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Algebraic independence of values of exponential type power series

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Abstract: In this paper we study variants of exponential type power series of the form $f_k(x) = \sum_{n=0}^{\infty} c_{k,n}\alpha^n/n!$ $(k = 1, \ldots, q)$ with real or complex coefficients $c_{k,n}$ and with a nonzero algebraic number α. We find all subsets of $\{f_1(\alpha), \ldots, f_q(\alpha)\}$ which are algebraically independent over $\mathbb{Q}$. We apply our method to series with periodic sequences, i. e., with $c_{k,n} = 1$ if $n \equiv k \pmod q$, $c_{k,n} = 0$ otherwise and also with $c_{k,n} = \{(an + k)/q\}$ for coprime integers a and q, where $\{x\}$ is the fractional part of a real number x. More applications deal with various series formed by Fibonacci and Lucas numbers, e. g., $c_{k,n} = F_{an+k}$. All the algebraic independence results are finally reduced to the Lindemann—Weierstrass theorem.

Keywords: Algebraic independence, power series, Lindemann—Weierstrass theorem

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1. Introduction

In this paper we consider the values of exponential type power series

$$f(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$$

at a nonzero algebraic point $z = \alpha$. In Section 2 we investigate the numbers

$$f_r(\alpha) = \sum_{n=0}^{\infty} a_{r,n} \frac{\alpha^n}{n!} \quad (r = 0, 1, \dots, q-1),$$

where $(a_{r,n})_{n \geqslant 0}$ are periodic with period $q \geqslant 3$. In particular, we prove that among q numbers

$$\sum_{\substack{n=0 \\ n \equiv r \pmod{q}}}^{\infty} \frac{\alpha^n}{n!} \quad (r = 0, 1, \dots, q-1)$$

any $\varphi(q)$ are algebraically independent over $\mathbb{Q}$ and not are any $\varphi(q)+1$, where $\varphi(q)$ is Euler's totient (Theorem 1). Section 3 is devoted to series with coefficients given by fractional parts $a_n = \{p(n)\}$ of polynomials $p(x) \in \mathbb{Q}[x]$. The transcendence of the number

$$\sum_{n=0}^{\infty} \{p(n)\} \frac{\alpha^n}{n!}$$

is given in Theorem 3. In the case of linear polynomials, we obtain the following (Theorem 4): Let $q \geqslant 3$ and a be integers with $0 < a < q$ and $(a, q) = 1$. Then among q numbers

$$\sum_{n=0}^{\infty} \left\{ \frac{an+b}{q} \right\} \frac{\alpha^n}{n!} \quad (b = 0, 1, \dots, q-1)$$

any $\varphi(q)$ are algebraically independent over $\mathbb{Q}$ and not are any $\varphi(q)+1$. In the final section 4 the coefficients $a_{r,n}$ are formed by Fibonacci numbers F_n and Lucas numbers L_n. We show that any two numbers in the set

$$\left\{ \sum_{n=0}^{\infty} F_n^s \frac{\alpha^n}{n!} \,\middle|\, s \in \mathbb{N} \right\} \cup \left\{ \sum_{n=0}^{\infty} L_n^s \frac{\alpha^n}{n!} \,\middle|\, s \in \mathbb{N} \right\}$$

are algebraically independent over $\mathbb{Q}$ and not are any three (Theorem 5), and the same holds in each of the sets

$$\left\{ \sum_{n=0}^{\infty} F_{an+b} \frac{\alpha^n}{n!} \,\middle|\, a \in \mathbb{N},\ b \in \mathbb{N}_0 \right\}, \quad \left\{ \sum_{n=0}^{\infty} L_{an+b} \frac{\alpha^n}{n!} \,\middle|\, a \in \mathbb{N},\ b \in \mathbb{N}_0 \right\}.$$

(Theorem 6). For the proof we express numbers in question as rational functions of some values of the exponential function that are algebraically independent over $\mathbb{Q}$ due to the Lindemann—Weierstrass theorem. Then we apply a criterion of algebraic independence (Lemma 1) to a set of such numbers and reduce the problem to the non-vanishing of a Jacobian corresponding to these expressions.

2. Exponential type power series with periodic coefficients

Let $q \geqslant 2$ be an integer and let $\xi = e^{2\pi i/q}$. We consider the power series

$$e_r(z) = e_{q,r}(z) = \sum_{\substack{n=0 \\ n \equiv r \pmod{q}}}^{\infty} \frac{z^n}{n!} \quad (r = 0, 1, \ldots, q-1)\,. \tag{1}$$

Trivially the relation

$$e_0(z) + e_1(z) + \ldots + e_{q-1}(z) = e^z$$

holds. Using the formula

$$\frac{1}{q} \sum_{k=0}^{q-1} \xi^{k(n-r)} = \begin{cases} 1 & \text{if } n \equiv r \pmod{q}\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

we have

$$\begin{aligned} e_r(z) &= \frac{1}{q} \sum_{k=0}^{q-1} \xi^{-kr} \sum_{n=0}^{\infty} \frac{\xi^{kn} z^n}{n!} \\ &= \frac{1}{q} \left(e^z + \xi^{-r} e^{\xi z} + \xi^{-2r} e^{\xi^2 z} + \ldots + \xi^{-(q-1)r} e^{\xi^{q-1} z} \right), \end{aligned}$$

or

$$\begin{pmatrix} e_0(z) \\ e_1(z) \\ e_2(z) \\ \vdots \\ e_{q-1}(z) \end{pmatrix} = C \begin{pmatrix} e^z \\ e^{\xi z} \\ e^{\xi^2 z} \\ \vdots \\ e^{\xi^{q-1} z} \end{pmatrix}, \tag{2}$$

where

$$C = \frac{1}{q}\begin{pmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,q} \\ c_{2,1} & c_{2,2} & \dots & c_{2,q} \\ \vdots & \vdots & & \vdots \\ c_{q,1} & c_{q,2} & \dots & c_{q,q} \end{pmatrix} = \frac{1}{q}\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \xi^{-1} & \xi^{-2} & \dots & \xi^{-(q-1)} \\ 1 & \xi^{-2} & \xi^{-2\cdot 2} & \dots & \xi^{-(q-1)\cdot 2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \xi^{-(q-1)} & \xi^{-2(q-1)} & \dots & \xi^{-(q-1)(q-1)} \end{pmatrix}. \tag{3}$$

Since ξ is a root of the q-th cyclotomic polynomial, $1, \xi, \xi^2, \dots, \xi^{\varphi(q)-1}$ are linearly independent over $\mathbb{Q}$ and every ξ^k can be written as a linear combination of these $\varphi(q)$ numbers over $\mathbb{Z}$. If α is a nonzero algebraic number, $e^{\alpha}, e^{\xi\alpha}, \dots, e^{\xi^{\varphi(q)-1}\alpha}$ are algebraically independent over $\mathbb{Q}$ by the Lindemann—Weierstrass theorem and in view of (2) each of the numbers $e_0(\alpha),\ e_1(\alpha), \dots,\ e_{q-1}(\alpha)$ is transcendental (cf. [2]).

THEOREM 1. *Let $q \geqslant 3$ be an integer. If α is a nonzero algebraic number, then among q numbers*

$$e_0(\alpha),\ e_1(\alpha), \dots,\ e_{q-1}(\alpha)$$

any $\varphi(q)$ are algebraically independent over $\mathbb{Q}$. Moreover, any $\varphi(q)+1$ of the q functions $e_0(z),\ e_1(z), \dots,\ e_{q-1}(z)$ are algebraically dependent over $\mathbb{Q}$.

COROLLARY 1. *Let $q \geqslant 3$ be an integer and let α be a nonzero algebraic number. Then any $\varphi(q)$ of the numbers*

$$\sum_{n=0}^{\infty} \frac{\alpha^n}{(qn+r)!} \quad (r = 0, 1, \dots, q-1)$$

are algebraically independent over $\mathbb{Q}$.

Example 1. In the case of $q = 2$, we have

$$e_{2,0}^2(z) - e_{2,1}^2(z) = \cosh^2(z) - \sinh^2(z) = 1\,,$$

and for $q = 3$

$$e_0^3(z) + e_1^3(z) + e_2^3(z) - 3e_0(z)e_1(z)e_2(z) = 1\,.$$

Remark. It is easy to check that the functions $e_r(z)$ satisfy very symmetric system of differential equations with constant coefficients:

$$e_r(z)' = e_{r-1}(z), \quad 0 \leqslant r < q,$$

where $e_{-1}(z) = e_{q-1}(z)$.

By the general theory of E-functions created by C.L.Siegel and A.B.Shidlovskii, see [4], the Theorem 1 can be reduced to the proof of algebraic independence over $\mathbb{C}(z)$ corresponding functions. This problem is simpler but not trivial one.

