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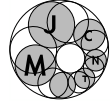
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On simultaneous approximations to zeta-values

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Abstract: We describe a new method to obtain simultaneous rational approximations to $\zeta(2)$ and $\zeta(3)$, based on the arithmetical study of triple integrals of mixed type of suitable rational functions.

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1. Introduction

The arithmetical structure of multiple integrals of rational functions is a major tool in the study of rational approximations to zeta-values, or to values of the logarithm or the dilogarithm at rational numbers (see, e. g., [2], [3], [4], [5], [6], [7], [8]). An interesting feature of this approach is that \mathbb{Q} -linear forms involving such values are given by suitable multiple integrals made over products of paths with distinct endpoints, whereas the corresponding multiple contour integrals around the poles yield the leading coefficients of the linear forms. Thus it is natural to ask which is the arithmetical meaning (if there is any) of allied multiple integrals of mixed type, i. e., partly over paths with distinct endpoints, and partly over closed contours in \mathbb{C} . An answer to this question in a special case can be found in [7], Theorem 2.1, where

the double integral of mixed type

$$\int_0^1 \left(\frac{1}{2\pi i} \oint_{\left|y - \frac{x}{x-z}\right|=\rho} \frac{x^j(1-x)^h y^k(1-y)^l}{(x(1-y) + yz)^{j+k-m+1}} dy \right) dx$$

is shown to play an essential role in the construction of simultaneous rational approximations to $\text{Li}_1(1/z)$ and $\text{Li}_2(1/z)$ for suitable $z \in \mathbb{Q}$ (here Li_r denotes the polylogarithm of integer order $r \geq 1$, defined in the unit disc $|t| < 1$ by the Taylor series $\text{Li}_r(t) = \sum_{n=1}^{\infty} t^n/n^r$).

The aim of the present note is to give a further instance of this approach, concerning triple integrals of mixed type related to simultaneous rational approximations to $\zeta(2)$ and $\zeta(3)$. Although the results obtained so far are not quantitatively sufficient to yield the \mathbb{Q} -linear independence of $1, \zeta(2)$ and $\zeta(3)$, we hope that our method might be improved to reach this goal in the future, or to suggest the treatment of similar simultaneous approximation problems in a different context.

2. Simultaneous approximations to $\zeta(2)$ and $\zeta(3)$

Let

$$h, j, k, l, q, r, s \geq 0 \tag{1}$$

be integers satisfying

$$h \leq k + r \tag{2}$$

and

$$j + q \leq l + s. \tag{3}$$

Define

$$I^{(3)} := \int_0^1 \int_0^1 \int_0^1 \frac{x^h(1-x)^l y^k(1-y)^s z^j(1-z)^q}{(1 - (1-xy)z)^{q+h-r+1}} dx dy dz$$

(here the superscript 3 stands for the dimension of the integral). From the results of [6], Theorem 2.1 and Remark 2.2, we know that

$$I^{(3)} = a + 2b^{(3)}\zeta(3), \tag{4}$$

with $a \in \mathbb{Q}$ and $b^{(3)} \in \mathbb{Z}$. Moreover (see [6], Theorem 3.1 and the linear decomposition in Remark 2.2) the integer $b^{(3)}$ can be expressed as the associated triple contour integral:

$$\begin{aligned}
 b^{(3)} &= \\
 &= \frac{1}{(2\pi i)^3} \oint_{|x|=\varrho} \oint_{\left|y-\frac{1}{x}\right|=\sigma} \oint_{\left|z-\frac{1}{1-xy}\right|=\tau} \frac{x^h(1-x)^l y^k(1-y)^s z^j(1-z)^q}{(1-(1-xy)z)^{q+h-r+1}} dz dy dx \quad (5)
 \end{aligned}$$

for any $\varrho, \sigma, \tau > 0$. We also define the triple integral of mixed type:

$$\begin{aligned}
 J^{(3)} &:= \\
 &:= \int_0^1 \int_0^1 x^h(1-x)^l y^k(1-y)^s \left(\frac{1}{2\pi i} \oint_{\left|z-\frac{1}{1-xy}\right|=\tau} \frac{z^j(1-z)^q}{(1-(1-xy)z)^{q+h-r+1}} dz \right) dx dy.
 \end{aligned}$$

THEOREM. *Under the assumptions (1), (2), (3) and*

$$j+r-h \geq 0, \quad q+h-r \geq 0, \quad (6)$$

we have

$$J^{(3)} = A - b^{(3)}\zeta(2), \quad (7)$$

with $A \in \mathbb{Q}$ and with the same integer $b^{(3)}$, given by (5), appearing in (4).

