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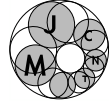
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On a mixed Khintchine problem in Diophantine approximation

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Abstract: We establish a ‘mixed’ version of a fundamental theorem of Khintchine [16] within the field of simultaneous Diophantine approximation. Via the notion of ‘ubiquity’ we are able to make significant progress towards the completion of the metric theory associated with mixed problems in this setting. This includes finding a natural mixed analogue of the classical Jarník–Besicovitch Theorem. Previous knowledge surrounding mixed problems in this area was almost entirely restricted to the multiplicative setup of de Mathan & Teulié [22], where the concept originated.

Keywords: Diophantine approximation, mixed Littlewood-type problems, Khintchine’s theorem, Furstenberg’s theorem, metric results, well approximable numbers

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1. Introduction

A classical result of Dirichlet implies that for any real vector $\mathbf{x} = (x_1, x_2) \in [0, 1)^2$ the system of inequalities

$$\left| x_1 - \frac{p_1}{q} \right| \leq \frac{1}{q^{3/2}}, \quad \left| x_2 - \frac{p_2}{q} \right| \leq \frac{1}{q^{3/2}}$$

is satisfied for infinitely many $p_1, p_2 \in \mathbb{Z}$ and $q \in \mathbb{N}$. This tells us that any real vector can be approximated by rational vectors $(p_1/q, p_2/q)$ at a ‘rate’ of $q^{-3/2}$. Moreover, this rate can be considered optimal in the obvious sense. This idea can

be generalised in the following manner. Choose any positive real numbers i and j satisfying

$$i, j > 0 \quad \text{and} \quad i + j = 1 \tag{1}$$

and let $\psi : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ be any non-negative arithmetic function. For reasons that will become apparent we refer to ψ as an *approximating function*. Consider the set $W(i, j, \psi)$ of real vectors $\mathbf{x} = (x_1, x_2) \in [0, 1]^2$ for which the system of inequalities

$$\left| x_1 - \frac{p_1}{q} \right| \leq \frac{\psi^i(q)}{q}, \quad \left| x_2 - \frac{p_2}{q} \right| \leq \frac{\psi^j(q)}{q} \tag{2}$$

is satisfied for infinitely many $p_1, p_2 \in \mathbb{Z}$ and $q \in \mathbb{N}$. Essentially, if a vector \mathbf{x} is contained in $W(i, j, \psi)$ then it can be approximated by rational points at a rate prescribed by the approximating function ψ . The exponents i and j act as ‘weights’, perturbing the speed of approximation across the two components of \mathbf{x} . Evidently, the set $W(i, j, \psi)$ is only interesting if the function ψ takes small values for large q . It is therefore reasonable to assume, as we will here, that $\psi(q) \rightarrow 0$ as $q \rightarrow \infty$.

In 1926, Khintchine [16] proved a remarkable result describing the rate at which real vectors can ‘typically’ be approximated by rational points in the case ‘ $i = j = 1/2$ ’. Khintchine’s result was later generalised by Schmidt [23], who proved that for any pair of real numbers i, j satisfying (1) and any approximating function ψ we have

$$\lambda_2(W(i, j, \psi)) = \begin{cases} 0, & \sum_{r=1}^{\infty} \psi(r) < \infty, \\ 1, & \sum_{r=1}^{\infty} \psi(r) = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

Here and throughout, λ_n denotes standard n -dimensional Lebesgue measure. It was subsequently shown that the monotonicity restriction imposed on ψ in the ‘divergent’ part of the above statement can be relaxed. For example, Theorem 3.8 of Harman’s book [17] suffices. This provides a stark contrast to the present situation in the classical one dimensional setting, as we discuss in Subsection 3.1.

The system of inequalities given by (2) can be rephrased in more compact notation to read

$$\max \left\{ \|qx_1\|^{1/i}, \|qx_2\|^{1/j} \right\} \leq \psi(q), \tag{3}$$

where $\|\cdot\|$ denotes the distance to the nearest integer. When $i = j = 1/2$ the left hand side of (3) reduces to the familiar supremum norm. In this case one can consider the following ‘multiplicative’ variant of the set $W(1/2, 1/2, \psi)$ conceived by replacing the supremum norm with the geometric mean. In particular, we may define

$$M(\psi) := \left\{ \mathbf{x} \in [0, 1)^2 : \|qx_1\| \|qx_2\| \leq \psi(q) \text{ for inf. many } q \in \mathbb{N} \right\}.$$

A criterion for the Lebesgue measure of the set $M(\psi)$, analogues to Khintchine result, was found by Gallagher [14] in 1962. For any approximating function ψ we have that

$$\lambda_2(M(\psi)) = \begin{cases} 0, & \sum_{r=1}^{\infty} \psi(r) \log(r) < \infty, \\ 1, & \sum_{r=1}^{\infty} \psi(r) \log(r) = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

It is an open question as to whether the monotonicity assumption can be safely removed from this statement (for recent progress, see [5]). We remark that the concept of (consciously) adding weights to the components of approximation is not prevalent in the study of problems in the multiplicative setting and we avoid its inclusion here for the sake of clarity.

In 2004, de Mathan & Teulié [22] introduced a closely related ‘mixed’ multiplicative setup realised by retaining the condition that $\|qx_1\|$ is small but replacing the condition on $\|qx_2\|$ with a condition of divisibility. To elaborate we require some notation. A sequence $\mathcal{D} = \{n_k\}_{k=0}^{\infty}$ of positive integers is said to be a *pseudo-absolute value sequence* if it is strictly increasing with $n_0 = 1$ and $n_k | n_{k+1}$ for all k . In this case \mathcal{D} will often be referred to as a *\mathcal{D} -adic sequence*. We say a \mathcal{D} -adic sequence has *bounded ratios* if the quotients n_{k+1}/n_k do not exceed some universal constant.

