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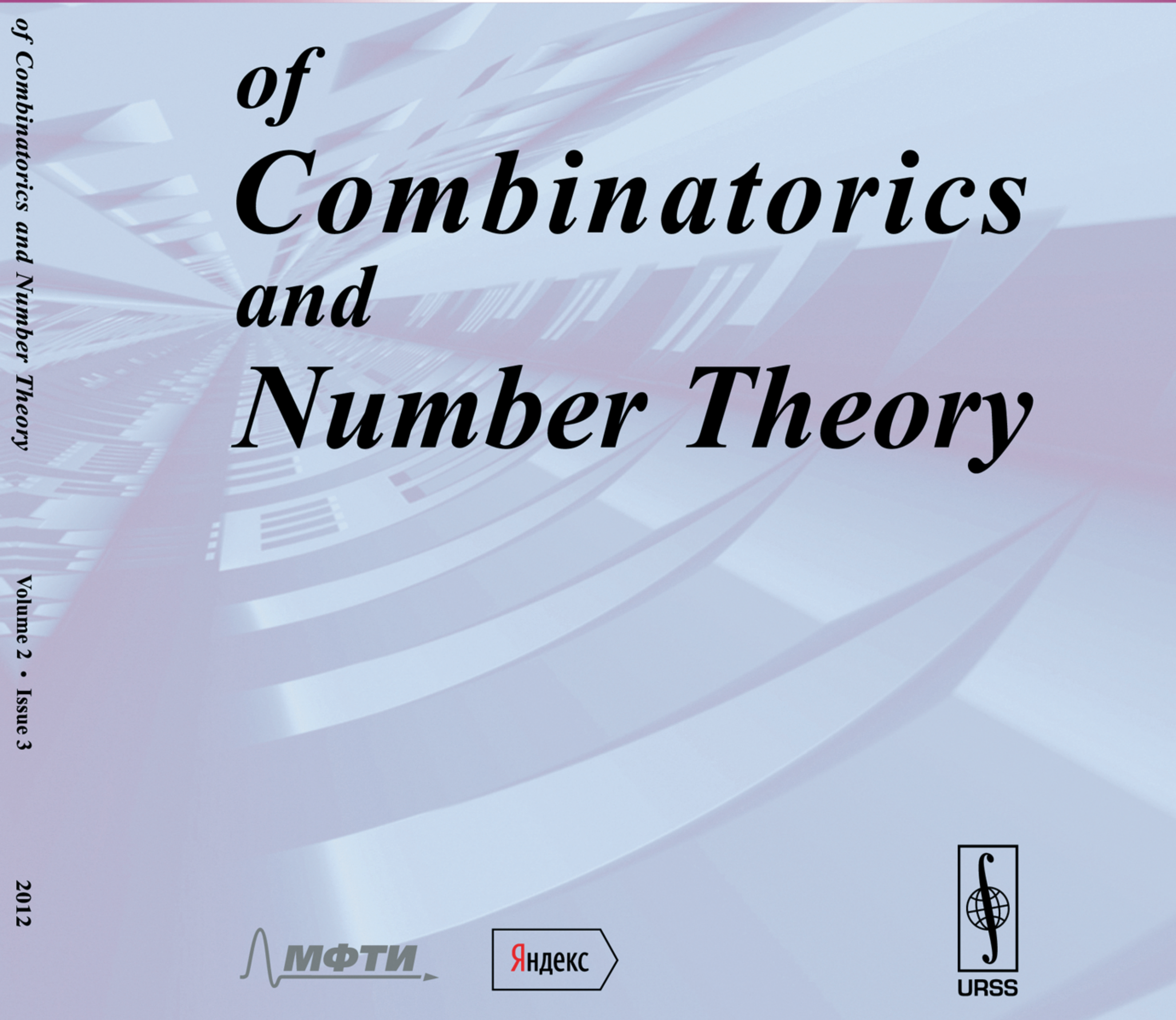