PROOF. By Cauchy’s integral formula we get, for $xy \neq 1$,

$$\begin{aligned}
 &\frac{1}{2\pi i} \oint_{\left|z-\frac{1}{1-xy}\right|=\tau} \frac{z^j(1-z)^q}{\left(z-\frac{1}{1-xy}\right)^{q+h-r+1}} dz = \\
 &= \frac{1}{(q+h-r)!} \left(\frac{d}{dz}\right)^{q+h-r} (z^j(1-z)^q) \Big|_{z=\frac{1}{1-xy}} = P_{j+r-h} \left(\frac{1}{1-xy}\right), \quad (8)
 \end{aligned}$$

where

$$P_{j+r-h}(z) = \sum_{\nu=0}^{\min\{q, j+r-h\}} \alpha_\nu z^{j+r-h-\nu} \quad (9)$$

is a polynomial of degree $j + r - h$ with coefficients

$$\alpha_\nu = (-1)^{q+\nu} \binom{q}{\nu} \binom{j + q - \nu}{j + r - h - \nu} \in \mathbb{Z}.$$

Let

$$w := l + s - j - q. \tag{10}$$

By (3) we have $w \geq 0$. Therefore

$$\begin{aligned} J^{(3)} &= \\ &= (-1)^{q+h-r+1} \int_0^1 \int_0^1 \frac{x^h(1-x)^l y^k(1-y)^s}{(1-xy)^{q+h-r+1}} P_{j+r-h} \left(\frac{1}{1-xy} \right) dx dy = \\ &= (-1)^{q+h-r+1} \sum_{\nu=0}^{\min\{q, j+r-h\}} \alpha_\nu \int_0^1 \int_0^1 \frac{x^h(1-x)^l y^k(1-y)^s}{(1-xy)^{j+q-\nu+1}} dx dy = \\ &= (-1)^{q+h-r+1} \sum_{\nu=0}^{\min\{q, j+r-h\}} \alpha_\nu \int_0^1 \int_0^1 \frac{x^h(1-x)^l y^k(1-y)^s}{(1-xy)^{l+s-(w+\nu)+1}} dx dy, \end{aligned}$$

where the last double integral converges since $w + \nu \geq 0$. Moreover, by [5], Theorem 2.1 and Lemma 2.6, we get

$$\int_0^1 \int_0^1 \frac{x^h(1-x)^l y^k(1-y)^s}{(1-xy)^{l+s-(w+\nu)+1}} dx dy = a_\nu - b_\nu^{(2)} \zeta(2) \tag{11}$$

with $a_\nu \in \mathbb{Q}$ and

$$b_\nu^{(2)} = \frac{1}{(2\pi i)^2} \oint_{|x|=\rho} \oint_{\left|y-\frac{1}{x}\right|=\sigma} \frac{x^h(1-x)^l y^k(1-y)^s}{(1-xy)^{l+s-(w+\nu)+1}} dy dx \in \mathbb{Z}. \tag{12}$$

Hence

$$J^{(3)} = (-1)^{q+h-r+1} \sum_{\nu=0}^{\min\{q, j+r-h\}} \alpha_\nu (a_\nu - b_\nu^{(2)} \zeta(2)) = A - B^{(2)} \zeta(2), \tag{13}$$

where

$$A = (-1)^{q+h-r+1} \sum_{\nu=0}^{\min\{q, j+r-h\}} \alpha_\nu a_\nu \in \mathbb{Q} \tag{14}$$

and

$$B^{(2)} = (-1)^{q+h-r+1} \sum_{\nu=0}^{\min\{q, j+r-h\}} \alpha_\nu b_\nu^{(2)} \in \mathbb{Z}. \tag{15}$$

By (10),

$$l + s - (w + \nu) + 1 = j + q - \nu + 1 = (q + h - r + 1) + (j + r - h - \nu).$$

