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The irrationality measures of $\zeta(2)$ and $\zeta(3)$ revisited

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Abstract: A family of double integrals over $(0,1)^2$ belonging to $\mathbb{Q}+\mathbb{Z}\zeta(2)$, whose rational parts have controlled denominators, yields the best known irrationality measure of $\zeta(2)$, namely 5.441242... (Rhin—Viola, 1996), while a family of triple integrals yields the best known irrationality measure of $\zeta(3)$, namely 5.513890... (Rhin—Viola, 2001). We obtain the arithmetical and algebraic structures of such integrals by means of repeated partial integrations, two transformations acting on certain Legendre-type polynomials, and, for $\zeta(3)$ only, a transformation arising from a classical integral representation of the first Appel hypergeometric function.

Keywords: irrationality measure, Legendre-type polynomials, first Appel function

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1. Introduction

The best known irrationality measure of $\zeta(2)$, namely

$$\mu\big(\zeta(2)\big) < 5.441242..., \tag{1}$$

was proved by Rhin and Viola [7] through the arithmetical study of 120 double integrals obtained from

$$\int_0^1\int_0^1 \frac{\big(x(1-x)\big)^{12n}\big(y(1-y)\big)^{14n}}{(1-xy)^{13n+1}}\,dx\,dy, \quad n \geqslant 0 \text{ integer}$$

by applying a suitable permutation group isomorphic to the symmetric group $\mathfrak{S}_5$. An irrational number α is said to have an irrationality measure μ if for all $\varepsilon > 0$ there exists a constant $v_0 = v_0(\varepsilon)$ for which

$$\left|\alpha - \frac{u}{v}\right| > v^{-\mu-\varepsilon}$$

for all integers u and v with $v \geqslant v_0$. We denote by $\mu(\alpha)$ the least irrationality measure of α.

Due to a descent argument developed for the first time in their paper, the authors of [7] prove that if

$$h, i, j, k, l \geqslant 0 \tag{2}$$

are integers, then

$$\int_0^1\int_0^1 \frac{x^h(1-x)^i y^k(1-y)^j}{(1-xy)^{i+j-l+1}}\,dx\,dy \in \mathbb{Q} + \mathbb{Z}\zeta(2). \tag{3}$$

We refer to the property (3) as the arithmetical structure of this double integral. An important step to achieve (3) is to prove that this integral is invariant under the action of the permutations $(h\ i\ j\ k\ l)$ and $(h\ k)(i\ j)$, and hence of a permutation group isomorphic to the dihedral group $\mathfrak{D}_5$. This arises by applying to the double integral in (3) a suitable birational transformation on the integration domain $(0,1)^2$.

If moreover

$$j+k-h,\ k+l-i,\ l+h-j,\ h+i-k,\ i+j-l \geqslant 0, \tag{4}$$

then the integral in (3) divided by $h!i!j!k!l!$ is invariant under the action of the symmetric group $\mathfrak{S}_5$ permuting the five sums

$$h+i,\ i+j,\ j+k,\ k+l,\ l+h,$$

and linearly extended to the ten integers (2) and (4). We refer to the last property as the algebraic structure of the double integral in (3). In [7] this follows from Euler's integral representation of the classical hypergeometric function, and is used to obtain crucial informations on the denominator of the rational part of (3).

By a highly non-trivial extension of the outlined method, Rhin and Viola also proved in [8] the best known irrationality measure of $\zeta(3)$,

$$\mu(\zeta(3)) < 5.513890\ldots. \tag{5}$$

The arithmetical structure of some special instances of (3) was studied by Hata in [3] and [4]. In these and in a subsequent paper [5] he applied to Beukers's integrals for $\zeta(2)$ and for $\zeta(3)$ (see [1]) the method used by Rukhadze [10] for the irrationality measure of $\log 2$. For $\zeta(2)$ Hata used repeated partial integration to transform the integral in (3) into an integral containing a suitable Legendre-type polynomial, under restrictions on the parameters h, i, j, k, l stronger than (4). Up to a linear transformation, these polynomials are also known in the literature as special cases of the Jacobi polynomials (see [2], Vol. II, p.169 formula (10)), and are important because the coefficients are integers having many prime factors in common. Hata also employed a similar method to obtain an irrationality measure of $\zeta(3)$ (see [3] and [5]).

In [11] Zudilin obtains new proofs of (1) and (5) in the spirit of Nesterenko's proof [6] of Apéry's theorem on the irrationality of $\zeta(3)$. Zudilin applies a theorem of Nesterenko (see Proposition 1 of [11]) in order to express Rhin and Viola's double (see (3) above) and triple (see (21) below) integrals in terms of the so-called Meyer G-functions. Zudilin uses an identity of Whipple and another identity, due to Bailey (see (6.6) and (4.1) of [11], respectively), thus obtaining a new interpretation of the Rhin—Viola permutation groups for $\zeta(2)$ and $\zeta(3)$.