Given a \mathcal{D} -adic sequence we define the *\mathcal{D} -adic pseudo-absolute value* $|\cdot|_{\mathcal{D}} : \mathbb{N} \rightarrow \{1/n_k : k \in \mathbb{N}\}$ by

$$|q|_{\mathcal{D}} := 1/n_{\omega_{\mathcal{D}}(q)} = \inf\{1/n_m : q \in n_m \mathbb{Z}\}.$$

In other words, the \mathcal{D} -adic value assigns to each natural number q the reciprocal of the largest member of \mathcal{D} dividing q . When $\{n_{k+1}/n_k\}_{k=0}^\infty$ is the constant sequence equal to a prime number p , the pseudo absolute value $|\cdot|_{\mathcal{D}}$ is the usual p -adic absolute value $|\cdot|_p$.

Within the setup of de Mathan & Teulié one may consider a ‘mixed’ version of the set $M(\psi)$. Namely, we define

$$M_{\mathcal{D}}(\psi) := \{x \in [0, 1) : |q|_{\mathcal{D}} ||qx|| \leq \psi(q) \text{ for inf. many } q \in \mathbb{N}\}.$$

Recently, in [18], an analogue of Gallagher’s statement was established concerning the set $M_{\mathcal{D}}(\psi)$. For any approximating function ψ and any \mathcal{D} -adic sequence with bounded ratios we have

$$\lambda_1(M_{\mathcal{D}}(\psi)) = \begin{cases} 0, & \sum_{r=1}^\infty \psi(r) \log(r) < \infty, \\ 1, & \sum_{r=1}^\infty \psi(r) \log(r) = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

Again, it is currently unknown whether the monotonicity assumption is necessary.

Somewhat surprisingly, a mixed analogue of the set $W(i, j, \psi)$ has not yet been studied. The intentions of the present paper are to do exactly that. In particular, a metric theorem is established concerning the one-dimensional set

$$W_{\mathcal{D}}(i, j, \psi) := \left\{ x \in [0, 1) : \max \left\{ |q|_{\mathcal{D}}^{1/i}, ||qx||^{1/j} \right\} \leq \psi(q) \text{ for inf. many } q \in \mathbb{N} \right\}.$$

As we have seen, for each monotonic approximating function ψ the Lebesgue measures of the multiplicative sets $M(\psi)$ and $M_{\mathcal{D}}(\psi)$ depend on the asymptotic behaviour of the same sum (assuming that \mathcal{D} has bounded ratios). We show that the sets $W(i, j, \psi)$ and $W_{\mathcal{D}}(i, j, \psi)$ enjoy a similar property.

For the case when $\psi(q) = 1/q$ and \mathcal{D} has bounded ratios the ‘badly approximable’ complement of the set $W_{\mathcal{D}}(i, j, \psi)$ was examined in [1] (see also [21]). This seems to constitute all previous knowledge of mixed problems in Diophantine approximation outside of the multiplicative setting. On the other hand, when $\psi(q) = 1/q$ the sets $M(\psi)$ and $M_{\mathcal{D}}(\psi)$ are strongly related to the famous Littlewood Conjecture and its mixed counterpart, both of which have received much recent attention.

2. Statement of Results

For notational purposes, let $\mathcal{A}_\psi := \mathcal{A}(\mathcal{D}, \psi, i) := \{r \in \mathbb{N} : |r|_{\mathcal{D}} < \psi^i(r)\}$.

The main result of this paper is the following analogue of the statement of Schmidt described above.

THEOREM 1. *For any pair of reals i, j satisfying (1), any decreasing approximating function ψ and any \mathcal{D} -adic sequence with bounded ratios we have*

$$\lambda_1(W_{\mathcal{D}}(i, j, \psi)) = \begin{cases} 0, & \sum_{r \in \mathbb{N}} \psi(r) < \infty, \\ 1, & \sum_{r \in \mathbb{N}} \psi(r) = \infty. \end{cases}$$

We remark that under the assumption that ψ is monotonic it is easy to show that the sums $\sum_{r \in \mathbb{N}} \psi(r)$ and $\sum_{r \in \mathcal{A}_\psi} \psi^j(r)$ are asymptotically equivalent. As a consequence, one is free to replace the former sum with the latter in the statement of Theorem 1. This feature is symptomatic of the fact that the problem at hand is essentially one of Diophantine approximation with restricted denominator. Indeed, we may write

$$W_{\mathcal{D}}(i, j, \psi) = \left\{ x \in [0, 1) : \|qx\| < \psi^j(q) \quad \text{for inf. many } q \in \mathcal{A}_\psi \right\}.$$

When ψ is not assumed monotonic the two sums described above are not necessarily asymptotically equivalent (see Subsection 3.2) Moreover, using standard techniques it quickly follows that

$$\lambda_1(W_{\mathcal{D}}(i, j, \psi)) = 0 \quad \text{if} \quad \sum_{r \in \mathcal{A}_\psi} \psi^j(r) < \infty. \quad (4)$$

However, as demonstrated in § 3.2, the analogous statement does not hold in general for the sum $\sum_{r \in \mathbb{N}} \psi(r)$. In this sense the monotonicity assumption is necessary in the ‘convergence’ part of Theorem 1 (in its stated form). The point is that $\sum_{r \in \mathcal{A}_\psi} \psi^j(r)$ should be considered the ‘genuine’ critical sum relating to the measure of the set $W_{\mathcal{D}}(i, j, \psi)$. In fact, it would not be unfair to view it as merely a coincidence that there is asymptotic equivalence between the two sums in question when monotonicity

is enforced. That said, to bring the similarity with Schmidt's classical result to the forefront we choose to present our statement in the current form, despite the possibly artificial nature of doing so. Regardless of which sum one chooses we prove in Subsection 3.3 that the monotonicity assumption in the 'divergent' part of Theorem 1 is absolutely necessary.

It is worth emphasising that the two degenerate cases ' $i = 0$ ' and ' $j = 0$ ' are not considered in this paper. On employing the convention that $x^{1/y} = 0$ when $y = 0$ for all real x , it is easily verified that in the former case Theorem 1 reduces to the classical one-dimensional result of Khintchine (see § 3.1), whilst in the latter case the measure of the set $W_{\mathcal{D}}(1, 0, \psi)$ trivially fulfils a 'zero-one' law. Indeed,

$$W_{\mathcal{D}}(1, 0, \psi) = \begin{cases} [0, 1), & \psi(q) > |q|_{\mathcal{D}} \text{ for infinitely many } q \in \mathbb{N}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Finally, we remark that whilst it would be desirable to generalise Theorem 1 to the case of pseudo-absolute value sequences whose ratios are not bounded, to do so would require more than trivial improvements over the techniques we present.