Thus, from (5), (8), (9), (12) and (15),

$$\begin{aligned} B^{(2)} &= \\ &= (-1)^{q+h-r+1} \sum_{\nu=0}^{\min\{q, j+r-h\}} \alpha_\nu \frac{1}{(2\pi i)^2} \oint_{|x|=\rho} \oint_{\left|y-\frac{1}{x}\right|=\sigma} \frac{x^h(1-x)^l y^k(1-y)^s}{(1-xy)^{l+s-(w+\nu)+1}} dy dx = \\ &= \frac{(-1)^{q+h-r+1}}{(2\pi i)^2} \oint_{|x|=\rho} \oint_{\left|y-\frac{1}{x}\right|=\sigma} \frac{x^h(1-x)^l y^k(1-y)^s}{(1-xy)^{q+h-r+1}} P_{j+r-h}\left(\frac{1}{1-xy}\right) dy dx = \\ &= \frac{1}{(2\pi i)^3} \oint_{|x|=\rho} \oint_{\left|y-\frac{1}{x}\right|=\sigma} \oint_{\left|z-\frac{1}{1-xy}\right|=\tau} \frac{x^h(1-x)^l y^k(1-y)^s z^j(1-z)^q}{(1-(1-xy)z)^{q+h-r+1}} dz dy dx = \\ &= b^{(3)}. \end{aligned}$$

By (13), this completes the proof of the theorem. □

3. The denominators

We now define the integer

$$m = k + r - h,$$

whence $h + m = k + r$ and $m \geq 0$ by (2), and we set

$$\begin{aligned} h' &= h + l - j, \\ j' &= j + m - k = j + r - h, \\ k' &= k + s - j, \\ l' &= l + r - m = l + h - k, \\ m' &= m + s - q, \\ q' &= q + h - r = q + k - m, \\ r' &= r + l - q, \\ s' &= s + k - h = s + m - r. \end{aligned} \tag{16}$$

We use the notation \max , \max' , \max'' , \dots to denote the successive maxima in a finite sequence of real numbers (see [6], p. 273, for a precise definition). Let

$$\mathcal{S} = (h', j', k', l', m', q', r', s')$$

be the sequence of the integers (16), and let

$$M = \max \mathcal{S}, \quad N = \max' \mathcal{S}, \quad Q = \max'' \mathcal{S}.$$

From [6], Theorem 2.1 and Remark 2.2, we get

$$d_M d_N d_Q a \in \mathbb{Z}, \tag{17}$$

where a is the rational part of the integral $I^{(3)}$ in (4) (we denote by d_n the least common multiple of the integers $1, 2, \dots, n$). As to the denominator of the rational part A of the integral $J^{(3)}$ in (7), we note that, for any integer ν satisfying

$$0 \leq \nu \leq \min\{q, j + r - h\},$$

the integers

$$s + k - h, k + w + \nu - l, h + w + \nu - s, l + h - k, l + s - (w + \nu) = j + q - \nu$$

do not exceed, respectively,

$$s', \min\{k', m'\}, \min\{h', r'\}, l', j + q = j' + q'.$$

Therefore, by (11), (14), and by [5], Theorem 2.2, we get

$$d_{M+N}d_Q A \in \mathbb{Z}. \tag{18}$$

For the integers (6) we now assume

$$j' = j + r - h > 0, \quad q' = q + h - r > 0,$$

whence

$$M \geq N > 0.$$

Let $b_n^{(3)}, I_n^{(3)} = a_n + 2b_n^{(3)}\zeta(3)$ and $J_n^{(3)} = A_n - b_n^{(3)}\zeta(2)$ denote the integrals (5), (4) and (7) with h, j, k, l, q, r, s changed to $hn, jn, kn, ln, qn, rn, sn$ respectively, where h, j, k, l, q, r, s are fixed and $n = 1, 2, 3, \dots$. Then (17) and (18) become

$$d_{Mn}d_{Nn}d_{Qn}a_n \in \mathbb{Z} \tag{19}$$

and

$$d_{(M+N)n}d_{Qn}A_n \in \mathbb{Z}. \tag{20}$$

Let D_n denote the least common multiple of $d_{Mn}d_{Nn}$ and $d_{(M+N)n}$. In view of a result of Hata (see [1], Lemma 2.1 with $m = 0$), trying to prove the \mathbb{Q} -linear independence of $1, \zeta(2)$ and $\zeta(3)$, we multiply $I_n^{(3)}$ and $J_n^{(3)}$ by $D_n d_{Qn}$. By (19) and (20),