Let $\alpha_1, \ldots, \alpha_m$ be complex numbers. It is well known that algebraic dependence of functions $e^{\alpha_1 z}, \ldots, e^{\alpha_m z}$ over the field $\mathbb{C}(z)$ implies the linear dependence of numbers $\alpha_1, \ldots, \alpha_m$ over the field of rational numbers $\mathbb{Q}$. And this property implies in turn the algebraic dependence of functions $e^{\alpha_1 z}, \ldots, e^{\alpha_m z}$ over the field $\mathbb{C}$. That's why

$$\operatorname{tr}\deg_{\mathbb{C}(z)}\mathbb{C}(z, e^z, e^{\xi z} \ldots, e^{\xi^{q-1} z}) = \operatorname{tr}\deg_{\mathbb{C}}\mathbb{C}(e^z, e^{\xi z} \ldots, e^{\xi^{q-1} z}).$$

Due to (2) we have $\mathbb{C}(e^z, e^{\xi z} \ldots, e^{\xi^{q-1} z}) = \mathbb{C}(e_0(z), \ldots, e_{q-1}(z))$ and consequently

$$\operatorname{tr}\deg_{\mathbb{C}(z)}\mathbb{C}(z, e_0(z), \ldots, e_{q-1}(z)) = \operatorname{tr}\deg_{\mathbb{C}}\mathbb{C}(e_0(z), \ldots, e_{q-1}(z)).$$

This equality and Theorem 8 from Chapter 4 of [4] reduce the proof of Theorem 1 to the following functional statement:

Among q functions

$$e_0(z),\ e_1(z), \ldots,\ e_{q-1}(z)$$

any $\varphi(q)$ are algebraically independent over $\mathbb{C}(z)$.

But simpler way to prove the Theorem 1 is the direct reduction of it to the Lindemann—Weierstrass theorem.

For the proof of Theorem 1 we need the following lemmas.

LEMMA 1 ([1, Theorem 1.2]). *Let K be an algebraic number field, let $x_1, \dots, x_n \in \mathbb{C}$ be algebraically independent over $\mathbb{Q}$, and let $y_1, \dots, y_n \in \mathbb{C}$ satisfying the system of equations*

$$P_j(x_1, \dots, x_n, y_1, \dots, y_n) = 0 \quad (j = 1, \dots, n),$$

where $P_j(X_1, \dots, X_n, Y_1, \dots, Y_n) \in K[X_1, \dots, X_n, Y_1, \dots, Y_n]$ $(j = 1, \dots, n)$. Assume that

$$\det\left(\frac{\partial P_j}{\partial X_i}(x_1, \dots, x_n, y_1, \dots, y_n)\right)_{1 \leqslant i,j \leqslant n} \neq 0\,.$$

Then the numbers $y_1, \dots, y_n$ are algebraically independent over $\mathbb{Q}$.

LEMMA 2. *Let $q \geqslant 3$ be an integer and let $\xi = e^{2\pi i/q}$. Then, for any $2s$ integers $m_1, \dots, m_s$, $n_1, \dots, n_s$ with $0 \leqslant m_1 < \dots < m_s < n_1 < \dots < n_s < q$, we have*

$$\xi^{m_1} + \xi^{m_2} + \dots + \xi^{m_s} \neq \xi^{n_1} + \xi^{n_2} + \dots + \xi^{n_s}\,.$$

As a special case, we obtain:

LEMMA 3. *Let q and ξ be as above. Then for any $\varphi(q)$ integers $n_1, \dots, n_{\varphi(q)}$ with $0 \leqslant n_1 < \dots < n_{\varphi(q)} < q$ and $\big(n_1, \dots, n_{\varphi(q)}\big) \neq \big(0, 1, \dots, \varphi(q) - 1\big)$ we have*

$$1 + \xi + \xi^2 + \dots + \xi^{\varphi(q)-1} \neq \xi^{n_1} + \xi^{n_2} + \dots + \xi^{n_{\varphi(q)}}\,.$$

PROOF OF LEMMA 2. We may assume that $s \geqslant 2$. We put $\lambda = (m_1 + m_s)/2$ and

$$M_1 := \left\{e^{2\pi i\theta/q} \,\middle|\, -\frac{m_s - m_1}{2} \leqslant \theta \leqslant \frac{m_s - m_1}{2}\right\}.$$

Since

$$-\frac{m_s - m_1}{2} \leqslant m_j - \lambda \leqslant \frac{m_s - m_1}{2} \quad (j = 1, \dots, s)\,,$$

we have

$$\xi^{m_j - \lambda} \in M_1, \quad \xi^{n_j - \lambda} \in M_2 := \big\{z \in \mathbb{C} \,\big|\, |z| = 1\big\} \setminus M_1\,.$$

It follows from the definition that

$$\operatorname{Re} z_1 \geqslant \cos\left(\frac{\pi(m_s - m_1)}{q}\right) > \operatorname{Re} z_2$$

holds for all $z_1 \in M_1$ and $z_2 \in M_2$. Thus we get

$$\mathrm{Re}\left(\xi^{m_1-\lambda}+\ldots+\xi^{m_s-\lambda}\right) \geqslant s\cos\left(\frac{\pi(m_s-m_1)}{q}\right) > \mathrm{Re}\left(\xi^{n_1-\lambda}+\ldots+\xi^{n_s-\lambda}\right).$$

In particular, we obtain

$$\xi^{m_1-\lambda}+\xi^{m_2-\lambda}+\ldots+\xi^{m_s-\lambda} \neq \xi^{n_1-\lambda}+\xi^{n_2-\lambda}+\ldots+\xi^{n_s-\lambda},$$

and the desired inequality follows. □

PROOF OF THEOREM 1. We put $d=\varphi(q)$ for brevity. Each ξ^{k-1} can be written as

$$\xi^{k-1}=n_{1,k}+n_{2,k}\xi+n_{3,k}\xi^2+\ldots+n_{d,k}\xi^{d-1} \quad (k=1,\ldots,q) \tag{4}$$

with $n_{i,k}\in\mathbb{Z}$, where $(n_{1,k},\ldots,n_{d,k})=(0,\ldots,0,1,0\ldots,0)$ with 1 at the k-th position if $k\leqslant d$. Let an algebraic number $\alpha\neq 0$ be given. We put

$$x_k=e^{\xi^{k-1}\alpha} \quad (k=1,\ldots,q). \tag{5}$$

Then $x_1,x_2,\ldots,x_d$ are algebraically independent over $\mathbb{Q}$ and the identities

$$x_k=x_1^{n_{1,k}}x_2^{n_{2,k}}\cdots x_d^{n_{d,k}} \quad (k=1,\ldots,q) \tag{6}$$

hold in view of (4), where for $k\leqslant d$, $n_{j,k}=1$ if $j=k$, and $n_{j,k}=0$ otherwise. Putting

$$y_k=e_{k-1}(\alpha) \quad (k=1,\ldots,q),$$

we have by (2)

$$\begin{pmatrix} y_1\\ y_2\\ \vdots\\ y_q\end{pmatrix}=C\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_q\end{pmatrix}. \tag{7}$$

We set

$$m_j:=\max_{d<k\leqslant q}\left\{|n_{j,k}| \,\middle|\, n_{j,k}\leqslant 0\right\} \quad (j=1,\ldots,d),$$

so that $m_k + n_{j,k} \geqslant 0$ $(j = 1, \ldots, d,\ k = 1, \ldots, q)$, and define

$$w_k = w_k(x_1, \ldots, x_d) = x_1^{m_1} \cdots x_d^{m_d} x_k \tag{8}$$

$$= x_1^{m_1+n_{1,k}} \cdots x_d^{m_d+n_{d,k}} \quad (k = 1, \ldots, q) \tag{9}$$

by (6). Then we get

$$x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{pmatrix} = C \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_q \end{pmatrix} . \tag{10}$$

Now we introduce the polynomials

$$P_k = P_k(X_1, \ldots, X_d, Y_k) \quad (k = 1, \ldots, q)$$

defined by

$$\begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_q \end{pmatrix} = X_1^{m_1} X_2^{m_2} \cdots X_d^{m_d} \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_q \end{pmatrix} - C \begin{pmatrix} w_1(X_1, \ldots, X_d) \\ w_2(X_1, \ldots, X_d) \\ \vdots \\ w_q(X_1, \ldots, X_d) \end{pmatrix} ,$$

so that

$$P_k(x_1, \ldots, x_d, y_k) = 0 \quad (k = 1, \ldots, q) .$$

Letting for brevity $w_k = w_k(x_1, \ldots, x_d)$ and

$$\frac{\partial P_j}{\partial x_i} := \Big[\frac{\partial}{\partial X_i} P_j(X_1, \ldots, X_d, y_j) \Big]_{(X_1, \ldots, X_d) = (x_1, \ldots, x_d)} ,$$

we have the $q \times d$ matrix

$$
\begin{pmatrix}
x_1\dfrac{\partial P_1}{\partial x_1} & x_2\dfrac{\partial P_1}{\partial x_2} & \dots & x_d\dfrac{\partial P_1}{\partial x_d} \\
x_1\dfrac{\partial P_2}{\partial x_1} & x_2\dfrac{\partial P_2}{\partial x_2} & \dots & x_d\dfrac{\partial P_2}{\partial x_d} \\
\vdots & \vdots & & \vdots \\
x_1\dfrac{\partial P_q}{\partial x_1} & x_2\dfrac{\partial P_q}{\partial x_2} & \dots & x_d\dfrac{\partial P_q}{\partial x_d}
\end{pmatrix} =
$$

$$
= x_1^{m_1}x_2^{m_2}\cdots x_d^{m_d}\left(m_1\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{pmatrix} \quad m_2\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{pmatrix} \quad \dots \quad m_d\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{pmatrix} \right) -
$$