Theorem 1 is proven as a consequence of a more general Hausdorff measure result. Throughout, \mathcal{H}^s denotes standard s -dimensional Hausdorff measure and 'dim' represents Hausdorff dimension. Recall that when $s = 1$ Hausdorff measure is comparable to one-dimensional Lebesgue measure.

THEOREM 2. *Fix any pair of reals i, j satisfying (1), any \mathcal{D} -adic sequence with bounded ratios and any real $s \in (i, 1]$. Then, for any approximating function ψ for which the related function $f_{\psi} : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0} : r \mapsto r^{1-s}\psi^{i+j s}(r)$ is decreasing we have*

$$\mathcal{H}^s(W_{\mathcal{D}}(i, j, \psi)) = \begin{cases} 0, & \sum_{r \in \mathbb{N}} f_{\psi}(r) < \infty, \\ \mathcal{H}^s([0, 1)), & \sum_{r \in \mathbb{N}} f_{\psi}(r) = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

We prove this result via the notion of ubiquitous systems, a fundamental tool for establishing measure theoretic statements in Diophantine approximation. A tailored account of the ubiquity setup is presented in Subsection 4.1.

We do not claim the conditions imposed on the function ψ in Theorem 2 are optimal. In fact, we suspect that the monotonicity assumption imposed on f_ψ may be unnecessary. Despite this constraint, we are able to deduce the following generalisation of a classical theorem of Jarník [20] and Besicovich [8]. Their fundamental result corresponds to the case ‘ $i = 0, j = 1$ ’ in our setup.

COROLLARY 1. *Choose any pair of reals i, j satisfying (1), any \mathcal{D} -adic sequence with bounded ratios and any decreasing approximating function ψ . Assume there exists a real number τ such that*

$$\tau = \lim_{r \rightarrow \infty} \frac{-\log \psi(r)}{\log r} < \frac{1}{i}.$$

Then,

$$\dim(W_{\mathcal{D}}(i, j, \psi)) = \min \left\{ 1, \frac{2 - i\tau}{1 + j\tau} \right\}.$$

It should be observed that if $\psi(r) \leq r^{-1/i}$ for all sufficiently large r then the set $W_{\mathcal{D}}(i, j, \psi)$ is empty.

3. Removing monotonicity

3.1. The work of Duffin and Schaeffer

For any approximating function ψ let

$$W(\psi) := \{x \in [0, 1) : \|qx\| < \psi(q) \text{ for infinitely many } q \in \mathbb{N}\}$$

denote the standard set of ψ -approximable numbers. The one-dimensional version of Khintchine’s theorem states that the Lebesgue measure of $W(\psi)$ is zero if the sum $\sum_{r=1}^{\infty} \psi(r)$ converges or one if the sum $\sum_{r=1}^{\infty} \psi(r)$ diverges and ψ is decreasing. In their seminal paper [12], Duffin & Schaeffer produced a counterexample showing that the monotonicity assumption in the ‘divergent’ part is necessary. In particular, they constructed a non-monotonic approximating function ψ for which $\lambda_1(W(\psi)) = 0$ but the sum $\sum_{r=1}^{\infty} \psi(r)$ diverges.

In an attempt to provide an alternative statement to that of Khintchine, free from any restrictions on the choice of approximating function, Duffin & Schaeffer considered the following variant of $W(\psi)$. For any approximating function ψ let

$W'(\psi)$ denote the set of real numbers $x \in [0, 1)$ for which

$$|qx - p| < \psi(q) \text{ for infinitely many } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ with } (p, q) = 1.$$

This set differs from $W(\psi)$ only by the coprimality restriction on p and q . The restriction ensures that the rational approximations p/q to x are in reduced form. In their paper, Duffin & Schaeffer were able to establish partial results concerning the measure of the set $W'(\psi)$ but fell short of proving their now famous conjecture. In what follows φ denotes Euler's totient function.

DUFFIN—SCHAEFFER CONJECTURE (1941). *For any approximating function ψ we have*

$$\lambda_1(W'(\psi)) = 1 \quad \text{if} \quad \sum_{r=1}^{\infty} \frac{\varphi(r)}{r} \psi(r) = \infty.$$

It is clear that $W'(\psi) \subset W(\psi)$, which immediately implies the complementary statement

$$\lambda_1(W'(\psi)) = 0 \quad \text{if} \quad \sum_{r=1}^{\infty} \frac{\varphi(r)}{r} \psi(r) < \infty$$

holds for every approximating function ψ . The Duffin—Schaeffer Conjecture represents one of the most profound unsolved problems in metric Diophantine approximation. For a thorough account, including recent progress made concerning the conjecture, see § 2 of [17] and [4].

3.2. The mixed setting

One might expect that similar properties to those suggested by Duffin & Schaeffer hold within the mixed simultaneous setting. In this section we discuss the necessity of the monotonicity assumption imposed in Theorem 1. Firstly, we present a simple example of a function demonstrating that the assumption is necessary in the convergence part of the result.

For any real pair i, j and any pseudo-absolute value sequence $\mathcal{D} = \{n_k\}_{k=0}^{\infty}$ let

$$\psi_0(q) := \begin{cases} k^{-(1+\delta)}, & q = n_k, \\ 0, & \text{otherwise,} \end{cases}$$

where $\delta := (1/\max(i, j) - 1)/2 > 0$. Then, for all $k \in \mathbb{N}$ we have $n_k \geq k > k^{i(1+\delta)}$ implying that $|n_k|_{\mathcal{D}} = 1/n_k < \psi_0^i(n_k)$ and so $\mathcal{A}_{\psi_0} = \mathcal{D}$. Moreover, it is easy to see that

$$\sum_{r=1}^{\infty} \psi_0(r) < \infty \quad \text{but} \quad W_{\mathcal{D}}(i, j, \psi_0) = [0, 1)$$

as required. Of course, statement (4) provides the relevant monotonicity-free criterion giving measure zero and is certainly not contravened here as

$$\sum_{r \in \mathcal{A}_{\psi_0}} \psi_0^j(r) = \sum_{r \in \mathcal{D}} \psi_0^j(r) = \sum_{k=1}^{\infty} k^{-j(1+\delta)} > \sum_{k=1}^{\infty} k^{-1},$$

which diverges.