$$D_n d_{Qn} I_n^{(3)} = D_n d_{Qn} a_n + 2D_n d_{Qn} b_n^{(3)} \zeta(3)$$

and

$$D_n d_{Qn} J_n^{(3)} = D_n d_{Qn} A_n - D_n d_{Qn} b_n^{(3)} \zeta(2)$$

are linear forms with integer coefficients. Thus one would get the \mathbb{Q} -linear independence of 1, $\zeta(2)$ and $\zeta(3)$, provided that

$$\begin{cases} \lim_{n \rightarrow \infty} \frac{1}{n} \log (D_n d_{Q_n} I_n^{(3)}) < 0, \\ \lim_{n \rightarrow \infty} \frac{1}{n} \log (D_n d_{Q_n} |J_n^{(3)}|) < 0, \end{cases} \tag{21}$$

with

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log I_n^{(3)} \neq \lim_{n \rightarrow \infty} \frac{1}{n} \log |J_n^{(3)}|.$$

Clearly the factorization of $d_n = \text{l.c.m.}\{1, 2, \dots, n\}$ into prime powers is

$$d_n = \prod_p p^{\left\lceil \frac{\log n}{\log p} \right\rceil}, \tag{22}$$

i. e.,

$$\log d_n = \sum_{p \leq n} \left\lceil \frac{\log n}{\log p} \right\rceil \log p = \sum_{p^a \leq n} \log p = \sum_{h \leq n} \Lambda(h) = \psi(n),$$

where ψ is the Chebyshev psi-function. Thus, by the prime number theorem, conditions (21) become

$$\begin{cases} \lim_{n \rightarrow \infty} \frac{1}{n} \log D_n + Q + \lim_{n \rightarrow \infty} \frac{1}{n} \log I_n^{(3)} < 0, \\ \lim_{n \rightarrow \infty} \frac{1}{n} \log D_n + Q + \lim_{n \rightarrow \infty} \frac{1}{n} \log |J_n^{(3)}| < 0. \end{cases} \tag{23}$$

We now compute

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log D_n.$$

Since $D_n = \text{l.c.m.}\{d_{Mn} d_{Nn}, d_{(M+N)n}\}$, by (22) we have

$$D_n = \prod_p p^{\max\left\{\left\lceil \frac{\log(Mn)}{\log p} \right\rceil + \left\lceil \frac{\log(Nn)}{\log p} \right\rceil, \left\lceil \frac{\log(Mn+Nn)}{\log p} \right\rceil\right\}}. \tag{24}$$

Since $N \leq M$, we have $1 + \frac{N}{M} \leq 2 \leq p$. Thus

$$\left\lceil \frac{\log(Mn + Nn)}{\log p} \right\rceil = \left\lceil \frac{\log(Mn)}{\log p} + \frac{\log(1 + N/M)}{\log p} \right\rceil \leq \left\lceil \frac{\log(Mn)}{\log p} \right\rceil + 1.$$

Hence, if $p \leq Nn$ we obtain

$$\left\lceil \frac{\log(Mn + Nn)}{\log p} \right\rceil \leq \left\lceil \frac{\log(Mn)}{\log p} \right\rceil + 1 \leq \left\lceil \frac{\log(Mn)}{\log p} \right\rceil + \left\lceil \frac{\log(Nn)}{\log p} \right\rceil.$$

On the other hand, if $p > Nn$ then $\left\lceil \frac{\log(Nn)}{\log p} \right\rceil = 0$, whence

$$\left\lceil \frac{\log(Mn)}{\log p} \right\rceil + \left\lceil \frac{\log(Nn)}{\log p} \right\rceil = \left\lceil \frac{\log(Mn)}{\log p} \right\rceil \leq \left\lceil \frac{\log(Mn + Nn)}{\log p} \right\rceil.$$

It follows that

$$\begin{aligned} \max \left\{ \left\lceil \frac{\log(Mn)}{\log p} \right\rceil + \left\lceil \frac{\log(Nn)}{\log p} \right\rceil, \left\lceil \frac{\log(Mn + Nn)}{\log p} \right\rceil \right\} = \\ = \begin{cases} \left\lceil \frac{\log(Mn)}{\log p} \right\rceil + \left\lceil \frac{\log(Nn)}{\log p} \right\rceil, & \text{if } p \leq Nn \\ \left\lceil \frac{\log(Mn + Nn)}{\log p} \right\rceil, & \text{if } p > Nn. \end{cases} \end{aligned}$$