In the present paper we show that the arithmetical structure of the integral (3) can actually be obtained, in a sense surprisingly, through repeated partial integration not only in the special cases presented in [1], [3] and [4] but even in the general case fully considered in [7] . Furthermore the algebraic structure follows from two transformation formulae (see (7) and (8) below) relating Legendre-type polynomials (in x or in y) appearing in the partial integration method alluded to above. Comparing with [7], all the four corresponding permutations are "hypergeometric".

We also apply similar considerations to a triple Euler-type integral related to $\zeta(3)$. A special instance of this integral appears in [3] and is related to Beukers's by means of a change of variables. The arithmetical structure of our triple integral is again obtained by repeated partial integration. On the other hand, in order to obtain the required algebraic structure, we shall use, together with (7) and (8) below, a classical integral representation of the first Appel hypergeometric function.

2. The arithmetic of 120 double integrals

The following lemma was stated by Hata (lemma 1.1 of [4]), who attributed it to Beukers (its proof being implicit in the proof of lemma 1:(a),(c) of [1]).

Lemma 1. *Let* $F \in \mathbb{Z}[x]$, $G \in \mathbb{Z}[y]$. *Then*

$$\int_0^1 \int_0^1 F(x)G(y)\frac{dx\,dy}{1-xy} = A - B\zeta(2),$$

where

$$B = -\frac{1}{2\pi\sqrt{-1}} \oint F(z)G\Big(\frac{1}{z}\Big)\frac{dz}{z} = -\frac{1}{2\pi\sqrt{-1}} \oint F\Big(\frac{1}{z}\Big)G(z)\frac{dz}{z} \in \mathbb{Z},$$

and $d_M d_N A \in \mathbb{Z}$, *with* $M = \max\{\deg F, \deg G\}$ *and*

$$N = \min\{\max\{\deg F, \deg G - \operatorname{ord}_0 F\}, \max\{\deg G, \deg F - \operatorname{ord}_0 G\}\} =$$
$$= \max\{\deg G - \operatorname{ord}_0 F, \deg F - \operatorname{ord}_0 G, \min\{\deg F, \deg G\}\}.$$

In particular, if $\deg F < \operatorname{ord}_0 G$ *or* $\deg G < \operatorname{ord}_0 F$ *then* $B = 0$.

Here and in the rest of the present paper,

$$d_m := \text{least common multiple of } 1, \ldots, m \text{ if } m > 0, \quad d_0 := 1,$$

and $\operatorname{ord}_{z_0} E$ denotes the order of vanishing of the polynomial $E(z)$ at the point $z = z_0$. For convenience we also write the integer B as

$$B = \frac{1}{(2\pi\sqrt{-1})^2} \oint_{|y|=\rho_1} \oint_{|x-\frac{1}{y}|=\rho_2} \frac{F(x)\,G(y)}{1-xy}\,dx\,dy. \tag{6}$$

As announced in the introduction we shall also need two transformation formulae relating Legendre-type polynomials. The formula (7) below is a reformulation of the formula (1.3) in [3] (see also lemma 3.1 in [9]). For a given $m \geqslant 0$ integer, let

$$D_m\big(Q(z)\big) := \frac{1}{m!}\Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big)^m\big(Q(z)\big)$$

for any analytic function $Q(z)$.

LEMMA 2. *Let $a_1, a_2, b_1, b_2 \geqslant 0$ be integers satisfying*

$$a_1 + a_2 = b_1 + b_2.$$

Then

$$D_{b_1}\big(z^{a_1}(1-z)^{a_2}\big) = (-z)^{a_1-b_1}\frac{a_1!\ a_2!}{b_1!\ b_2!}D_{a_1}\big(z^{b_1}(1-z)^{b_2}\big) \tag{7}$$

and

$$D_{b_1}\big(z^{a_2}(1-z)^{a_1}\big) = (1-z)^{a_1-b_1}\frac{a_1!\ a_2!}{b_1!\ b_2!}D_{a_1}\big(z^{b_2}(1-z)^{b_1}\big). \tag{8}$$

PROOF. We decompose $(1-z)^{a_2}$ and $(1-z)^{b_2}$ in (7) by using the binomial theorem:

$$D_{b_1}\big(z^{a_1}(1-z)^{a_2}\big) = \sum_{l=\max\{0,b_1-a_1\}}^{a_2} (-1)^l \binom{a_2}{l}\binom{l+a_1}{b_1} z^{l+a_1-b_1}$$

and

$$z^{a_1-b_1}D_{a_1}\big(z^{b_1}(1-z)^{b_2}\big) = \sum_{r=\max\{0,a_1-b_1\}}^{b_2} (-1)^r \binom{b_2}{r}\binom{r+b_1}{a_1} z^r.$$