It is easier still to show that the monotonicity assumption is necessary in the divergence part of Theorem 1. For example, one may take

$$\psi_1(q) := \begin{cases} 1/2, & (n_k, q) = 1 \text{ for all } k \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$

in which case the set $W_{\mathcal{D}}(i, j, \psi_1)$ is empty but the sum $\sum_{r=1}^{\infty} \psi_1(r)$ diverges.

The simple nature of the examples ψ_0 and ψ_1 is indicative of the fact that the volume sum in question is not the morally correct choice. As discussed earlier, a more interesting problem is removing monotonicity from Theorem 1 when the critical sum is taken to be $\sum_{r \in \mathcal{A}_{\psi}} \psi^j(r)$. Here also, one can provide a counterexample in the divergence case, albeit with a degree more of ingenuity. Evidently the convergence case is completely covered by statement (4).

THEOREM 3. *For any pair of reals i, j satisfying (1) and any \mathcal{D} -adic sequence there exists an approximating function $\Psi : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ for which*

$$\lambda_1(W_{\mathcal{D}}(i, j, \Psi)) = 0 \quad \text{but} \quad \sum_{r \in \mathcal{A}_{\Psi}} \Psi^j(r) = \infty.$$

Our counterexample is constructed via direct modification of the method of Duffin & Schaeffer and for completion we include a full exposition in Subsection 3.3.

With regards to a mixed simultaneous analogue of the Duffin—Schaeffer Conjecture, one begins by imposing a coprimality condition on the rational approximations as before. Here, this equates to considering the set $W'_D(i, j, \psi)$ of points $x \in [0, 1)$ for which the conditions

$$\max \left\{ |q|_D^{1/i}, |qx - p|^{1/j} \right\} \leq \psi(q), \quad (p, q) = 1,$$

are satisfied for infinitely many $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. It should now be obvious that it would be naive to propose that we have

$$\lambda_1(W'_D(i, j, \psi)) = 1 \quad \text{if} \quad \sum_{r=1}^{\infty} \frac{\varphi(r)}{r} \psi(r) = \infty$$

for any pair of reals i, j satisfying (1), any approximating function ψ and any D -adic sequence with bounded ratios. Indeed, it is not difficult to see this statement is false; the function ψ_1 above provides a counterexample. A more astute and natural proposal for a mixed Duffin—Schaeffer Conjecture is that

$$\lambda_1(W'_D(i, j, \psi)) = 1 \quad \text{if} \quad \sum_{r \in \mathcal{A}_\psi} \frac{\varphi(r)}{r} \psi^j(r) = \infty.$$

The function ψ_1 certainly does not contradict this statement as the set \mathcal{A}_{ψ_1} is empty in this case. Along with its classical counterpart, a proof (or disproof) of this statement remains out of reach.

3.3. Proof of Theorem 3

In the vein of Duffin & Schaeffer we first show for any $R \geq 1$ and $\epsilon > 0$ that there exists an approximating function ψ such that

$$\sum_{r \in \mathcal{A}_\psi} \psi^j(r) > 1, \quad \psi(r) = 0 \quad \text{when} \quad r \leq R,$$

but the set of $x \in (0, 1)$ such that

$$\|qx\| < \psi^j(q) \quad \text{for some} \quad q \in \mathcal{A}_\psi, \tag{5}$$

has Lebesgue measure strictly less than ϵ .

Let α be a positive number such that $0 < \alpha < \epsilon/2$ and choose primes p_1, p_2, \dots, p_s with $p_t > R$ ($t = 1, \dots, s$) for some natural number s to be specified later. Since \mathcal{D} has bounded ratios we may choose the primes in such a way that $(p_t, n_k) = 1$ for all t and k . Next, let

$$K = K(s, \alpha) := \min \left\{ k \in \mathbb{N} : n_k \geq (p_1 \cdots p_s / \alpha)^{i/j} \right\}$$

and set

$$N := n_K p_1 \cdots p_s.$$

Finally, define

$$\psi(q) := \begin{cases} (q\alpha/N)^{1/j}, & n_K \mid q, \quad q \mid N, \quad q \neq n_K, \\ 0, & \text{otherwise.} \end{cases}$$

We claim that ψ satisfies the desired properties. Let $A_q \subset (0, 1)$ denote the set consisting of the $q - 1$ open intervals of length $2\psi^j(q)/q$ with centres at the rationals p/q ($p = 1, \dots, q - 1$) and the open intervals $(0, \psi^j(q)/q)$ and $(1 - \psi^j(q)/q, 1)$. For small enough ϵ these intervals are disjoint and so the Lebesgue measure of A_q is given by $2\psi^j(q) = 2q\alpha/N$ if $1 \leq q \leq N$. Furthermore, since $\psi^j(q)/q = \alpha/N = \psi^j(N)/N$ we have

$$A_N = \bigcup_{\substack{q \mid N: \\ n_K \mid q \\ q \neq n_K}} A_q$$

and for all q in this union

$$|q|_{\mathcal{D}} \leq \frac{1}{n_K} \leq \left(\frac{\alpha}{p_1 \cdots p_s} \right)^{i/j} = \left(\frac{n_K \alpha}{N} \right)^{i/j} < \left(\frac{q\alpha}{N} \right)^{i/j} = \psi^i(q);$$

i. e., $q \in \mathcal{A}_\psi$. Hence, as $\psi(q) = 0$ for all q not in the union we may deduce that property (5) will be satisfied by irrational $x \in (0, 1)$ if and only if $x \in A_N$. Moreover, we have $\lambda_1(A_N) = 2\alpha < \epsilon$.