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Faculty of Innovations
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Sums with convolution of Dirichlet characters

Dmitry Ushanov (Moscow)

Abstract: Let χ_1 and χ_2 be two primitive Dirichlet characters with conductors q_1 and q_2 respectively. We prove new upper bounds for the sum

$$S_{\chi_1,\chi_2}(T) = \sum_{0<xy\leqslant T} \chi_1(x)\chi_2(y).$$

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1. Introduction

Let χ_1 and χ_2 be two primitive Dirichlet characters with conductors q_1 and q_2, respectively. Recently Banks and Shparlinski [1] considered the sum

$$S_{\chi_1,\chi_2}(T) = \sum_{0<xy\leqslant T} \chi_1(x)\chi_2(y).$$

For this sum they established an upper bound

$$S_{\chi_1,\chi_2}(T) \ll T^{13/18} q_1^{2/27} q_2^{1/9+o(1)}$$

for $T \geqslant q_2^{2/3} \geqslant q_1^{2/3}$, and a bound

$$S_{\chi_1,\chi_2}(T) \ll T^{5/8} q_1^{3/32} q_2^{3/16+o(1)}$$

for $T \geqslant q_2^{3/4} \geqslant q_1^{3/4}$.

In this paper we obtain estimates for S_{χ_1,χ_2} which are stronger than the ones given above.

2. Statement of results

THEOREM 1. *Let χ_1 and χ_2 be two primitive Dirichlet characters with conductors q_1 and q_2, respectively. If $q_1 \leqslant q_2$ and $T > 1$ then for every $\epsilon > 0$ one has*

$$\sum_{0<xy\leqslant T} \chi_1(x)\chi_2(y) \ll \begin{cases} T^{2/3}(q_1q_2)^{1/9+\epsilon} & \text{if} \quad (q_1q_2)^{1/3} \leqslant T \leqslant q_1^{4/3}q_2^{1/3}, \\ T^{3/4}q_2^{1/12+\epsilon} & \text{if} \quad q_1^{4/3}q_2^{1/3} \leqslant T, \end{cases} \tag{1}$$

$$\sum_{0<xy\leqslant T} \chi_1(x)\chi_2(y) \ll \begin{cases} T^{1/2}(q_1q_2)^{3/16+\epsilon} & \text{if} \quad (q_1q_2)^{3/8} \leqslant T \leqslant q_1^{9/8}q_2^{3/8}, \\ T^{2/3}q_2^{1/8+\epsilon} & \text{if} \quad q_1^{9/8}q_2^{3/8} \leqslant T. \end{cases} \tag{2}$$

(*Constants implied by $\ll$ depend only on ϵ.*)

COROLLARY 1. *Suppose within the assumption of Theorem 1 that $q_1 = q_2 = q$. Then*

$$\sum_{0<xy\leqslant T} \chi_1(x)\chi_2(y) \ll \begin{cases} T^{2/3}q^{2/9+\epsilon} & \text{if} \quad q^{2/3} \leqslant T \leqslant q^{11/12}, \\ T^{1/2}q^{3/8+\epsilon} & \text{if} \quad q^{11/12} \leqslant T \leqslant q^{3/2}, \\ T^{2/3}q^{1/8+\epsilon} & \text{if} \quad q^{3/2} \leqslant T \leqslant q^{9/4}, \\ T^{1/2}q^{1/2+\epsilon} & \text{if} \quad q^{9/4} \leqslant T. \end{cases}$$

Remark. It is shown in [1] that for $q^{2/3} \leqslant T \leqslant q^{83/84}$ the assumption of Corollary 1 implies the estimate

$$\sum_{0<xy\leqslant T} \chi_1(x)\chi_2(y) \ll T^{13/18}q^{5/27+o(1)}.$$

Notice that our estimate is more precise for such T.