Comparing the two sums above we get (7). If indeed $l + a_1 - b_1 = r$, then

$$(-1)^l\binom{a_2}{l}\binom{l+a_1}{b_1} = (-1)^r\binom{b_2}{r}\binom{r+b_1}{a_1}\ (-1)^{a_1-b_1}\frac{a_1!\ a_2!}{b_1!\ b_2!}.$$

One obtains (8) by changing z into $1-z$ in (7). □

Let $h, i, j, k, l \geqslant 0$ be integers such that $i + j - l, l + h - j \geqslant 0$. Let

$$\alpha = \max\{0, j-l\}, \quad \beta = \max\{0, i+j-h-l, i+j-k-l-\alpha\},$$

$$\gamma = k + l - i - j + \alpha + \beta. \quad (9)$$

With the notations (9) we define

$$I(h, i, j, k, l) := (-1)^{i+j+l+\alpha} \times \\ \times \int_0^1 \int_0^1 x^{\beta}(1-x)^{\alpha} D_{i+j-l}(x^h(1-x)^i) y^{\gamma}(1-y)^{j-\alpha} \frac{dx\, dy}{1-xy}.$$

We may apply the Lemma 1 with

$$F(x) = x^{\beta}(1-x)^{\alpha} D_{i+j-l}(x^h(1-x)^i) \quad (10)$$

and

$$G(y) = y^{\gamma}(1-y)^{j-\alpha}, \quad (11)$$

so that

$$\deg F = \alpha + \beta + h + i - (i + j - l) = \max\{h, i, l + h - j, h + i - k, i + j - l\},$$

$$\deg G = \gamma + j - \beta = \alpha + k + l - i = \max\{k + l - i, j + k - l, \min\{j, l\}\},$$

$$\mathrm{ord}_0 F = \alpha + \max\{h + l - i - j\}, \quad \mathrm{ord}_0 G = \gamma = k + l - i - j + \alpha + \beta,$$

$$\deg G - \mathrm{ord}_0 F = \min\{j + k - h, k + l - i\}, \quad \deg F - \mathrm{ord}_0 G = h + i - k.$$

It follows that $d_M d_N I(h, i, j, k, l) = A - B\zeta(2)$, where $B \in \mathbb{Z}$ and $d_M d_N A \in \mathbb{Z}$, with

$$M = \max\{j + k - h, k + l - i, l + h - j, h + i - k, i + j - l, h, i, \min\{j, l\}\}$$

and

$$N = \min\{\max\{h, i, l + h - j, h + i - k, i + j - l, \min\{j + k - h, k + l - i\}\}, \\ \max\{j + k - h, k + l - i, h + i - k, \min\{j, l\}\}\}. \quad (12)$$

The next step is to apply (7) with $a_1 = h$, $a_2 = i$, $b_1 = i + j - l$ and $b_2 = l + h - j$, to obtain

$$(-1)^{i+j+l+\alpha} x^\beta (1-x)^\alpha D_{i+j-l}(x^h(1-x)^i) = \\ = \frac{h!i!}{(l+h-j)!(i+j-l)!}(-1)^{h+\alpha} x^{\beta+h+l-i-j}(1-x)^\alpha D_h(x^{i+j-l}(1-x)^{l+h-j}),$$

which readily implies

$$I(h,i,j,k,l) = \frac{h!i!}{(l+h-j)!(i+j-l)!} I(i+j-l, l+h-j, j, k, l).$$

Similarly, by applying (8) with $a_1 = i$, $a_2 = h$, $b_1 = i + j - l$ and $b_2 = l + h - j$, we get

$$(1-x)^\alpha D_{i+j-l}(x^h(1-x)^i) = \\ = \frac{h!i!}{(l+h-j)!(i+j-l)!}(1-x)^{\alpha+l-j} D_i(x^{l+h-j}(1-x)^{i+j-l}),$$

which plainly implies

$$I(h,i,j,k,l) = \frac{h!i!}{(l+h-j)!(i+j-l)!} I(l+h-j, i+j-l, l, k, j).$$

By (9), $\alpha + i \geqslant i + j - l$ and $\beta + h \geqslant i + j - l$. Hence, by a $(i + j - l)$-fold partial integration with respect to x,

$$I(h,i,j,k,l) = (-1)^\alpha \int_0^1 \int_0^1 x^h(1-x)^i D_{i+j-l} \frac{x^\beta(1-x)^\alpha}{1-xy} y^\gamma (1-y)^{j-\alpha}\, dx\, dy.$$