$$
- C\begin{pmatrix}
(m_1+n_{1,1})w_1 & (m_2+n_{2,1})w_1 & \dots & (m_d+n_{d,1})w_1 \\
(m_1+n_{1,2})w_2 & (m_2+n_{2,2})w_2 & \dots & (m_d+n_{d,2})w_2 \\
\vdots & \vdots & & \vdots \\
(m_1+n_{1,q})w_q & (m_2+n_{2,q})w_q & \dots & (m_d+n_{d,q})w_q
\end{pmatrix} =
$$

$$
= - C\begin{pmatrix}
n_{1,1}w_1 & n_{2,1}w_1 & \dots & n_{d,1}w_1 \\
n_{1,2}w_2 & n_{2,2}w_2 & \dots & n_{d,2}w_2 \\
\vdots & \vdots & & \vdots \\
n_{1,q}w_q & n_{2,q}w_q & \dots & n_{d,q}w_q
\end{pmatrix} =
$$

$$
= -\frac{1}{q}\begin{pmatrix}
c_{1,1} & c_{1,2} & \dots & c_{1,q} \\
c_{2,1} & c_{2,2} & \dots & c_{2,q} \\
\vdots & \vdots & & \vdots \\
c_{q,1} & c_{q,2} & \dots & c_{q,q}
\end{pmatrix}
\begin{pmatrix}
n_{1,1}w_1 & n_{2,1}w_1 & \dots & n_{d,1}w_1 \\
n_{1,2}w_2 & n_{2,2}w_2 & \dots & n_{d,2}w_2 \\
\vdots & \vdots & & \vdots \\
n_{1,q}w_q & n_{2,q}w_q & \dots & n_{d,q}w_q
\end{pmatrix} =
$$

$$= -\frac{1}{q}\left(\vec{v}_1\ \vec{v}_2 \ldots \vec{v}_d\right) \tag{11}$$

with

$$\vec{v}_j = n_{j,1} w_1 \begin{pmatrix} c_{1,1} \\ c_{2,1} \\ \vdots \\ c_{q,1} \end{pmatrix} + n_{j,2} w_2 \begin{pmatrix} c_{1,2} \\ c_{2,2} \\ \vdots \\ c_{q,2} \end{pmatrix} + \ldots + n_{j,q} w_q \begin{pmatrix} c_{1,q} \\ c_{2,q} \\ \vdots \\ c_{q,q} \end{pmatrix} \quad (j = 1, \ldots, d)\,,$$

using (9), (10), and (3).

We prove first the algebraic independence over $\mathbb{Q}$ of the numbers $y_1, y_2, \ldots, y_d$. In this case we may regard $P_1, \ldots, P_d$ as polynomials of $X_1, \ldots, X_d, Y_1, \ldots, Y_d$ over $K = \mathbb{Q}(\xi)$. (The proof for arbitrarily chosen numbers $y_{l_1}, y_{l_2}, \ldots, y_{l_d}$ is similar and will be discussed later.) By Lemma 1 it is enough to show that

$$\Delta := \begin{vmatrix} \frac{\partial P_1}{\partial x_1} & \frac{\partial P_1}{\partial x_2} & \cdots & \frac{\partial P_1}{\partial x_d} \\ \frac{\partial P_2}{\partial x_1} & \frac{\partial P_2}{\partial x_2} & \cdots & \frac{\partial P_2}{\partial x_d} \\ \vdots & \vdots & & \vdots \\ \frac{\partial P_d}{\partial x_1} & \frac{\partial P_d}{\partial x_2} & \cdots & \frac{\partial P_d}{\partial x_d} \end{vmatrix} \neq 0\,.$$

It follows from (11) that

$$(-q)^d x_1 x_2 \cdots x_d \Delta =$$

$$= \sum_{1 \leqslant k_1 < \ldots < k_d \leqslant q} D(k_1, \ldots, k_d) \begin{vmatrix} c_{1,k_1} & c_{1,k_2} & \ldots & c_{1,k_d} \\ c_{2,k_1} & c_{2,k_2} & \ldots & c_{2,k_d} \\ \vdots & \vdots & & \vdots \\ c_{d,k_1} & c_{d,k_2} & \ldots & c_{d,k_d} \end{vmatrix} w_{k_1} \cdots w_{k_d} =$$

$$= \begin{vmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,d} \\ c_{2,1} & c_{2,2} & \dots & c_{2,d} \\ \vdots & \vdots & & \vdots \\ c_{d,1} & c_{d,2} & \dots & c_{d,d} \end{vmatrix} w_1 \cdots w_d +$$

$$+ \sum_{\substack{1 \leqslant k_1 < \ldots < k_d \leqslant q \\ (k_1, \ldots, k_d) \neq (1, \ldots, d)}} D(k_1, \ldots, k_d) \begin{vmatrix} c_{1,k_1} & c_{1,k_2} & \dots & c_{1,k_d} \\ c_{2,k_1} & c_{2,k_2} & \dots & c_{2,k_d} \\ \vdots & \vdots & & \vdots \\ c_{d,k_1} & c_{d,k_2} & \dots & c_{d,k_d} \end{vmatrix} w_{k_1} \cdots w_{k_d}\,, \qquad (12)$$

where $D(k_1, \ldots, k_d) = \pm n_{1,k_1} n_{2,k_2} \cdots n_{d,k_d}$, and especially $D(1, \ldots, d) = 1$. Here we prove that

$$w_1 w_2 \cdots w_d \neq w_{k_1} w_{k_2} \cdots w_{k_d} \qquad (13)$$

for all $(k_1, \ldots, k_d) \neq (1, \ldots, d)$ with $1 \leqslant k_1 < \ldots < k_d \leqslant q$. Suppose on the contrary that

$$w_1 w_2 \cdots w_d = w_{k_1} w_{k_2} \cdots w_{k_d}$$

holds for some $(k_1, \ldots, k_d) \neq (1, \ldots, d)$. Then we have by (8) that

$$x_1 x_2 \cdots x_d = x_{k_1} x_{k_2} \cdots x_{k_d}\,,$$

or

$$e^{(1+\xi+\xi^2+\ldots+\xi^{d-1})\alpha} = e^{(\xi^{k_1-1}+\xi^{k_2-1}+\ldots+\xi^{k_d-1})\alpha}$$

by (5). Since the exponents are algebraic, it follows that

$$1 + \xi + \xi^2 + \ldots + \xi^{d-1} = \xi^{k_1-1} + \xi^{k_2-1} + \ldots + \xi^{k_d-1}\,,$$

which contradicts Lemma 3 and (13) follows.

Now, from (12) with (3) and (9), $(-q)^d x_1 x_2 \cdots x_d \Delta$ is a nonzero polynomial in algebraically independent numbers $x_1, x_2, \ldots, x_d$ over $K = \mathbb{Q}(\xi)$. Indeed, in the right-hand side of (12), the coefficient of the first term is

$$\begin{vmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,d} \\ c_{2,1} & c_{2,2} & \dots & c_{2,d} \\ \vdots & \vdots & & \vdots \\ c_{d,1} & c_{d,2} & \dots & c_{d,d} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \xi^{-1} & \xi^{-2} & \dots & \xi^{-(d-1)} \\ 1 & \xi^{-2} & \xi^{-2\cdot 2} & \dots & \xi^{-(d-1)\cdot 2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \xi^{-(d-1)} & \xi^{-2(d-1)} & \dots & \xi^{-(d-1)(d-1)} \end{vmatrix} =$$

$$= \prod_{0\leqslant i<j\leqslant d-1} \left(\xi^{-j} - \xi^{-i}\right) \neq 0$$

and by (13) the first term is not cancelled by the remaining terms. Therefore, we have $(-q)^d x_1 x_2 \cdots x_d \Delta \neq 0$, namely, $\Delta \neq 0$.

Let $y_{l_1}, \dots, y_{l_d}$ be arbitrarily chosen d numbers from $y_1, \dots, y_q$. Our arguments up to the formula (11) are independent of this choice. We have to show the non-vanishing of the determinant

$$\Delta = \Delta(l_1, l_2, \dots, l_d) = \begin{vmatrix} \frac{\partial P_{l_1}}{\partial x_1} & \frac{\partial P_{l_1}}{\partial x_2} & \dots & \frac{\partial P_{l_1}}{\partial x_d} \\ \frac{\partial P_{l_2}}{\partial x_1} & \frac{\partial P_{l_2}}{\partial x_2} & \dots & \frac{\partial P_{l_2}}{\partial x_d} \\ \vdots & \vdots & & \vdots \\ \frac{\partial P_{l_d}}{\partial x_1} & \frac{\partial P_{l_d}}{\partial x_2} & \dots & \frac{\partial P_{l_d}}{\partial x_d} \end{vmatrix}.$$