THEOREM 2. *Let χ_1 and χ_2 be two primitive Dirichlet characters with prime conductors q_1 and q_2, respectively. Suppose that $q_1 \leqslant q_2$, $T > 1$ and let $r \geqslant 2$ be an integer. Put*

$$\nu_r = \begin{cases} 1 & \textit{if} \quad r = 2, \\ 0 & \textit{otherwise.} \end{cases}$$

Set

$$T_r = q_1^{\frac{(r+1)^2}{4r}} q_2^{\frac{r+1}{4r}} (\log q_1)^{r+1} (\log q_2)^{\nu_r r(r+1)+r^2+1}.$$

Then

$$\sum_{0<xy\leqslant T} \chi_1(x)\chi_2(y) \ll \begin{cases} T^{1-\frac{1}{r}} (q_1 q_2)^{\frac{r+1}{4r^2}} \log^{\frac{1}{r}} q_1 \log^{\frac{1}{r}+\nu_r+1} q_2 & \textit{if} \quad (q_1q_2)^{\frac{r+1}{4r}} \leqslant T \leqslant T_r, \\ T^{\frac{r}{r+1}} q_2^{\frac{1}{4r}} (\log q_2)^{\frac{2}{r+1}} & \textit{if} \quad T_r \leqslant T. \end{cases}$$

3. Basic notation

Let χ_1 and χ_2 be two primitive Dirichlet characters with conductors q_1 and q_2, respectively. Suppose that $q_1 \leqslant q_2$. Set

$$Q = q_1 q_2.$$

Let us fix $T > 0$ and consider the hyperbola

$$\Gamma = \{(x, y) \in \mathbb{R}_+^2 \mid xy = T\}.$$

For each subset $\Omega \subset \mathbb{R}^2$ let us define the character sum

$$S(\Omega) = \sum_{(x,y)\in\Omega\cap\mathbb{Z}^2} \chi_1(x)\chi_2(y).$$

In our proofs we will use the following result (see [2]).

THEOREM 3 (Burgess). *For any primitive Dirichlet character χ of conductor q and any nonnegative integers M, N we have*

$$\left| \sum_{M<n\leqslant M+N} \chi(n) \right| \leqslant c_\epsilon N^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}+\epsilon},$$

where $r \in \{1, 2, 3\}$. *If* q *is a prime number then*

$$\left| \sum_{M<n\leqslant M+N} \chi(n) \right| \leqslant c'_\epsilon N^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}} (\ln q)^{\frac{1}{r}}$$

for every $r \geqslant 1$.

Suppose I_1 and I_2 are two intervals. We define a rectangle $I_1 \times I_2$ as follows:

$$I_1 \times I_2 = \{(x, y) \in \mathbb{R}^2 \mid x \in I_1, y \in I_2\}.$$

We write $|\Pi|$ for the area of a rectangle Π of the above type and $\delta(\Pi) = \mathit{length}(I_1)$ for its width.

Consider the rectangles

$$U_0 = (0; \sqrt{T}) \times [0; \sqrt{T}), \tag{3}$$

$$U_k = \left(0; \frac{\sqrt{T}}{2^k}\right) \times [2^{k-1}\sqrt{T}; 2^k\sqrt{T}),$$

where $k = 1, 2, \ldots$ Obviously, one of the vertices of each U_k belongs to Γ.

Suppose that the vertex (x_1, y_1) of the rectangle

$$\Pi = [x_0, x_1) \times [y_0, y_1)$$

lies on Γ, i. e. $x_1 y_1 = T$. Define two new rectangles $r(\Pi)$ and $u(\Pi)$ as follows:

$$r(\Pi) = \left[x_1, \frac{3x_1 - x_0}{2}\right) \times \left[y_0, \frac{2T}{3x_1 - x_0}\right),$$

$$u(\Pi) = \left[x_0, \frac{x_0 + x_1}{2}\right) \times \left[y_1, \frac{2T}{x_0 + x_1}\right).$$

For $k = 1, 2, 3, \ldots$ let us consider the rectangles

$$\Pi_k = \left[\frac{\sqrt{T}}{2^k}; \frac{3\sqrt{T}}{2^{k+1}}\right) \times \left[2^{k-1}\sqrt{T}; \frac{2^{k+1}}{3}\sqrt{T}\right) = r(U_k). \tag{4}$$

Let $\mathcal{F}_k$ be the set of all the rectangles of the form $\sigma_1 \cdots \sigma_n \Pi_k$, where $n \geqslant 0$, $\sigma_i \in \{r, u\}$, $i = 1, \ldots, n$.