All that remains is to show that

$$\sum_{r \in \mathcal{A}_\psi} \psi^j(r) > 1.$$

Via the change of variables $\ell := rn_K^{-1}$ and $M := Nn_K^{-1}$ we have

$$\sum_{r \in \mathcal{A}_\psi} \psi^j(r) = \frac{\alpha}{N} \sum_{\substack{r|N: \\ n_K | r \\ r \neq n_K}} r = \frac{\alpha}{M} \sum_{\substack{\ell > 1: \\ \ell | M}} \ell.$$

We may now follow the original argument of Duffin & Schaeffer. Choose s large enough so that

$$\prod_{t=1}^s (1 + 1/p_t) > 1 + 1/\alpha.$$

This is always possible because the above product diverges when extended over all primes. Then, since $M = p_1 \cdots p_s$ it follows by standard arithmetic techniques that

$$\begin{aligned} \frac{\alpha}{M} \sum_{\substack{\ell > 1: \\ \ell | M}} \ell &= \frac{\alpha}{M} \left(\prod_{t=1}^s (1 + p_t) - 1 \right) > \\ &> \alpha \left(\prod_{t=1}^s (1 + 1/p_t) - 1 \right) > 1, \end{aligned}$$

as required. To complete the construction of our counterexample we proceed as follows. Let ψ_1 satisfy the above properties with $R = R_1 := 1$ and $\epsilon = \epsilon_1 := 2^{-2}$. Then, for some R_2 we must have that $\psi_1(q) = 0$ for all $q > R_2$. Let ψ_2 satisfy the above properties with $R = R_2$ and $\epsilon = \epsilon_2 := 2^{-2}$. Continue in this fashion to choose numbers R_t and construct functions ψ_t satisfying the above properties for $R = R_t$ and $\epsilon = \epsilon_t := 2^{-t}$. Then, define

$$\Psi(q) := \begin{cases} \psi_1(q), & q \leq R_2, \\ \psi_t(q), & R_t < q \leq R_{t+1}, \quad t \geq 2. \end{cases}$$

It is clear that

$$\sum_{r \in \mathcal{A}_\Psi} \Psi^j(r) = \infty,$$

but for $x \in (0, 1)$ the system

$$\|qx\| < \Psi^j(q), \quad q \in \mathcal{A}_\Psi, \quad q > R_t$$

can be satisfied only if x belongs to a set of measure at most

$$\sum_{r=t}^{\infty} 2^{-r} = 2^{-t+1},$$

as desired.

4. Ubiquitous Systems

Ubiquity is a fundamental tool for establishing measure theoretic statements in Diophantine approximation and will be utilised in the proof of Theorem 2. This section comprises of a brief description of a restricted form of ubiquity tailored to our needs.

The concept of ubiquitous systems was first introduced by Dodson, Rynne & Vickers in [11] as a method of determining lower bounds for the Hausdorff dimension of limsup sets. Recently, this idea was developed by Beresnevich, Dickinson & Velani in [2] to provide a very general framework for establishing the Hausdorff measure of a large class of limsup sets. A slightly more simplified account is presented in [7].

4.1. The ubiquity setup

Let (Ω, d) be a compact metric space supporting a non-atomic probability measure μ and assume that any open subset of Ω is μ -measurable. Throughout, $B(c, r)$ will denote a ball in Ω centred at a point c and of radius $r > 0$. The following regularity condition will be imposed on the measure of balls: There exist positive constants a, b, δ and r_0 such that for any $c \in \Omega$ and $r \leq r_0$

$$ar^\delta \leq \mu(B(c, r)) \leq br^\delta.$$

If this power law holds then we say μ is δ -Ahlfors regular. It easily follows that if μ is δ -Ahlfors regular then $\dim \Omega = \delta$ and that μ is comparable to δ -dimensional Hausdorff measure \mathcal{H}^δ . For details see Chapter 4 of [13].

Let $\mathcal{R} = \{R_a \in \Omega : a \in J\}$ be a collection of points R_a in Ω indexed by some infinite, countable set J . The points R_a are referred to as the *resonant points*. Next, let $\beta : J \rightarrow \mathbb{R}_{>0} : a \rightarrow \beta_a$ be a positive function defined on J for which the number of $a \in J$ with β_a bounded above is always finite. The latter condition imposed on the function β is very natural and is often referred to as the *Northcott property* in reference to Northcott’s famous result that the number of algebraic numbers of bounded degree and bounded height is finite. Finally, given an approximating function Ψ define

$$\Lambda(\Psi) := \{x \in \Omega : x \in B(R_a, \Psi(\beta_a)) \text{ for infinitely many } a \in J\}.$$

It is the measure of this set in which we are interested.

To demonstrate the ‘limsup’ nature of $\Lambda(\Psi)$ first choose any two positive increasing sequences $l := \{l_k\}$ and $u := \{u_k\}$ such that $l_k < u_k$ and $\lim_{k \rightarrow \infty} l_k = \infty$. These sequences will be referred to as the *lower* and *upper* sequences respectively. For $k \in \mathbb{N}$ let

$$\Lambda_l^u(\Psi, k) := \bigcup_{a \in J_l^u(k)} B(R_a, \Psi(\beta_a)),$$

where $J_l^u(k) := \{a \in J : l_k < \beta_a \leq u_k\}$. Then, it is easily seen that

$$\Lambda(\Psi) = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \Lambda_l^u(\Psi, k).$$

We can now define what it means to be a ubiquitous system. Let $\rho : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ be any function with $\rho(r) \rightarrow 0$ as $r \rightarrow \infty$ and let

$$\Delta_l^u(\rho, k) := \bigcup_{a \in J_l^u(k)} B(R_a, \rho(u_k)).$$

DEFINITION (LOCAL μ -UBIQUITY). *Let $B = B(c, r)$ be an arbitrary ball in Ω of radius $r \leq r_0$. Suppose there exists a function ρ , sequences l and u and an absolute constant $\kappa > 0$ such that*

$$\mu(B \cap \Delta_l^u(\rho, k)) \geq \kappa \mu(B) \quad \forall k \geq k_0(B). \tag{6}$$

Then the pair (\mathcal{R}, β) is said to be a local μ -ubiquitous system relative to (ρ, l, u) . The function ρ is referred to as the ubiquitous function.