Now, in place of (12), we have

$$(-q)^d x_1 x_2 \cdots x_d \Delta =$$

$$= \begin{vmatrix} c_{l_1,1} & c_{l_1,2} & \dots & c_{l_1,d} \\ c_{l_2,1} & c_{l_2,2} & \dots & c_{l_2,d} \\ \vdots & \vdots & & \vdots \\ c_{l_d,1} & c_{l_d,2} & \dots & c_{l_d,d} \end{vmatrix} w_1 \cdots w_d +$$

$$+ \sum_{\substack{1 \leqslant k_1 < \ldots < k_d \leqslant q \\ (k_1, \ldots, k_d) \neq (1, \ldots, d)}} D(k_1, \ldots, k_d) \begin{vmatrix} c_{l_1,k_1} & c_{l_1,k_2} & \cdots & c_{l_1,k_d} \\ c_{l_2,k_1} & c_{l_2,k_2} & \cdots & c_{l_2,k_d} \\ \vdots & \vdots & & \vdots \\ c_{l_d,k_1} & c_{l_d,k_2} & \cdots & c_{l_d,k_d} \end{vmatrix} w_{k_1} \cdots w_{k_d} ,$$

where $D(k_1, \ldots, k_d) = \pm n_{l_1,k_1} \cdots n_{l_d,k_d}$. By the same reason as above, we have only to show that

$$\begin{vmatrix} c_{l_1,1} & c_{l_1,2} & \cdots & c_{l_1,d} \\ c_{l_2,1} & c_{l_2,2} & \cdots & c_{l_2,d} \\ \vdots & \vdots & & \vdots \\ c_{l_d,1} & c_{l_d,2} & \cdots & c_{l_d,d} \end{vmatrix} = \begin{vmatrix} 1 & \xi^{-l_1} & \xi^{-2l_1} & \ldots & \xi^{-dl_1} \\ 1 & \xi^{-l_2} & \xi^{-2l_2} & \ldots & \xi^{-dl_2} \\ 1 & \xi^{-l_3} & \xi^{-2l_3} & \ldots & \xi^{-dl_3} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & \xi^{-l_d} & \xi^{-2l_d} & \ldots & \xi^{-dl_d} \end{vmatrix} \neq 0 ,$$

which however is obviously true. The proof of the first statement is now completed.

Next we replace α in (5) by z. Then, by (6) and (7), we have

$$y_k(z) \in \mathbb{Q}(\xi)(x_1, \ldots, x_d) \quad (k = 1, \ldots, q) .$$

Thus, any $d+1$ functions from the set $\{y_1(z), \ldots, y_q(z)\}$ are algebraically dependent over $\mathbb{Q}(\xi)$, and therefore over $\mathbb{Q}$. This proves the second statement. □

Let $q \geqslant 3$ be an integer, $\xi = e^{2\pi i/q}$, and K be a number field. For brevity put $d = \varphi(q)$. We consider the functions $f_1(z), \ldots, f_d(z)$ defined by

$$\begin{pmatrix} f_1(z) \\ f_2(z) \\ \vdots \\ f_d(z) \end{pmatrix} = A \begin{pmatrix} e^z \\ e^{\xi z} \\ \vdots \\ e^{\xi^{q-1} z} \end{pmatrix} , \tag{14}$$

where

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,q} \\ a_{2,1} & a_{2,2} & \dots & a_{2,q} \\ \vdots & \vdots & & \vdots \\ a_{d,1} & a_{d,2} & \dots & a_{d,q} \end{pmatrix} \quad (a_{i,j} \in K) .$$

For a given nonzero algebraic number α, we define x_k $(k = 1, \dots, q)$ by (5) and $y_k = f_k(\alpha)$ $(k = 1, \dots, d)$, so that we have

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_d \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{pmatrix} .$$

By the same arguments as above with A in place of C, we deduce:

THEOREM 2. *Let $f_1(z), \dots, f_d(z)$ be defined by* (14). *If*

$$\begin{vmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,d} \\ a_{2,1} & a_{2,2} & \dots & a_{2,d} \\ \vdots & \vdots & & \vdots \\ a_{d,1} & a_{d,2} & \dots & a_{d,d} \end{vmatrix} \neq 0 , \tag{15}$$

then the values of these series at any nonzero algebraic point z are algebraically independent over $\mathbb{Q}$.

As a typical example, we consider exponential type power series with periodic coefficients. Let $\{b_n\}_{n\geqslant 0}$ be a periodic sequence of algebraic numbers of period q and let

$$f(z) = \sum_{n=0}^{\infty} b_n \frac{z^n}{n!} .$$

In studying arithmetic properties of the values of such series at algebraic points, we may assume that $\{b_n\}_{n\geqslant 0}$ is purely periodic with a period $(b_0, b_1, \dots, b_{q-1}) \neq$

$\neq (0, \ldots, 0)$. Then we have

$$f(z) = \sum_{k=0}^{q-1} b_k e_k(z) \,, \tag{16}$$

where $e_k(z)$ is defined by (1), and hence the value of the series at any nonzero algebraic point is transcendental (cf. [2, 2.3]).

Let $\{b_{k,n}\}_{n\geqslant 0}$ $(k = 1, \ldots, d)$ be d purely periodic sequences of algebraic numbers with periods $(b_{k,0}, b_{k,1}, \ldots, b_{k,q-1}) \neq (0, \ldots, 0)$ and let

$$f_k(z) = \sum_{n=0}^{\infty} b_{k,n} \frac{z^n}{n!} \quad (k = 1, \ldots, d) \,.$$

Then, from (2) and (16), we deduce the expression (14) with

$$A = \begin{pmatrix} b_{1,0} & b_{1,1} & \ldots & b_{1,q-1} \\ b_{2,0} & b_{2,1} & \ldots & b_{2,q-1} \\ \vdots & \vdots & & \vdots \\ b_{d,0} & b_{d,1} & \ldots & b_{d,q-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \xi^{-1} & \xi^{-2} & \ldots & \xi^{-(q-1)} \\ 1 & \xi^{-2} & \xi^{-2\cdot 2} & \ldots & \xi^{-(q-1)\cdot 2} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & \xi^{-(q-1)} & \xi^{-2(q-1)} & \ldots & \xi^{-(q-1)(q-1)} \end{pmatrix} , \tag{17}$$

and so we can apply Theorem 2 to these functions.

Example 2. Let $\chi_0, \chi_1, \chi_2, \chi_3$ be the Dirichlet characters modulo 5, where χ_0 denotes the principle character. Set $q = 5,\ d = 4$,

$$\xi = e^{2\pi i/5} = \frac{1}{2\rho} + \frac{i}{2}\sqrt{2+\rho} \quad \left(\rho = \frac{1+\sqrt{5}}{2}\right) ,$$

and

$$b_{k,n} := \chi_{k-1}(n) \quad (k = 1, 2, 3, 4,\ n = 0, 1, \ldots) \,.$$

Then

$$A=\begin{pmatrix} a_{1,1} & \dots & a_{1,5} \\ \vdots & \vdots & \vdots \\ a_{4,1} & \dots & a_{4,5} \end{pmatrix}=\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & i & -i & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 1 & -i & i & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \xi^{-1} & \xi^{-2} & \xi^{-3} & \xi^{-4} \\ 1 & \xi^{-2} & \xi^{-4} & \xi^{-6} & \xi^{-8} \\ 1 & \xi^{-3} & \xi^{-6} & \xi^{-9} & \xi^{-12} \\ 1 & \xi^{-4} & \xi^{-8} & \xi^{-12} & \xi^{-16} \end{pmatrix},$$

and

$$\begin{vmatrix} a_{1,1} & \dots & a_{1,4} \\ \vdots & \vdots & \vdots \\ a_{4,1} & \dots & a_{4,4} \end{vmatrix}=-i\,\frac{1280(1+\sqrt{5})^{6}\sqrt{10+2\sqrt{5}}\sqrt{92274650+41266478\sqrt{5}}}{(3+\sqrt{5})^{12}}\neq 0\,.$$

Therefore, by Theorem 2, the values of the functions

$$f_{k+1}(z)=\sum_{n=0}^{\infty}\chi_k(n)\frac{z^n}{n!}\quad (k=0,1,2,3)$$

at a given nonzero algebraic point are algebraically independent over $\mathbb{Q}$.

3. Series involving fractional parts of polynomials

For any real number α we denote by $[\alpha]$ and $\{\alpha\}$ the integer and the fractional parts of α respectively.

THEOREM 3. *Let $f(x)\in\mathbb{Q}[x]$, α be a nonzero algebraic number, and*

$$S=\sum_{n=0}^{\infty}\frac{\{f(n)\}}{n!}\alpha^n\neq 0\,.$$

Then S is a transcendental number.

PROOF. Let q be the common denominator of all coefficients of the polynomial $f(x)$. Then $f(x)=h(x)/q$ with $h(x)\in\mathbb{Z}[x]$. For any two integer numbers n,m such that $n\equiv m \pmod q$ we have $h(n)\equiv h(m) \pmod q$ and $\{f(n)\}=\{f(m)\}$.