If $\Pi \in \mathcal{F}_k$ and $\Pi = \sigma_1 \cdots \sigma_l \Pi_k$ then we say that Π is a rectangle of order l.

We say that $\Pi \in \mathcal{F}_k$ has parameters (δ, x) if $\Pi = [x_0, x) \times [y_0, y)$ and $\delta = \delta(\Pi) = x - x_0$.

4. Lemmata

LEMMA 1. *Consider a rectangle* Π. *Suppose that* $|\Pi| = P$ *and that both height and width of* Π *are greater than* 1. *Then for every real* $\epsilon > 0$ *one has*

$$|S(\Pi)| \ll \begin{cases} P^{2/3} Q^{1/9+\epsilon}, \\ P^{1/2} Q^{3/16+\epsilon}. \end{cases}$$

PROOF. It suffices to apply Burgess' theorem with $r = 3$ in the first case and with $r = 2$ in the second case. ☐

LEMMA 2. *Consider a rectangle* $\Pi = [x_0; x) \times [y_0; y) \in \mathcal{F}_k$ *with vertex* (x, y) *on* Γ, *so that* $xy = T$. *Let* $\delta = \delta(\Pi)$. *Then the following statements hold*:

a) $|\Pi| = T\delta\left(\frac{1}{x} - \frac{1}{x+\delta}\right)$;

b) $|r(\Pi)| \leqslant \frac{|\Pi|}{4}$;

c) $|u(\Pi)| \leqslant \frac{|\Pi|}{4\left(1 - \frac{3\delta}{2x}\right)}$.

PROOF.

a) It can be shown by induction on the order of Π that

$$y_0 = \frac{T}{x+\delta}.$$

Therefore

$$|\Pi| = (x - x_0)(y - y_0) = \delta\left(\frac{T}{x} - \frac{T}{x+\delta}\right).$$

b) First of all we notice that for the parameters (δ', x') of $r(\Pi)$ we have

$$\delta' = \frac{\delta}{2}, \quad x' = x + \frac{\delta}{2}.$$

Therefore it is sufficient to show that

$$4T\frac{\delta}{2}\left(\frac{1}{x+\delta/2}-\frac{1}{x+\delta}\right)\leqslant T\delta\left(\frac{1}{x}-\frac{1}{x+\delta}\right)$$

or, equivalently,

$$\frac{1}{x+\delta/2}\leqslant\frac{1}{2}\left(\frac{1}{x}+\frac{1}{x+\delta}\right).$$

But the latter follows from the convexity of the map $x\to\dfrac{1}{x}$.

c) Denote by α the ratio δ/x. It is obvious that $0<\alpha<1/2$. We note that the parameters (δ',x') of $u(\Pi)$ satisfy the relations

$$\delta'=\frac{\delta}{2},\quad x'=x-\frac{\delta}{2}.$$

We see that

$$\frac{|u(\Pi)|}{|\Pi|}=\frac{T\frac{\delta}{2}\left(\frac{1}{x-\delta/2}-\frac{1}{x}\right)}{T\delta\left(\frac{1}{x}-\frac{1}{x+\delta}\right)}=\frac{\frac{1}{2x-\delta}-\frac{1}{2x}}{\frac{1}{x}-\frac{1}{x+\delta}}=\frac{\frac{1}{2-\alpha}-\frac{1}{2}}{1-\frac{1}{1+\alpha}}=\frac{1+\alpha}{2(2-\alpha)}.$$