Finally, a function h is said to be u -regular if there exists a strictly positive constant $\lambda < 1$, which may depend on u , such that for k sufficiently large

$$h(u_{k+1}) \leq \lambda h(u_k).$$

We now present two powerful results associated with ubiquitous systems, which have been tailored to our needs. The first theorem (see [2, Corollary 2]) concerns the μ -measure of the limsup set $\Lambda(\Psi)$ and in our setup corresponds to the case ' $s = 1$ ' in Theorem 1. The second (see [3, Theorem 10]) deals with the s -dimensional Hausdorff measure \mathcal{H}^s of $\Lambda(\Psi)$ for $0 < s < 1$. Due to the nature of the ubiquity framework it is necessary to deal with the two scenarios separately.

THEOREM BDV1 (2006). *Let (Ω, d) be a compact metric space equipped with a δ -Ahlfors regular measure μ . Suppose that (\mathcal{R}, β) is a local μ -ubiquitous system relative to (ρ, l, u) and that Ψ is a decreasing approximation function. Furthermore, suppose that either Ψ or ρ is u -regular and that*

$$\sum_{k=1}^{\infty} \left(\frac{\Psi(u_k)}{\rho(u_k)} \right)^{\delta} = \infty.$$

Then,

$$\mu(\Lambda(\Psi)) = 1.$$

THEOREM BDV2 (2006). *Let (Ω, d) be a compact metric space equipped with a δ -Ahlfors regular measure μ . Suppose that (\mathcal{R}, β) is a local μ -ubiquitous system relative to (ρ, l, u) and that Ψ is a decreasing approximation function. Furthermore, suppose that $0 < s < \delta$. Let g be the positive function given by $g(r) := \Psi^s \rho^{-\delta}$ and let $G := \limsup_{k \rightarrow \infty} g(u_k)$.*

(i) *Suppose that $G = 0$ and Ψ is u -regular. Then,*

$$\mathcal{H}^s(\Lambda(\Psi)) = \infty \quad \text{if} \quad \sum_{k=1}^{\infty} g(u_k) = \infty.$$

(ii) *Suppose that $0 < G < \infty$. Then, $\mathcal{H}^s(\Lambda(\Psi)) = \infty$.*

Before proceeding, we recall a generalisation of the Cauchy condensation test attributed to Oscar Schlömilch, which can be found in [9, Theorem 2.4]. We will appeal to this result multiple times in our proof.

SCHLÖMILCH’S THEOREM (LATE 19TH CENTURY). *Let $\sum_{r=0}^{\infty} a_r$ be an infinite real series whose terms are positive and decreasing and let $m_0 < m_1 < \dots$ be a strictly increasing sequence of positive integers for which there exists a constant $M > 0$ such that*

$$\frac{m_{k+1} - m_k}{m_k - m_{k-1}} \leq M \quad \text{for every } k \in \mathbb{N}. \tag{7}$$

Then the series $\sum_{r=0}^{\infty} a_r$ converges if and only if the series $\sum_{k=0}^{\infty} (m_{k+1} - m_k)a_{m_k}$ converges.

It should be noted that, taking $m_k = n_k$ for some \mathcal{D} -adic sequence $\{n_k\}$, condition (7) is satisfied for some $M \geq 2$ if and only if the sequence \mathcal{D} has bounded ratios.

5. Proof of Theorem 2

For the divergence part of Theorem 2 we will appeal to the ubiquity framework described in the previous section. The convergence part follows by well-known arguments stemming from the Borel—Cantelli Lemma. For completeness we include a short proof here.

Firstly, note that we may assume $\psi(r) < 1$ for all sufficiently large r , for otherwise the sum $\sum_{r \in \mathbb{N}} f_{\psi}(r)$ would surely diverge. So, for each $k \in \mathbb{N}$ sufficiently large we can find a unique natural number m_k for which

$$\frac{1}{n_{m_k}} < \psi^i(n_k) \leq \frac{1}{n_{m_{k-1}}}. \tag{8}$$

This is possible since ψ is decreasing and \mathcal{D} is an increasing sequence. The pseudo-absolute value is discrete, so for sufficiently large $k \in \mathbb{N}$ it follows from (8) that

$$\begin{aligned} \# \{q \in (n_k, n_{k+1}] : q \in \mathcal{A}_{\psi}\} &\leq \# \left\{ q \in (n_k, n_{k+1}] : |q|_{\mathcal{D}} < \psi^i(n_k) \right\} = \\ &= \# \left\{ q \in (n_k, n_{k+1}] : |q|_{\mathcal{D}} \leq \frac{1}{n_{m_k}} \right\} = \\ &= \# \{q \in (n_k, n_{k+1}] : n_{m_k} | q\} = \end{aligned}$$

$$\begin{aligned}
 &= \frac{n_{k+1} - n_k}{n_{m_k}} < \\
 &< (n_{k+1} - n_k)\psi^i(n_k).
 \end{aligned}$$

Next, for each $q \in \mathcal{A}_\psi$ let W_q denote the set of real numbers $x \in (0, 1)$ satisfying

$$\max \left\{ |qx|_{\mathcal{D}}^{1/i}, \|qx\|^{1/j} \right\} < \psi(q)$$

and let $M \geq 2$ be an upper bound for the ratios of consecutive elements of \mathcal{D} ; i.e., $n_{k+1}/n_k \leq M$ for all $k \in \mathbb{N}$. Each set W_q is covered by the $q - 1$ open intervals of length $2\psi^j(q)/q$ with centres at the rationals p/q ($p = 1, \dots, q - 1$) and the two open intervals $(0, \psi^j(q)/q)$ and $(1 - \psi^j(q)/q, 1)$. Let us denote by E_q this collection of covering intervals. For any $k_0 \in \mathbb{N}$ we have that the countable collection

$$\bigcup_{\substack{q \in \mathcal{A}_\psi \\ q > n_{k_0}}} E_q$$

is a ρ -cover for $W_{\mathcal{D}}(i, j, \psi)$ for $\rho = 2\psi^j(n_{k_0})/n_{k_0}$. Thus, the value $\mathcal{H}_\rho^s(W_{\mathcal{D}}(i, j, \psi))$ is at most

$$\begin{aligned}
 2^s \sum_{\substack{q \in \mathcal{A}_\psi \\ q > n_{k_0}}} q^{1-s} \psi^{js}(q) &\leq 2^s M^{1-s} \sum_{k=k_0}^{\infty} n_k^{1-s} \psi^{js}(n_k) \sum_{\substack{q \in \mathcal{A}_\psi \\ q \in (n_k, n_{k+1}]}} 1 < \\
 &< 2^s M^{1-s} \sum_{k=k_0}^{\infty} (n_{k+1} - n_k) n_k^{1-s} \psi^{i+js}(n_k). \tag{9}
 \end{aligned}$$