This means that the sequence $\{f(n)\}$, $n = 0, 1, 2, \ldots$, is periodic with the period q. The polynomial $x^q - 1$ has distinct roots ξ^k $(k = 0, 1, \ldots, q-1)$, where $\xi = e^{2\pi i/q}$. These roots are the characteristic roots of the recurrence equation $u_{n+q} - u_n = 0$. Since the sequence $\{f(n)\}$ satisfies this equation, we have

$$\{f(n)\} = b_0 + b_1\xi^n + \cdots + b_{q-1}\xi^{(q-1)n}$$

with some algebraic numbers $b_0, b_1, \ldots, b_{q-1}$. That's why

$$S = \sum_{n=0}^{\infty} \frac{\{f(n)\}}{n!}\alpha^n = \sum_{n=0}^{\infty}\sum_{k=0}^{q-1} b_k \frac{\xi^{kn}\alpha^n}{n!} = \sum_{k=0}^{q-1} b_k e^{\xi^k\alpha}. \tag{18}$$

Due to the Lindemann—Weierstrass theorem the number S is transcendental. □

Denote $\delta = z\dfrac{d}{dz}$. Since $\delta(z^n) = nz^n$, then $f(\delta)(z^n) = f(n)z^n$. Applying the operator $f(\delta)$ to the identity $\sum_{n=0}^{\infty} \dfrac{z^n}{n!} = e^z$, we derive

$$\sum_{n=0}^{\infty} \frac{f(n)}{n!} z^n = f(\delta)e^z = g(z)e^z\,,$$

where $g(z)$ is a polynomial with rational coefficients. Substituting $z = \alpha$ into this identity, we have

$$\sum_{n=0}^{\infty} \frac{f(n)}{n!}\alpha^n = g(\alpha)e^{\alpha}\,. \tag{19}$$

It follows from (18) and (19) that

$$\sum_{n=0}^{\infty} \frac{[f(n)]}{n!}\alpha^n = \sum_{n=0}^{\infty} \frac{f(n)}{n!}\alpha^n - \sum_{n=0}^{\infty} \frac{\{f(n)\}}{n!}\alpha^n =$$
$$= (g(\alpha) - b_0)e^{\alpha} - b_1 e^{\xi\alpha} - \cdots - b_{q-1}e^{\xi^{q-1}\alpha}\,.$$

Applying again the Lindemann—Weierstrass theorem, we have:

Corollary 2. *Let $f(x) \in \mathbb{Q}[x]$, α be a nonzero algebraic number, and*

$$S = \sum_{n=0}^{\infty} \frac{[f(n)]}{n!}\alpha^n \neq 0\,.$$

Then S is a transcendental number.

In the case of linear polynomials, we obtain the following results.

THEOREM 4. *Let q and a are coprime integers with $q \geqslant 3$ and $0 < a < q$. Let*

$$f_b(z) = \sum_{n=0}^{\infty} \left\{ \frac{an+b}{q} \right\} \frac{z^n}{n!} \quad (b = 0, 1, \dots, q-1).$$

If α is a nonzero algebraic number, then among q numbers $f_0(\alpha), \dots, f_{q-1}(\alpha)$ any $\varphi(q)$ are algebraically independent over $\mathbb{Q}$. Moreover, any $\varphi(q)+1$ of the functions $f_0(z), \dots, f_{q-1}(z)$ are algebraically dependent over $\mathbb{Q}$.

PROOF. We have

$$\begin{aligned} f_b(z) &= \sum_{m=0}^{\infty} \sum_{s=0}^{q-1} \left\{ \frac{a(mq+s)+b}{q} \right\} \frac{z^{mq+s}}{(mq+s)!} = \\ &= \sum_{s=0}^{q-1} \left\{ \frac{as+b}{q} \right\} e_s(z) = \frac{1}{q} \sum_{s=0}^{q-1} \overline{sa+b}\, e_s(z)\,, \end{aligned}$$

where $\overline{n} = r$ if $n \equiv r \pmod{q}$ and $0 \leqslant r < q$, namely,

$$\begin{pmatrix} f_0(z) \\ f_1(z) \\ f_2(z) \\ \vdots \\ f_{q-1}(z) \end{pmatrix} = D \begin{pmatrix} e_0(z) \\ e_1(z) \\ e_2(z) \\ \vdots \\ e_{q-1}(z) \end{pmatrix} = DC \begin{pmatrix} e^{z} \\ e^{\xi z} \\ e^{\xi^2 z} \\ \vdots \\ e^{\xi^{q-1} z} \end{pmatrix}, \tag{20}$$

where C is defined by (3) and

$$D = \frac{1}{q}\begin{pmatrix} 0 & \overline{a} & \overline{2a} & \dots & \overline{(q-1)a} \\ 1 & \overline{a+1} & \overline{2a+1} & \dots & \overline{(q-1)a+1} \\ 2 & \overline{a+2} & \overline{2a+2} & \dots & \overline{(q-1)a+2} \\ \vdots & \vdots & \vdots & & \vdots \\ q-1 & \overline{a+(q-1)} & \overline{2a+(q-1)} & \dots & \overline{(q-1)a+(q-1)} \end{pmatrix}.$$

Since $(a, q) = 1$ by assumption, the mapping $\sigma(s) := \overline{sa} \quad (s = 0, 1, \dots, q-1)$ defines a permutation of $(0, 1, \dots, q-1)$ with $\sigma(0) = 0$. So putting

$$\begin{pmatrix} d_{00} & d_{01} & \dots & d_{0,q-1} \\ d_{10} & d_{11} & \dots & d_{1,q-1} \\ \vdots & \vdots & & \vdots \\ d_{s0} & d_{s1} & \dots & d_{s,q-1} \\ \vdots & \vdots & & \vdots \\ d_{q-1,0} & d_{q-1,1} & \dots & d_{q-1,q-1} \end{pmatrix} := \begin{pmatrix} 0 & 1 & \dots & q-1 \\ 1 & 2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ s & s+1 & \dots & s-1 \\ \vdots & \vdots & & \vdots \\ q-1 & 0 & \dots & q-2 \end{pmatrix}, \tag{21}$$

we may write the matrix D as

$$D = \frac{1}{q}\begin{pmatrix} d_{0\sigma(0)} & d_{0\sigma(1)} & \cdots & d_{0,\sigma(q-1)} \\ d_{1\sigma(0)} & d_{1\sigma(1)} & \cdots & d_{1,\sigma(q-1)} \\ \vdots & \vdots & & \vdots \\ d_{s\sigma(0)} & d_{s\sigma(1)} & \cdots & d_{s,\sigma(q-1)} \\ \vdots & \vdots & & \vdots \\ d_{q-1,\sigma(0)} & d_{q-1,\sigma(1)} & \cdots & d_{q-1,\sigma(q-1)} \end{pmatrix}. \tag{22}$$

Again by $(a, q) = 1$, we can choose a q-th root of unity ρ such that $\xi = \rho^a$, so that $\xi^{-st} = \rho^{-sat} = \rho^{-\overline{sa}t} = \rho^{-\sigma(s)t}$ $(s, t = 0, 1, \dots, q-1)$ in the definition (3) of C.

Thus we have

$$C = \frac{1}{q}\begin{pmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ 1 & \rho^{-\sigma(1)} & \cdots & \rho^{-\sigma(1)t} & \cdots & \rho^{-\sigma(1)(q-1)} \\ 1 & \rho^{-\sigma(2)} & \cdots & \rho^{-\sigma(2)t} & \cdots & \rho^{-\sigma(2)(q-1)} \\ \vdots & \vdots & & \vdots & & \vdots \\ 1 & \rho^{-\sigma(q-1)} & \cdots & \rho^{-\sigma(q-1)t} & \cdots & \rho^{-\sigma(q-1)(q-1)} \end{pmatrix}. \tag{23}$$

It follow from (21), (22), and (23) that

$$\begin{aligned} DC &= \frac{1}{q^2}\left(\sum_{r=0}^{q-1} d_{s\sigma(r)}\rho^{-\sigma(r)t}\right)_{s,t=0,1,\ldots,q-1} = \\ &= \frac{1}{q^2}\left(\sum_{r=0}^{q-1} d_{sr}\rho^{-rt}\right)_{s,t=0,1,\ldots,q-1}. \end{aligned}$$

Since $(d_{s0}, d_{s1}, \ldots, d_{s,q-1}) = (s, s+1, \ldots, q-1, 0, 1, \ldots, s-1)$, we get

$$\begin{aligned} &\sum_{r=0}^{q-1} d_{sr}\rho^{-rt} = \\ &= s + (s+1)\rho^{-t} + \ldots + (q-1)\rho^{-(q-s-1)t} + \rho^{-(q-s+1)t} + (s-1)\rho^{-(q-1)t} = \\ &= \rho^{(s-1)t}\left(1 + 2\rho^{-t} + \ldots + (q-1)\rho^{-(q-2)t}\right) = \\ &= \begin{cases} q\dfrac{(q-1)}{2} & \text{if } t = 0\,, \\ q\dfrac{\rho^{(s+1)t}}{1-\rho^t} & \text{if } t = 1, 2, \ldots, q-1, \end{cases} \end{aligned}$$

using the formula

$$1 + 2x + 3x^2 + \ldots + (q-1)x^{q-2} = \frac{1 - qx^{q-1} + (q-1)x^q}{(1-x)^2}$$

with $x = \rho^{-t} \neq 1$ and $x^q = \rho^{-qt} = 1$ in the case $t \neq 0$. Thus we obtain

$$DC = \frac{1}{q} \begin{pmatrix} \frac{q-1}{2} & \frac{\rho}{1-\rho} & \frac{\rho^2}{1-\rho^2} & \cdots & \frac{\rho^{q-1}}{1-\rho^{q-1}} \\ \frac{q-1}{2} & \frac{\rho^2}{1-\rho} & \frac{\rho^4}{1-\rho^2} & \cdots & \frac{\rho^{2(q-1)}}{1-\rho^{q-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{q-1}{2} & \frac{\rho^{q-1}}{1-\rho} & \frac{\rho^{2(q-1)}}{1-\rho^2} & \cdots & \frac{\rho^{(q-1)(q-1)}}{1-\rho^{q-1}} \\ \frac{q-1}{2} & \frac{1}{1-\rho} & \frac{1}{1-\rho^2} & \cdots & \frac{1}{1-\rho^{q-1}} \end{pmatrix}. \tag{24}$$