It remains to show that

$$\frac{1+\alpha}{2(2-\alpha)}\leqslant\frac{1}{4\left(1-\frac{3}{2}\alpha\right)},$$

or, equivalently,

$$\frac{\alpha^2}{(2-3\alpha)(2-\alpha)}\geqslant 0.$$

The latter follows from the inequality $0<\alpha<1/2$. This concludes the proof. □

LEMMA 3. *Let* $\Pi=\sigma_1\cdots\sigma_l\Pi_k$. *Then*

$$|\Pi|\leqslant\frac{|\Pi_k|}{4^l\prod_{j=1}^{l}\left(1-\frac{3}{2}\left(\frac{2}{3}\right)^l\right)}.$$

PROOF.

We will show that the ratio δ/x is reduced by a factor $\geqslant 3/2$ every time Π is replaced by $u(\Pi)$ or $r(\Pi)$.

Case 1. Suppose that the parameters of Π and $u(\Pi)$ are (δ, x) and (δ', x'), respectively. Then $\delta' = \delta/2$ and $x' = x - \delta/2$. Therefore

$$\frac{\delta'}{x'} = \frac{\delta/2}{x - \delta/2} = \frac{\delta}{2x}\frac{1}{1 - \frac{\delta}{2x}} \leqslant \frac{2\delta}{3x},$$

because $\delta/x \leqslant 1/2$ for each rectangle in $\mathcal{F}_k$.

Case 2. Suppose that the parameters of Π and $r(\Pi)$ are (δ, x) and (δ', x'), respectively. Then $\delta' = \delta/2$ and $x' = x + \delta/2$. Therefore

$$\frac{\delta'}{x'} = \frac{\delta/2}{x + \delta/2} = \frac{\delta}{2x}\frac{1}{1 + \frac{\delta}{2x}} \leqslant \frac{\delta}{2x} \leqslant \frac{2\delta}{3x}.$$

It remains to apply Lemma 2. ☐

Lemma 4. *Given* $\delta, t, T \in \mathbb{R}$ *such that* $1/2 < \delta < 1$ *and* $1 < t < T^{\delta}$, *denote*

$$\Xi_t = \{(x, y) \in \mathbb{R}_+^2 \mid T - 2t \leqslant xy \leqslant T\}.$$

Then $\#(\Xi_t \cap \mathbb{Z}^2) \ll t \ln T$.

Proof. The number of integer points under the hyperbola can be estimated as

$$\sum_{x=1}^{T} \left[\frac{T}{x}\right] = T \ln T + (2\gamma - 1)T + O(T^{1/2}).$$

Therefore

$$\#(\Xi_t \cap \mathbb{Z}^2) = T \ln T + (2\gamma - 1)T - (T - 2t) \ln \big(T(1 - 2t/T)\big) - (2\gamma - 1)(T - 2t) + O(T^{1/2}),$$

which proves the lemma. ☐

5. Proof of Theorem 1 for small T

Set $\Omega = \{(x, y) \in \mathbb{R}_+^2 \mid xy < T\}$ and $\Omega_1 = \{(x, y) \in \mathbb{R}_+^2 \mid xy < T, x < \sqrt{T}\}$. Without loss of generality we may confine our argument to $S(\Omega_1)$.

It is obvious that the rectangles from $\mathcal{F}_k$, $k = 1, 2, \dots$ together with U_k, $k = 0, 1, 2, \dots$ cover Ω_1.

Set $t = T^{3/4} q_2^{1/12}$ and consider a real $\eta > 0$. Set

$$W_t = \{(x, y) \in \mathbb{R}_+^2 \mid xy < T,\ y \leqslant t,\ x \leqslant \sqrt{T}\}, \tag{5}$$

$$W_t' = \{(x, y) \in \mathbb{R}_+^2 \mid xy < T,\ y \geqslant t\}, \tag{6}$$

$$\Xi_t = \{(x, y) \in \mathbb{R}_+^2 \mid T - 2t \leqslant xy \leqslant T\}. \tag{7}$$

The number of integer points in Ξ_t is $\ll tQ^{\eta}$.