However, the function $f_\psi = r^{1-s} \psi^{i+js}(r)$ is assumed decreasing and \mathcal{D} is assumed to have bounded ratios and so we may apply Schlömilch’s Theorem. The sum $\sum_{r=1}^{\infty} f_\psi(r)$ converges so we may take (9) to be as small as we wish. In particular, as $\rho \rightarrow 0$ (or equivalently as $k_0 \rightarrow \infty$) we have $\mathcal{H}_\rho^s(W_{\mathcal{D}}(i, j, \psi)) \rightarrow 0$ and the ‘convergence’ part of Theorem 2 is complete.

We now demonstrate how the ubiquity framework can be applied to the set $W_{\mathcal{D}}(i, j, \psi)$. Firstly, choose a natural number c . It is easy to see that $W_{\mathcal{D}}(i, j, \psi)$ can be expressed in the form $\Lambda(\Psi)$ with

$$\Omega := [0, 1], \quad \Psi(r) := \psi^j(r)/r, \quad J := \{(p, q) \in \mathbb{N} \times \mathbb{N} : q \in \mathcal{A}_\psi, 0 \leq p \leq q\},$$

$$a := (p, q) \in J, \quad \beta_a := q, \quad R_a := p/q, \quad u_k := l_{k+1} := n_{ck}, \quad \mu := \lambda_1, \quad \delta := 1, \\ J_l^u(k) := \{(p, q) \in J : n_{c(k-1)} < q \leq n_{ck}\}, \quad \Lambda_l^u(\Psi, k) := \bigcup_{(p,q) \in J_l^u(k)} B(p/q, \psi^j(q)/q),$$

so that

$$W_{\mathcal{D}}(i, j, \psi) = \limsup_{k \rightarrow \infty} \Lambda_l^u(\Psi, k).$$

The natural number c above is introduced for technical reasons and its appearance will be qualified later, suffice to say we may not take $c = 1$.

We now show that this system is locally λ_1 -ubiquitous relative to (ρ, l, u) , for l and u as chosen above and some real positive function ρ satisfying with $\rho(r) \rightarrow 0$ as $r \rightarrow \infty$. It is apparent that an appropriate choice of ubiquitous function is $\rho(q) := \gamma/q^2 \psi^i(q)$ for some constant $\gamma > 0$ for then the sum

$$\sum_{k=1}^{\infty} \left(\frac{\Psi(u_k)}{\rho(u_k)} \right)^\delta = \sum_{k=1}^{\infty} \frac{n_{ck}^2 \psi^i(n_{ck}) \psi^j(n_{ck})}{\gamma n_{ck}} = \frac{1}{\gamma} \sum_{k=1}^{\infty} n_{ck} \psi(n_{ck})$$

diverges if and only if the sum $\sum_{r=1}^{\infty} \psi(r)$ diverges by the result of Schlömilch.

Next, we point out an important observation. When $\sum_{r \in \mathbb{N}} r^{1-s} \psi^{i+js}(r) = \infty$ and $s \in (i, 1]$ we may assume that

$$\psi^i(r) > 1/r \quad \text{for all } r \in \mathbb{N}. \tag{10}$$

To see this, let $\mathcal{R} := \{r_k\}_{r \in \mathbb{N}}$ be an increasing sequence of integers for which $\psi^i(r_k) \leq 1/r_k$. Then, for $s \in (i, 1]$ we have

$$\sum_{k \in \mathbb{N}} r_k^{1-s} \psi^{i+js}(r_k) \leq \sum_{k \in \mathbb{N}} r_k^{-(1+j/s)s} < \infty \quad \text{and} \quad \sum_{r \in \mathbb{N} \setminus \mathcal{R}} r^{1-s} \psi^{i+js}(r) = \infty.$$

Moreover, for each $k \in \mathbb{N}$ we have

$$\psi^i(r_k) \leq \frac{1}{r_k} \leq |r_k|_{\mathcal{D}}$$

and so $r_k \in \mathcal{A}_\psi$. The upshot is that we may choose $J \subset \mathbb{N} \times (\mathbb{N} \setminus \mathcal{R})$ in the ubiquity setup and neither the set $W_{\mathcal{D}}(i, j, \psi)$ nor the divergence of the corresponding volume sum is affected by the removal of the integers r_k .

Observation (10) immediately implies $\rho(r) \rightarrow 0$ as $r \rightarrow \infty$ as required in the ubiquity setup. Furthermore, let $M \geq 2$ be an upper bound for the ratios of consecutive elements of \mathcal{D} . Then, the monotonicity of ψ immediately implies that

$$\frac{\psi^j(n_{c(k+1)})}{n_{c(k+1)}} \leq \frac{\psi^j(n_{ck})}{n_{c(k+1)}} \leq \frac{\psi^j(n_{ck})}{M^c n_{ck}}$$

and so Ψ is trivially u -regular. Hence, to prove the ‘divergence’ part of Theorem 2 it suffices to show the following holds.

PROPOSITION 1. *Let $\rho(q) := \gamma/q^2 \psi^i(q)$ for some $\gamma > 0$. Then, the system defined above is a locally λ_1 -ubiquitous relative to the triple $(\rho, n_{c(k-1)}, n_{ck})$ for some $c \in \mathbb{N}$.*

We begin by modifying the sequence specified in (8). Once more we may assume that $\psi(r) < 1$ for large r and so for any sufficiently large $k \in \mathbb{N}$ and any $c \in \mathbb{N}$ we can find a unique natural number $m_k := m_k(c)$ for which

$$\frac{1}{n_{cm_k}} < \psi^i(n_{ck}) \leq \frac{1}{n_{c(m_k-1)}}. \quad (11)$$

To prove Proposition 1 we require a the following consequence of a classical theorem of Dirichlet.