Now we choose any $d := \varphi(q)$ functions $f_{b_1}(z), f_{b_2}(z), \ldots, f_{b_d}(z)$ given in (20). Then by (20) and (24) we have the expressions

$$\begin{pmatrix} f_{b_1}(z) \\ f_{b_2}(z) \\ \vdots \\ f_{b_d}(z) \end{pmatrix} = \frac{1}{q} \begin{pmatrix} \frac{q-1}{2} & \frac{\rho^{(b_1+1)}}{1-\rho} & \frac{\rho^{2(b_1+1)}}{1-\rho^2} & \cdots & \frac{\rho^{(q-1)(b_1+1)}}{1-\rho^{q-1}} \\ \frac{q-1}{2} & \frac{\rho^{(b_2+1)}}{1-\rho} & \frac{\rho^{2(b_2+1)}}{1-\rho^2} & \cdots & \frac{\rho^{(q-1)(b_2+1)}}{1-\rho^{q-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{q-1}{2} & \frac{\rho^{(b_d+1)}}{1-\rho} & \frac{\rho^{2(b_d+1)}}{1-\rho^2} & \cdots & \frac{\rho^{(q-1)(b_d+1)}}{1-\rho^{q-1}} \end{pmatrix} \begin{pmatrix} e^{z} \\ e^{\xi z} \\ e^{\xi^2 z} \\ \vdots \\ e^{\xi^{q-1} z} \end{pmatrix}, \tag{25}$$

where the determinant of the attached matrix is

$$\frac{1}{q^d} \begin{vmatrix} \frac{q-1}{2} & \frac{\rho^{(b_1+1)}}{1-\rho} & \frac{\rho^{2(b_1+1)}}{1-\rho^2} & \cdots & \frac{\rho^{(d-1)(b_1+1)}}{1-\rho^{d-1}} \\ \frac{q-1}{2} & \frac{\rho^{(b_2+1)}}{1-\rho} & \frac{\rho^{2(b_2+1)}}{1-\rho^2} & \cdots & \frac{\rho^{(d-1)(b_2+1)}}{1-\rho^{d-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{q-1}{2} & \frac{\rho^{(b_d+1)}}{1-\rho} & \frac{\rho^{2(b_d+1)}}{1-\rho^2} & \cdots & \frac{\rho^{(d-1)(b_d+1)}}{1-\rho^{d-1}} \end{vmatrix} =$$

$$= \frac{q-1}{2q^d(1-\rho)(1-\rho^2)\cdots(1-\rho^{d-1})} \cdot \begin{vmatrix} 1 & \rho^{b_1+1} & \rho^{2(b_1+1)} & \dots & \rho^{(d-1)(b_1+1)} \\ 1 & \rho^{b_2+1} & \rho^{2(b_2+1)} & \dots & \rho^{(d-1)(b_2+1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \rho^{b_d+1} & \rho^{2(b_d+1)} & \dots & \rho^{(d-1)(b_d+1)} \end{vmatrix} \neq 0\,.$$

Hence the first statement in Theorem 4 follows from Theorem 2.

To prove the second statement, we set $x_k = \exp(\rho^{k-1}z)$ $(k = 1, \dots, q)$. It follows from (6) (with α replaced by z) and (20), (24) that

$$f_b(z) \in \mathbb{Q}\big(\rho\big)(x_1, \dots, x_d\big) \quad (b = 0, \dots, q-1)\,.$$

So any $d+1$ functions in the set $\big\{f_0(z), \dots, f_{q-1}(z)\big\}$ are algebraically dependent over $\mathbb{Q}$. □

4. Series involving Fibonacci numbers

In this section we set $\rho := (1+\sqrt{5})/2$. Let

$$F_n = \frac{1}{\sqrt{5}}\left(\rho^n - \left(-\frac{1}{\rho}\right)^n\right), \quad L_n = \rho^n + \left(-\frac{1}{\rho}\right)^n \tag{26}$$

denote the *Fibonacci numbers* and the *Lucas numbers*, respectively.

THEOREM 5. *Let $f_s(\alpha)$ and $g_s(\alpha)$ be power series defined by*

$$f_s(z) = \sum_{n=0}^{\infty} F_n^s \frac{z^n}{n!}\,, \quad g_s(z) = \sum_{n=0}^{\infty} L_n^s \frac{z^n}{n!}.$$

If α is a nonzero algebraic number, then all the numbers in the set $\{f_s(\alpha) \,\big|\, s \in \mathbb{N}\} \cup$ $\cup \{g_s(\alpha) \mid s \in \mathbb{N}\}$ are distinct and any two are algebraically independent over $\mathbb{Q}$. Moreover, any three functions in the set $\{f_s(z) \mid s \in \mathbb{N}\} \cup \{g_s(z) \mid s \in \mathbb{N}\}$ are algebraically dependent over $\mathbb{Q}$.

The Fibonacci numbers F_n may be defined for negative integers $-n$ via Binet's formula, namely

$$F_{-n} = \frac{1}{\sqrt{5}}\Big(\rho^{-n} - \Big(-\frac{1}{\rho}\Big)^{-n}\Big) = (-1)^{n-1}F_n \quad (n \geqslant 0) \tag{27}$$

(cf. [3, p. 84, (5.18)]). Moreover, the following two identities are known:

$$\rho^n = \rho F_n + F_{n-1} \quad (n \geqslant 0)\,,$$

$$\rho^{-n} = \begin{cases} \rho F_n - F_{n+1} & \text{if } n \equiv 1 \pmod 2 \\ F_{n+1} - \rho F_n & \text{if } n \equiv 0 \pmod 2 \end{cases} \qquad (n \geqslant 1)$$

(cf. [3, p. 78, Lemma 5.1] and [3, p. 84, (5.20)]). Using (27), these formulas can be combined into

$$\rho^n = \rho F_n + F_{n-1} \quad (n \in \mathbb{Z})\,. \tag{28}$$

Proof of Theorem 5. We express first $g_s(z)$ as a finite sum involving the exponential function using (26) and (28), namely

$$\begin{aligned} g_s(z) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{s} \binom{s}{k} \rho^{n(s-k)} \Big(-\frac{1}{\rho}\Big)^{nk} = \\ &= \sum_{k=0}^{s} \binom{s}{k} \sum_{n=0}^{\infty} \frac{1}{n!}\Big((-1)^k \rho^{s-2k} z\Big)^n = \sum_{k=0}^{s} \binom{s}{k} e^{(-1)^k \rho^{s-2k} z} = \\ &= \sum_{k=0}^{s} \binom{s}{k} e^{(-1)^k (\rho F_{s-2k} + F_{s-2k-1}) z}\,. \end{aligned} \tag{29}$$

By similar arguments we obtain

$$f_s(z) = \frac{1}{5^{s/2}} \sum_{k=0}^{s} (-1)^k \binom{s}{k} e^{(-1)^k (\rho F_{s-2k} + F_{s-2k-1}) z}\,. \tag{30}$$

To prove the first statement, we set $x_1 = e^{\alpha}$ and $x_2 = e^{\rho\alpha}$, where α is a nonzero algebraic number. We know that x_1 and x_2 are algebraically independent over $\mathbb{Q}$. Expressing $g_s(\alpha)$ and $f_s(\alpha)$ by x_1, x_2, we have from (29) and (30) the rational functions

$$\left.\begin{aligned} g_s(\alpha) &= \sum_{k=0}^{s} \binom{s}{k} x_1^{(-1)^k F_{s-2k-1}} x_2^{(-1)^k F_{s-2k}}\,, \\ f_s(\alpha) &= \frac{1}{5^{s/2}} \sum_{k=0}^{s} (-1)^k \binom{s}{k} x_1^{(-1)^k F_{s-2k-1}} x_2^{(-1)^k F_{s-2k}}\,. \end{aligned}\right\} \tag{31}$$

First we show that $y_1 = g_{s_1}(\alpha)$ and $y_2 = f_{s_2}(\alpha)$ are algebraically independent over $\mathbb{Q}$ for arbitrary integers $s_1, s_2 \geqslant 0$. By (31) we have

$$y_i = R_i(x_1, x_2) := \frac{T_i(x_1, x_2)}{U_i(x_1, x_2)}\,, \tag{32}$$

where $T_i, U_i \in \mathbb{Q}(\rho)[X_1, X_2] \setminus \{0\}$ $(i = 1, 2)$ are coprime. We apply Lemma 1 introducing $P_i \in \mathbb{Q}(\rho)[X_1, X_2, Y_i]$ $(i = 1, 2)$ defined by

$$P_i(X_1, X_2, Y_i) := Y_i U_i(X_1, X_2) - T_i(X_1, X_2) \quad (i = 1, 2)\,.$$

We have by definition that $P_i(x_1, x_2, y_i) = 0$ $(i = 1, 2)$ and by (32)

