Soc. **127** (1999), 117–131.
12. **A. Ivić**, **J. Wu**, *On the general additive divisor problem*, Proceedings of the Steklov Institute of Mathematics **276** (2012), 140–148.
13. **M. Jutila**, *On the divisor problem for short intervals*, Ann. Univer. Turkuensis Ser. AI **186** (1984), 23–30.
14. **A. Kaczorowski**, **A. Perelli**, *The Selberg class: a survey*, Number Theory in Progress, Proc. Conf. in honour of A. Schinzel (K. Györy et al. eds) (1999), de Gruyter, Berlin, 953–992.
15. **E. Landau**, *Über die Anzahl der Gitterpunkte in gewissen Bereichen II*, Nachr. Ges. Wiss. Göttingen (1915), 209–243.
16. **R. A. Rankin**, *Contributions to the theory of Ramanujan’s function  $\tau(n)$  and similar arithmetical functions II. The order of the Fourier coefficients of integral modular forms*, Proc. Cambridge Phil. Soc. **35** (1939), 357–372.
17. **R. A. Rankin**, *Modular forms and functions*, Cambridge Univ. Press, Cambridge (1977).

18. **A. Sankaranarayanan**, *Fundamental properties of symmetric square  $L$ -functions I*, Illinois J. Math. **46** (2002), 23–43.
19. **A. Selberg**, *Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist*, Arch. Math. Naturvid. **43** (1940), 47–50.
20. **A. Selberg**, *Old and new conjectures and results about a class of Dirichlet series*, Proc. Amalfi Conf. Analytic Number Theory 1989 (E. Bombieri et al. eds.) (1992), University of Salerno, Salerno, 367–385.
21. **G. Shimura**, *On the holomorphy of certain Dirichlet series*, Proc. London Math. Soc. **31** (1975), 79–98.
22. **E. C. Titchmarsh**, *The theory of the Riemann zeta-function* (2nd ed.), University Press, Oxford (1986).

ALEKSANDAR IVIĆ

Katedra Matematike RGF-a,  
Univerzitet u Beogradu,  
Dušina 7,  
11 000 Beograd, Serbia,  
ivic@rgf.bg.ac.rs,  
aivic\_2000@yahoo.com