PROPOSITION 2. *Fix $c \in \mathbb{N}$. Then, for every $x \in \mathbb{R}$ and every $k \in \mathbb{N}$ there exists $p/q \in \mathbb{Q}$ with $n_{cm_k} \leq q \leq n_{ck}$ such that*

$$\left| x - \frac{p}{q} \right| < \frac{n_{cm_k}}{qn_{ck}} \quad \text{and} \quad |q|_{\mathcal{D}} \leq \frac{1}{n_{cm_k}}. \quad (12)$$

PROOF OF PROPOSITION 2. Dirichlet’s theorem states that for all $x' \in \mathbb{R}$ and for all $N \in \mathbb{N}$ there exists $p/q' \in \mathbb{Q}$ with $q' \leq N$ such that

$$|x' - p/q'| < 1/q'N.$$

Let $N := n_{ck}/n_{cm_k}$. Observation (10) guarantees that $N \geq 1$. Next, set $x := x'n_{cm_k}$ and $q = n_{cm_k}q'$. Then, for all $x \in \mathbb{R}$ we have

$$\left| xn_{cm_k} - \frac{pn_{cm_k}}{q} \right| < \frac{n_{cm_k}^2}{qn_{ck}}$$

whereby upon division by n_{cm_k} the desired inequality is reached. Furthermore, $n_{cm_k} \leq q \leq n_{cm_k} n_{ck} / n_{cm_k} = n_{ck}$ and $|q|_{\mathcal{D}} \leq 1/n_{cm_k}$ as required. \square

In what follows, for $r \in \mathbb{N}$ we denote by $K^-(r)$ the set of $q \in \mathbb{N}$ with $|q|_{\mathcal{D}} \leq 1/n_{cm_k}$ for which $q \leq n_{c(r-1)}$, whereas $K^+(r)$ will denote the set of $q \in \mathbb{N}$ with $|q|_{\mathcal{D}} \leq 1/n_{cm_k}$ that satisfy $n_{c(r-1)} < q \leq n_{cr}$. Recall that $\rho(r) := \gamma/r^2 \psi^i(r)$ for some $\gamma > 0$.

To prove Proposition 1 it now suffices to show that for every interval $I \subset [0, 1)$ there exists an absolute constant $\kappa > 0$ such that

$$\lambda_1 \left(I \cap \bigcup_{\substack{q \in \mathcal{A}_\psi: \\ q \in (n_{c(k-1)}, n_{ck}]}} \bigcup_{p=0}^{q-1} B \left(\frac{p}{q}, \rho(n_{ck}) \right) \right) \geq \kappa \lambda_1(I) \tag{13}$$

for all k sufficiently large. Assume $M \geq 2$ is an upper bound for the ratios of consecutive elements of \mathcal{D} . Upon setting $\gamma = M^{2c}$ it is easily verified that the LHS of (13) is bounded below by

$$\lambda_1 \left(I \cap \bigcup_{K^+(k)} \bigcup_{p=0}^{q-1} B \left(\frac{p}{q}, \frac{n_{cm_k}}{q n_{ck}} \right) \right). \tag{14}$$

To see this simply note that for $n_{c(k-1)} < q \leq n_{ck}$ we have

$$n_{ck} < q \prod_{t=c(k-1)+1}^{ck} \frac{n_t}{n_{t-1}} \leq q M^{ck - (c(k-1)+1) + 1} = q M^c$$

and by definition

$$n_{cm_k} = n_{c(m_k-1)} \prod_{s=c(m_k-1)+1}^{cm_k} \frac{n_s}{n_{s-1}} \leq n_{c(m_k-1)} M^c \leq \psi^{-i}(n_{ck}) M^c.$$

Proposition 2 implies that the value in (14) exceeds $\lambda_1(I) - \lambda_1(\mathcal{J})$, where

$$\mathcal{J} := \bigcup_{K^-(k)} \bigcup_{p=0}^{q-1} B \left(\frac{p}{q}, \frac{n_{cm_k}}{q n_{ck}} \right).$$

However, for each q there are at most $\lambda_1(I)q + 3$ possible choices for p and so

$$\begin{aligned} \lambda_1(\mathcal{J}) &\leq 2 \sum_{K^-(k)} \frac{n_{cm_k}}{qn_{ck}} (\lambda_1(I)q + 3) = \\ &= 2\lambda_1(I) \frac{n_{cm_k}}{n_{ck}} \sum_{K^-(k)} 1 + \frac{6n_{cm_k}}{n_{ck}} \sum_{K^-(k)} \frac{1}{q}. \end{aligned}$$

For each $r \in \mathbb{N}$ the cardinality of $K^+(r)$ is bounded above by $(n_{cr} - n_{c(r-1)})/n_{cm_k}$. Therefore,

$$\begin{aligned} \frac{6n_{cm_k}}{n_{ck}} \sum_{K^-(k)} \frac{1}{q} &\leq \frac{6n_{cm_k}}{n_{ck}} \sum_{r=1}^{k-1} \sum_{K^+(r)} \frac{1}{q} < \\ &< \frac{6n_{cm_k}}{n_{ck}} \sum_{r=1}^{k-1} \frac{(n_{cr} - n_{c(r-1)})}{n_{c(r-1)}n_{cm_k}} < \\ &< \frac{6(M^c - 1)(k-1)}{n_{ck}} < \\ &< \frac{\lambda_1(I)}{4}, \end{aligned}$$

for k large enough. Moreover, the cardinality of $K^-(r)$ is bounded above by $n_{c(r-1)}/n_{cm_k}$ for $r \in \mathbb{N}$ and so

$$\begin{aligned} 2\lambda_1(I) \frac{n_{cm_k}}{n_{ck}} \sum_{K^-(k)} 1 &\leq 2\lambda_1(I) \frac{n_{c(k-1)}}{n_{ck}} \leq \\ &\leq 2\lambda_1(I) 2^{-(ck-c(k-1))} = \\ &= 2^{1-c} \lambda_1(I). \end{aligned}$$

It follows that for $c \geq 2$ and for sufficiently large k we have $\lambda_1(\mathcal{J}) \leq 3\lambda_1(I)/4$, and inequality (13) indeed holds with $\kappa = 1/4$.

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