Consider a rectangle $\Pi \in \mathcal{F}_k$ such that $\Pi \subset W$ and let (x_0, y_0) be its left lower vertex. Set $\delta = \delta(\Pi)$.

Let $2\delta = T/y_0 - x_0 \leqslant 2$. Then $\Pi \subset \Xi_t$. Indeed, $T - x_0 y_0 \leqslant 2y_0 \leqslant 2t$, whence $T - 2t \leqslant x_0 y_0$.

Let us now estimate $S(W_t)$.

For the rectangles $\Pi \in \mathcal{F}_k$ such that $\Pi \subset W_t$ and $\delta(\Pi) \geqslant 1$ we apply Lemma 1. All the other rectangles are contained in Ξ_t.

The sum $S(\Xi_t)$ is trivially bounded by the number of integer points in Ξ_t.

The number of rectangles Π of order l is equal to 2^l. If Π is such a rectangle then $|\Pi| \ll |\Pi_k|/4^l$.

Thus we get the following estimate for the character sum over all Π of order l with $\delta(\Pi) \geqslant 1$:

$$S(\Pi^l) \ll T^{2/3} Q^{1/9+\eta} 4^{-2l/3} 2^l.$$

Since

$$\sum_{l=0}^{\infty} 4^{-2l/3} 2^l$$

converges, the character sum over all Π with $\delta(\Pi) \geqslant 1$ is $\ll T^{2/3} Q^{1/9+\eta}$.

Therefore

$$S(W_t) \ll \max(T^{2/3} Q^{1/9+\eta}, tQ^{\eta}). \tag{8}$$

In order to estimate $S(W_t')$ we use Burgess' Lemma with $r = 3$:

$$S(W_t') \ll \sum_{x=1}^{T/t} \left(\frac{T}{x}\right)^{2/3} q_2^{1/9+\eta} \ll T^{2/3} \left(\frac{T}{t}\right)^{1/3} q_2^{1/9+\eta},$$

whence

$$S(W_t') \ll T t^{-1/3} q_2^{1/9+\eta}. \tag{9}$$

So,

$$S(\Omega_1) \ll \max(T^{2/3}Q^{1/9+\eta}, tQ^{\eta}, Tt^{-1/3}q_2^{1/9+\eta}).$$

Using the definition of t we obtain the following result. If

$$T \leqslant q_1^{4/3}q_2^{1/3} \quad \text{then} \quad S(\Omega) \ll T^{2/3}Q^{1/9+\eta}.$$

If $T \geqslant q_1^{4/3}q_2^{1/3}$ then $S(\Omega) \ll tQ^{\eta} = T^{3/4}q_2^{1/12+\eta}$.

6. Proof of Theorem 1 for large T

Set

$$t = T^{2/3}q_2^{1/8}. \tag{10}$$

As before, we use the sets W_t, W_t' and Ξ_t defined by (5), (6) and (7).

The only difference between this case and the previous one is the convergence argument. The sum over all the rectangles of order l can be estimated by $T^{1/2}Q^{3/16+\eta/2}$. Therefore, the sum

$$\sum_{l=0}^{\infty} T^{1/2}Q^{3/16+\eta/2}$$

does not converge. But it is easy to see that if $l \gg \log T$ then each rectangle Π of order l is contained in the set Ξ_t. So we have

$$S(W_t) \ll \max(T^{1/2}Q^{3/16+\eta}, tQ^{\eta}). \tag{11}$$

Applying Burgess' Lemma with $r = 2$ we obtain

$$S(W_t') \ll \sum_{x=1}^{T/t} \left(\frac{T}{x}\right)^{1/2} q_2^{3/16+\eta} \ll T^{1/2}\left(\frac{T}{t}\right)^{1/2} q_2^{3/16+\eta},$$

whence

$$S(W_t') \ll Tt^{-1/2}q_2^{3/16+\eta}. \tag{12}$$

Substituting (10) into (11) and (12) we get the estimates

$$S(\Omega) \ll T^{1/2}Q^{3/16+\eta} \quad \text{for} \quad T \leqslant q_1^{9/8}q_2^{3/8}$$

and $S(\Omega) \ll tQ^{\eta} = T^{2/3}q_2^{1/8+\eta}$ for $T \geqslant q_1^{9/8}q_2^{3/8}$.