Therefore, by (24),

$$\begin{aligned} \log D_n = \\ = \sum_{p \leq Nn} \left\lceil \frac{\log(Mn)}{\log p} \right\rceil \log p + \sum_{p \leq Nn} \left\lceil \frac{\log(Nn)}{\log p} \right\rceil \log p + \\ + \sum_{Nn < p \leq Mn + Nn} \left\lceil \frac{\log(Mn + Nn)}{\log p} \right\rceil \log p = \\ = \sum_{p \leq Nn} \left\lceil \frac{\log(Nn)}{\log p} \right\rceil \log p + \sum_{p \leq Mn + Nn} \left\lceil \frac{\log(Mn + Nn)}{\log p} \right\rceil \log p - \end{aligned}$$

$$\begin{aligned}
& - \sum_{p \leq Nn} \left(\left[\frac{\log(Mn + Nn)}{\log p} \right] - \left[\frac{\log(Mn)}{\log p} \right] \right) \log p = \\
& = \psi(Nn) + \psi(Mn + Nn) - \sum_{p \leq Nn} \left(\left[\frac{\log(Mn + Nn)}{\log p} \right] - \left[\frac{\log(Mn)}{\log p} \right] \right) \log p.
\end{aligned}$$

For $\sqrt{M+N}\sqrt{n} < p \leq Nn$ we have

$$1 \leq \frac{\log(Mn)}{\log p} \leq \frac{\log(Mn + Nn)}{\log p} < 2,$$

whence

$$\left[\frac{\log(Mn)}{\log p} \right] = \left[\frac{\log(Mn + Nn)}{\log p} \right] = 1.$$

Thus

$$\begin{aligned}
& \sum_{p \leq Nn} \left(\left[\frac{\log(Mn + Nn)}{\log p} \right] - \left[\frac{\log(Mn)}{\log p} \right] \right) \log p = \\
& = \sum_{p \leq \sqrt{M+N}\sqrt{n}} \left(\left[\frac{\log(Mn + Nn)}{\log p} \right] - \left[\frac{\log(Mn)}{\log p} \right] \right) \log p \leq \\
& \leq \sum_{p \leq \sqrt{M+N}\sqrt{n}} \frac{\log(Mn + Nn)}{\log p} \log p = \\
& = (\log(M + N) + \log n) \sum_{p \leq \sqrt{M+N}\sqrt{n}} 1 \ll (\log n) \frac{\sqrt{n}}{\log n} = \sqrt{n}.
\end{aligned}$$

Again by the prime number theorem we obtain

$$\log D_n = \psi(Nn) + \psi(Mn + Nn) + O(\sqrt{n}) = Mn + 2Nn + o(n),$$

whence

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log D_n = M + 2N.$$

Then conditions (23) become

$$M + 2N + Q + \lim_{n \rightarrow \infty} \frac{1}{n} \log I_n^{(3)} < 0$$

and

$$M + 2N + Q + \lim_{n \rightarrow \infty} \frac{1}{n} \log |J_n^{(3)}| < 0. \quad (25)$$

At present, the inequality (25) looks out of reach, though some quantitative improvements can be obtained by applying the Rhin—Viola permutation group method (see [5], [6] and [7]). We remark that if h, j, k, l, q, r, s satisfy the inequalities (1), (2), (3) and (6) in strict form (i. e., with \leq and \geq replaced by $<$ and $>$), then a straightforward computation shows that the function

$$f(x, y, z) := \frac{x^h(1-x)^l y^k(1-y)^s z^j(1-z)^q}{(1-(1-xy)z)^{q+h-r}}$$

has exactly three stationary points (x_0, y_0, z_0) , (x_1, y_1, z_1) and (x_2, y_2, z_2) in \mathbb{C}^3 such that $f(x, y, z) \neq 0$. These points actually lie in \mathbb{R}^3 , and their coordinates satisfy

$$x_2 < 0 < x_0 < x_1 < 1,$$

$$y_2 < 0 < y_0 < y_1 < 1,$$

$$z_2 < 0 < z_0 < 1 < \frac{1}{1-x_1 y_1} < z_1.$$

Then a suitable version of the saddle-point method in \mathbb{C}^3 yields

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log I_n^{(3)} = \log f(x_0, y_0, z_0),$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |J_n^{(3)}| = \log |f(x_1, y_1, z_1)|,$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |b_n^{(3)}| = \log |f(x_2, y_2, z_2)|.$$

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