$$\begin{aligned} \begin{vmatrix} \dfrac{\partial P_1}{\partial x_1} & \dfrac{\partial P_1}{\partial x_2} \\[2mm] \dfrac{\partial P_2}{\partial x_1} & \dfrac{\partial P_2}{\partial x_2} \end{vmatrix} &= \begin{vmatrix} y_1 \dfrac{\partial U_1}{\partial x_1} - \dfrac{\partial T_1}{\partial x_1} & y_1 \dfrac{\partial U_1}{\partial x_2} - \dfrac{\partial T_1}{\partial x_2} \\[2mm] y_2 \dfrac{\partial U_2}{\partial x_1} - \dfrac{\partial T_2}{\partial x_1} & y_2 \dfrac{\partial U_2}{\partial x_2} - \dfrac{\partial T_2}{\partial x_2} \end{vmatrix} = \\ = U_1 U_2 &\begin{vmatrix} \dfrac{1}{U_1^2}\left(\dfrac{\partial T_1}{\partial x_1} U_1 - T_1 \dfrac{\partial U_1}{\partial x_1}\right) & \dfrac{1}{U_1^2}\left(\dfrac{\partial T_1}{\partial x_2} U_1 - T_1 \dfrac{\partial U_1}{\partial x_2}\right) \\[2mm] \dfrac{1}{U_2^2}\left(\dfrac{\partial T_2}{\partial x_1} U_2 - T_2 \dfrac{\partial U_2}{\partial x_1}\right) & \dfrac{1}{U_2^2}\left(\dfrac{\partial T_2}{\partial x_2} U_2 - T_2 \dfrac{\partial U_2}{\partial x_2}\right) \end{vmatrix} = \\ = U_1 U_2 &\begin{vmatrix} \dfrac{\partial R_1}{\partial x_1} & \dfrac{\partial R_1}{\partial x_2} \\[2mm] \dfrac{\partial R_2}{\partial x_1} & \dfrac{\partial R_2}{\partial x_2} \end{vmatrix} =: U_1 U_2 \Delta\,. \end{aligned} \tag{33}$$

To apply Lemma 1 we have to show that $\Delta = \Delta(x_1, x_2) \neq 0$.

From (31) and (32) we obtain

$$\left.\begin{aligned}
\frac{\partial R_1}{\partial x_1} &= \sum_{k=0}^{s_1} (-1)^k \binom{s_1}{k} F_{s_1-2k-1} x_1^{(-1)^k F_{s_1-2k-1}-1} x_2^{(-1)^k F_{s_1-2k}},\\
\frac{\partial R_1}{\partial x_2} &= \sum_{k=0}^{s_1} (-1)^k \binom{s_1}{k} F_{s_1-2k} x_1^{(-1)^k F_{s_1-2k-1}} x_2^{(-1)^k F_{s_1-2k}-1},\\
\frac{\partial R_2}{\partial x_1} &= \frac{1}{5^{s_2/2}} \sum_{k=0}^{s_2} \binom{s_2}{k} F_{s_2-2k-1} x_1^{(-1)^k F_{s_2-2k-1}-1} x_2^{(-1)^k F_{s_2-2k}},\\
\frac{\partial R_2}{\partial x_2} &= \frac{1}{5^{s_2/2}} \sum_{k=0}^{s_2} \binom{s_2}{k} F_{s_2-2k} x_1^{(-1)^k F_{s_2-2k-1}} x_2^{(-1)^k F_{s_2-2k}-1}.
\end{aligned}\right\} \tag{34}$$

Since x_1 and x_2 are algebraically independent over $\mathbb{Q}$, it suffices to show that the function

$$5^{s_2/2} X^2 \Delta(X,X) = \\
= \begin{vmatrix}
\sum_{k=0}^{s_1} (-1)^k \binom{s_1}{k} F_{s_1-2k-1} X^{(-1)^k F_{s_1-2k+1}} & \sum_{k=0}^{s_1} (-1)^k \binom{s_1}{k} F_{s_1-2k} X^{(-1)^k F_{s_1-2k+1}} \\
\sum_{k=0}^{s_2} \binom{s_2}{k} F_{s_2-2k-1} X^{(-1)^k F_{s_2-2k+1}} & \sum_{k=0}^{s_2} \binom{s_2}{k} F_{s_2-2k} X^{(-1)^k F_{s_2-2k+1}}
\end{vmatrix}, \tag{35}$$

does not vanish identically. For two integers $m_1 \geqslant m_2$ and real numbers $a_{m_2}, a_{m_2+1}, \ldots, a_{m_1}$, we denote by $l(H) = a_{m_1} X^{m_1}$ with $a_{m_1} \neq 0$ the *leading term* of the rational function

$$H = H(X) = \sum_{\mu=m_2}^{m_1} a_\mu X^\mu .$$

To show that $5^{s_2/2} X^2 \Delta(X, X)$ does not vanish identically we compute the leading term

$$\mathcal{L} := l\big(5^{s_2/2} X^2 \Delta(X, X)\big) .$$

Case 1: $s_1 \neq s_2$. This is the simplest case, because the leading term $\mathcal{L}$ only depends on the terms corresponding to $k=0$:

$$\mathcal{L} = \begin{vmatrix} F_{s_1-1}X^{F_{s_1+1}} & F_{s_1}X^{F_{s_1+1}} \\ F_{s_2-1}X^{F_{s_2+1}} & F_{s_2}X^{F_{s_2+1}} \end{vmatrix} = \big(F_{s_1-1}F_{s_2} - F_{s_1}F_{s_2-1}\big)X^{F_{s_1+1}+F_{s_2+1}} =$$

$$= (-1)^{s_2-1}F_{s_1-s_2}X^{F_{s_1+1}+F_{s_2+1}} \not\equiv 0\,,$$

where the identity $F_{s_1-1}F_{s_2} - F_{s_1}F_{s_2-1} = (-1)^{s_2-1}F_{s_1-s_2}$ can be proven by induction on s_1 and by (27).

Case 2: $s := s_1 = s_2$. We have to distinguish whether $s \equiv 1 \pmod 2$ or $s \equiv 0 \pmod 2$.

Case 2.1: $s \equiv 1 \pmod 2$. Here the leading term $\mathcal{L}$ depends on all terms corresponding to $k=0$ *and* $k=s$. Using the relation

$$L_n = F_{n-1} + F_{n+1} \quad (n \geqslant 1)$$

(cf. [3, Corollary 5.5]), we obtain

$$\mathcal{L} = \begin{vmatrix} F_{s-1}X^{F_{s+1}} + (-1)^s F_{-s-1}X^{(-1)^s F_{-s+1}} & F_sX^{F_{s+1}} + (-1)^s F_{-s}X^{(-1)^s F_{-s+1}} \\ F_{s-1}X^{F_{s+1}} + F_{-s-1}X^{(-1)^s F_{-s+1}} & F_sX^{F_{s+1}} + F_{-s}X^{(-1)^s F_{-s+1}} \end{vmatrix} =$$

$$= \begin{vmatrix} F_{s-1}X^{F_{s+1}} + F_{s+1}X^{F_{s-1}} & F_sX^{F_{s+1}} - F_sX^{F_{s-1}} \\ F_{s-1}X^{F_{s+1}} - F_{s+1}X^{F_{s-1}} & F_sX^{F_{s+1}} + F_sX^{F_{s-1}} \end{vmatrix} =$$

$$= 2F_s\big(F_{s-1} + F_{s+1}\big)X^{F_{s-1}+F_{s+1}} = 2F_sL_sX^{L_s} \not\equiv 0\,.$$

Case 2.2: $s \equiv 0 \pmod 2$. We separate the cases $s \equiv 2 \pmod 4$ and $s \equiv 0 \pmod 4$.

Case 2.2.1: $s \equiv 2 \pmod 4$. This is the most complicate case, because the leading term $\mathcal{L}$ of the determinant in (35), say

$$5^{s_2/2} X^2 \Delta(X, X) = \begin{vmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{vmatrix},$$

depends only on the terms corresponding to $k = s/2$ in $S_{1,1}$ and $S_{2,1}$, and on the terms for $k = 0$ in $S_{1,2}$ and $S_{2,2}$:

$$\mathcal{L} = \begin{vmatrix} (-1)^{s/2} \binom{s}{s/2} F_{-1} X^{(-1)^{s/2} F_1} & F_s X^{F_{s+1}} \\ \binom{s}{s/2} F_{-1} X^{(-1)^{s/2} F_1} & F_s X^{F_{s+1}} \end{vmatrix} = F_s X^{F_{s+1}} \begin{vmatrix} -\binom{s}{s/2} X^{-1} & 1 \\ \binom{s}{s/2} X^{-1} & 1 \end{vmatrix} =$$

$$= -2 \binom{s}{s/2} F_s X^{F_{s+1}-1} \not\equiv 0 \,.$$

Case 2.2.2: $s \equiv 0 \pmod 4$. The leading term $\mathcal{L}$ depends on all terms corresponding to $k = 0$ *and* $k = 1 + s/2$. Let

$$\mathcal{L} = \begin{vmatrix} \mathcal{L}_{1,1} & \mathcal{L}_{1,2} \\ \mathcal{L}_{2,1} & \mathcal{L}_{2,2} \end{vmatrix}$$

with

$$\mathcal{L}_{1,1} = F_{s-1} X^{F_{s+1}} + (-1)^{1+s/2} \binom{s}{1+s/2} F_{-3} X^{(-1)^{1+s/2} F_{-1}} =$$

$$= F_{s-1} X^{F_{s+1}} - 2 \binom{s}{1+s/2} X^{-1} \,,$$

$$\mathcal{L}_{1,2} = F_s X^{F_{s+1}} + (-1)^{1+s/2} \binom{s}{1+s/2} F_{-2} X^{(-1)^{1+s/2} F_{-1}} =$$

$$= F_s X^{F_{s+1}} + \binom{s}{1+s/2} X^{-1},$$

$$\mathcal{L}_{2,1} = F_{s-1} X^{F_{s+1}} + \binom{s}{1+s/2} F_{-3} X^{(-1)^{1+s/2} F_{-1}} =$$

$$= F_{s-1} X^{F_{s+1}} + 2\binom{s}{1+s/2} X^{-1},$$

$$\mathcal{L}_{2,2} = F_s X^{F_{s+1}} + \binom{s}{1+s/2} F_{-2} X^{(-1)^{1+s/2} F_{-1}} =$$

$$= F_s X^{F_{s+1}} - \binom{s}{1+s/2} X^{-1}.$$

Hence, applying the identity $F_{n+2} = 2F_n + F_{n-1}$ for $n \geqslant 1$, we obtain

$$\mathcal{L} = \mathcal{L}_{1,1}\mathcal{L}_{2,2} - \mathcal{L}_{1,2}\mathcal{L}_{2,1} = 2\binom{s}{1+s/2}\big(-F_{s-1} - 2F_s\big) X^{F_{s+1}-1} =$$

$$= -2\binom{s}{1+s/2} F_{s+2} X^{F_{s+1}-1} \not\equiv 0,$$

since $s \geqslant 4$. Thus the numbers $g_{s_1}(\alpha)$ and $f_{s_2}(\alpha)$ are algebraically independent for any positive integers s_1, s_2.