7. Prime moduli

Fix $r \geqslant 2$ and set

$$t = T^{\frac{r}{r+1}} q_2^{\frac{1}{4r}} \log^{\frac{1-r}{r+1}} q_2.$$

Let us use the sets defined by (5), (6) and (7), again.

Let us perform a similar argument to estimate $S(W_t)$. For each $l \geqslant 1$ there exist 2^l rectangles of order l. It follows from Lemma 3 that for the character sum over all the rectangles of order l we have the following estimate:

$$S(\Pi^l) \ll T^{1-\frac{1}{r}} Q^{\frac{r+1}{4r^2}} (\log q_1)^{\frac{1}{r}} (\log q_2)^{\frac{1}{r}} 4^{-(1-\frac{1}{r})l} 2^l.$$

Let us consider two cases.

Case 1 $(r \geqslant 3)$. The sum

$$\sum_{l=0}^{\infty} 4^{-(1-\frac{1}{r})l} 2^l$$

converges, so

$$\sum_{l=0}^{\infty} S(\Pi^l) \ll T^{1-\frac{1}{r}} Q^{\frac{r+1}{4r^2}} (\log q_1)^{\frac{1}{r}} (\log q_2)^{\frac{1}{r}}.$$

Case 2 $(r = 2)$. The sum

$$\sum_{l=0}^{\infty} 4^{-(1-\frac{1}{r})l} 2^l$$

diverges. In this case it suffices to take the first

$$\ll \log(T) \ll \log(q_2)$$

values of l.

There are $\ll \log T \ll \log q_2$ rectangles in $\mathcal{F}_k$ lying below the line $y = t$. Hence

$$S(W_t') \ll \max\left(T^{1-\frac{1}{r}} Q^{\frac{r+1}{4r^2}} \log^{1/r} q_1 \log^{\frac{1}{r}+\nu_r+1} q_2, t \log q_2\right).$$

By Burgess' Lemma we have

$$S(W_t') \ll \sum_{x=1}^{T/t} \left(\frac{T}{x}\right)^{1-\frac{1}{r}} q_2^{\frac{r+1}{4r^2}} (\log q_2)^{\frac{1}{r}} \ll T t^{-\frac{1}{r}} q_2^{\frac{r+1}{4r^2}} (\log q_2)^{\frac{1}{r}}.$$

8. Proof of Corollary 1

The first three inequalities immediately follow from Theorem 1.

Let us derive the fourth inequality from Burgess' Lemma with $r = 1$. We begin with splitting the sum over the points under the hyperbola into three parts:

$$\Omega_1 = \big\{(x, y) \in \mathbb{R}_+^2 \mid xy < T, x < \sqrt{T}\big\},$$

$$\Omega_2 = \big\{(x, y) \in \mathbb{R}_+^2 \mid xy < T, y < \sqrt{T}\big\},$$

and U_0 defined by (3).

Applying Burgess' Lemma with $r = 1$ we get

$$S(\Omega_1) \ll \sqrt{T} q^{1/2+\epsilon}$$

and

$$S(U_0) \ll q^{1+\epsilon}.$$

We are interested in the case $T \geqslant q^{3/2}$. So,

$$\sqrt{T} q^{1/2+\epsilon} \geqslant q^{1+\epsilon},$$

which implies the desired statement.

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Dmitry Ushanov

Lomonosov Moscow State University,
Fac. Mathematics and Mechanics,
Dept. Number Theory,
119991 Moscow, Russia
ushanov.dmitry@gmail.com