Next we show that $y_1 := g_{s_1}(\alpha)$ and $y_2 := g_{s_2}(\alpha)$ are also algebraically independent for any distinct positive integers s_1, s_2. Setting R_1, R_2 as in (32), we have $\partial R_1/\partial x_1$ and $\partial R_1/\partial x_2$ as in (34), and

$$\frac{\partial R_2}{\partial x_1} = \sum_{k=0}^{s_2} (-1)^k \binom{s_2}{k} F_{s_2-2k-1} x_1^{(-1)^k F_{s_2-2k-1}-1} x_2^{(-1)^k F_{s_2-2k}},$$
$$\frac{\partial R_2}{\partial x_2} = \sum_{k=0}^{s_2} (-1)^k \binom{s_2}{k} F_{s_2-2k} x_1^{(-1)^k F_{s_2-2k-1}} x_2^{(-1)^k F_{s_2-2k}-1}.$$

We proceed as above by showing that $\Delta(X, X) \not\equiv 0$. Here (35) becomes

$$X^2 \Delta(X, X) =$$

$$= \begin{vmatrix} \sum_{k=0}^{s_1} (-1)^k \binom{s_1}{k} F_{s_1-2k-1} X^{(-1)^k F_{s_1-2k+1}} & \sum_{k=0}^{s_1} (-1)^k \binom{s_1}{k} F_{s_1-2k} X^{(-1)^k F_{s_1-2k+1}} \\ \sum_{k=0}^{s_2} (-1)^k \binom{s_2}{k} F_{s_2-2k-1} X^{(-1)^k F_{s_2-2k+1}} & \sum_{k=0}^{s_2} (-1)^k \binom{s_2}{k} F_{s_2-2k} X^{(-1)^k F_{s_2-2k+1}} \end{vmatrix}.$$

The leading term $\mathcal{L} = l\big(X^2\Delta(X,X)\big)$ only depends on the terms corresponding to $k=0$, so that $\mathcal{L} \not\equiv 0$ follows by the same arguments as used in Case 1.

In the same way one proves the algebraic independence of $y_1 := f_{s_1}(\alpha)$ and $y_2 := f_{s_2}(\alpha)$ for distinct positive integers s_1, s_2 by showing the non-vanishing of the leading term $\mathcal{L} = l\big(5^{(s_1+s_2)/2} X^2 \Delta(X,X)\big)$. The proof of the first statement is now completed.

Setting $x_1 = e^z$ and $x_2 = e^{\rho z}$, it follows from (31) that $f_s(z), g_s(z) \in \mathbb{Q}(\sqrt{5})(x_1, x_2)$ for positive number s. The second statement follows immediately. □

THEOREM 6. *Let $f_{a,b}(z)$ and $g_{a,b}(z)$ be power series defined by*

$$f_{a,b}(z) = \sum_{n=0}^{\infty} F_{an+b} \frac{z^n}{n!}, \quad g_{a,b}(z) = \sum_{n=0}^{\infty} L_{an+b} \frac{z^n}{n!}.$$

If α is a nonzero algebraic number, then any two numbers in the set $\{f_{a,b}(\alpha) \mid a \in \mathbb{N}, b \in \mathbb{N}_0\}$ are algebraically independent over $\mathbb{Q}$. Moreover, any three functions in the set $\{f_{a,b}(z) \mid a \in \mathbb{N},\ b \in \mathbb{N}_0\}$ are algebraically dependent over $\mathbb{Q}$. The same statements hold also for the power series $g_{a,b}(z)$.

PROOF. By similar arguments as in the proof of Theorem 5 we get the formula

$$f_{a,b}(\alpha) = \frac{1}{\sqrt{5}} \Big(\rho^b e^{(\rho F_a + F_{a-1})\alpha} + (-1)^{b+1} \rho^{-b} e^{(-1)^a (\rho F_{-a} + F_{-a-1})\alpha} \Big),$$

so that

$$f_{a,b}(\alpha) = \frac{1}{\sqrt{5}} \Big(\rho^b x_1^{F_{a-1}} x_2^{F_a} + (-1)^{b+1} \rho^{-b} x_1^{F_{a+1}} x_2^{-F_a} \Big) \in \mathbb{Q}(\sqrt{5})(x_1, x_2) \qquad (36)$$

with $x_1 = e^\alpha$ and $x_2 = e^{\rho\alpha}$. Set $R_1(x_1,x_2) := f_{a_1,b_1}(\alpha)$ and $R_2(x_1,x_2) := f_{a_2,b_2}(\alpha)$.

Case 1: $a_1 \neq a_2$. By the same arguments for the application of Lemma 1 as in the proof of Theorem 5 we conclude for

$$\Delta(x_1, x_2) := \begin{vmatrix} \dfrac{\partial R_1}{\partial x_1} & \dfrac{\partial R_1}{\partial x_2} \\ \dfrac{\partial R_2}{\partial x_1} & \dfrac{\partial R_2}{\partial x_2} \end{vmatrix}$$

that

$$l\big(X^2\Delta(X,X)\big) = \frac{(-1)^{a_2-1}}{5}\rho^{b_1+b_2}F_{a_1-a_2}X^{F_{a_1+1}+F_{a_2+1}} \not\equiv 0\,,$$

since $F_{a_1-a_2}$ does not vanish by the condition of case 1. This proves that for $a_1 \neq a_2$ the numbers $f_{a_1,b_1}(\alpha)$ and $f_{a_2,b_2}(\alpha)$ are algebraically independent over $\mathbb{Q}$.
Case 2: $a := a_1 = a_2$. We have $b_1 \neq b_2$, since otherwise $f_{a_1,b_1}(\alpha) = f_{a_2,b_2}(\alpha)$. It follows that

$$l\big(X^2\Delta(X,X)\big) = \frac{1}{5}F_{2a}\big((-1)^{b_2}\rho^{b_1-b_2} - (-1)^{b_1}\rho^{b_2-b_1}\big)X^{F_{a+1}+F_{a-1}}$$

with

$$(-1)^{b_2}\rho^{b_1-b_2} - (-1)^{b_1}\rho^{b_2-b_1} = $$
$$= 2(-1)^{b_2}\begin{cases}\sinh\big((b_1-b_2)\log\rho\big), \text{ if } & b_1 \equiv b_2 \pmod 2\,, \\ \cosh\big((b_1-b_2)\log\rho\big), \text{ if } & b_1 \not\equiv b_2 \pmod 2\end{cases} \neq 0\,.$$

Hence, $f_{a,b_1}(\alpha)$ and $f_{a,b_2}(\alpha)$ are algebraically independent over $\mathbb{Q}$.
The statement on the algebraic dependence of three functions in the set

$$\Big\{\sum_{n=0}^{\infty} F_{an+b}\frac{z^n}{n!}\,\Big|\, a \in \mathbb{N},\ b \in \mathbb{N}_0\Big\}$$

follows immediately from (36) with α replaced by z.

For the function $g_{a,b}(z)$ we obtain

$$g_{a,b}(\alpha) = \rho^b x_1^{F_{a-1}}x_2^{F_a} + (-1)^b\rho^{-b}x_1^{F_{a+1}}x_2^{-F_a} \in \mathbb{Q}(\sqrt{5})(x_1,x_2)\,, \tag{37}$$

from which we deduce for $R_1(x_1, x_2) := g_{a_1,b_1}(\alpha)$ and $R_2(x_1, x_2) := g_{a_2,b_2}(\alpha)$ the identities

$$l\big(X^2\Delta(X, X)\big) = (-1)^{a_2-1}\rho^{b_1+b_2}F_{a_1-a_2}X^{F_{a_1+1}+F_{a_2+1}} \tag{38}$$

for $a_1 \neq a_2$, and

$$l\big(X^2\Delta(X, X)\big) = F_{2a}\big((-1)^{b_1}\rho^{b_2-b_1} - (-1)^{b_2}\rho^{b_1-b_2}\big)X^{F_{a+1}+F_{a-1}} \tag{39}$$

for $a = a_1 = a_2$. (37), (38), and (39) prove the statements of the theorems in the case of Lucas numbers. □

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