Since $x^\beta(1-x)^\alpha$ is a polynomial in x of degree not exceeding $i + j - l$, we have

$$D_{i+j-l} \frac{x^\beta(1-x)^\alpha}{1-xy} = \frac{\frac{1}{y^\beta}(1-\frac{1}{y})^\alpha y^{i+j-l}}{(1-xy)^{i+j-l+1}},$$

whence $I(h, i, j, k, l)$ equals the integral in (3). According to the introduction, we may now transform the integral (3) into an integral analogue to our definition of

$I(h,i,j,k,l)$ by repeated partial integration with respect to the variable y, and then apply (7) and (8) to the latter integral. However it turns out that this is equivalent to remark that by exchanging the rules of x and y in (3) we also have

$$I(h,i,j,k,l)=I(i,h,k,j,l).$$

From now on let us assume (2) and (4). Let $p,q,r,s,t\geqslant 0$ be integers such that

$$h=p+q,\ i=r+s,\ j=t+p,\ k=q+r,\ l=s+t.$$

From what we proved, the quotient

$$\frac{I(p+q,r+s,t+p,q+r,s+t)}{(p+q)!(q+r)!(r+s)!(s+t)!(t+p)!}$$

is invariant under the action of the permutations $(q\ r)$, $(p\ s)$ and $(p\ r)(s\ t)$. These three permutations generate the whole symmetric group $\mathfrak{S}_5$ on p,q,r,s,t, because

$$(p\ r)(s\ t)(q\ r)(p\ r)(s\ t)=(p\ q) \quad \text{and} \quad (p\ r)(s\ t)(p\ s)(p\ r)(s\ t)=(r\ t).$$

The permutations $(p\ q)$ and $(q\ r)$ correspond to the transformation (7), and $(p\ s)$ and $(r\ t)$ to the transformation (8). In particular

$$I(h,i,j,k,l)=I(p+q,r+s,t+p,q+r,s+t)$$

is invariant under the action of the dihedral group $\mathfrak{D}_5$ of all the permutations on p,q,r,s,t (or, equivalently, on h,i,j,k,l) preserving or reversing the cyclic order of the elements. This group is transitive on $p+r,\ r+t,\ t+q,\ q+s,\ s+p$, i. e. on the five integers in (4), whence we may assume in (12) that

$$h+i-k=\min\{j+k-h,k+l-i,l+h-j,h+i-k,i+j-l\}.$$

In this case

$$N=\min\{\max\{h,i,l+h-j,i+j-l,\min\{j+k-h,k+l-i\}\},\\ \max\{j+k-h,k+l-i,\min\{j,l\}\}\}.$$

Let

$$H = \max\{h, i, j, k, l, j+k-h, k+l-i, l+h-j, h+i-k, i+j-l\}$$

and

$$K = \max{}'\{h, i, j, k, l, j+k-h, k+l-i, l+h-j, h+i-k, i+j-l\}$$

(as usual, max, max$'$, max$''$, ... are the successive maxima in a finite sequence of real numbers). Then

$$I(h, i, j, k, l) = A - B\zeta(2),$$

where $d_H d_K A \in \mathbb{Z}$ and $B \in \mathbb{Z}$. This quantitative version of (3) must be combined with the algebraic structure described above.

Let $h = i = 12n$, $j = k = 14n$, $l = 13n$ where $n \geqslant 0$ is an integer, and let $I(12n, 12n, 14n, 14n, 13n) = A_n - B_n\zeta(2)$. By sections 4 and 5 of [7] there exists a sequence Δ_n such that Δ_n divides B_n and $d_{16n} d_{15n} A_n$, and

$$\lim_{n\to\infty} \frac{1}{n} \log \Delta_n = 9.29787398... + 1.76782442... = 11.0656984... .$$

By standard computations

$$\lim_{n\to\infty} \frac{1}{n} \log I(12n, 12n, 14n, 14n, 13n) = -31.27178857... .$$

By a straightforward application of Cauchy's integral formula to (6) with $F(x)$ and $G(y)$ given by (10) and (11) respectively,

$$B = \frac{1}{(2\pi\sqrt{-1})^2} \oint_{|y|=\rho_1} \oint_{|x-\frac{1}{y}|=\rho_2} \frac{x^h(1-x)^i y^k (1-y)^j}{(1-xy)^{i+j-l+1}}\, dx\, dy,$$

whence

$$\lim_{n\to\infty} \frac{1}{n} \log |B_n| \leqslant 30.41828189... .$$

By a classical lemma one obtains $\mu(\zeta(2)) < 5.441242...$.

3. The arithmetic of 1920 triple integrals

By lemma 1:(b),(d) of [1], and taking into account that

$$-\frac{\log((1-x)(1-y))}{1-(1-x)(1-y)} = \int_0^1 \frac{dz}{(1-x(1-z))(1-yz)}$$

(see [3], p.122), we have

LEMMA 3. *Let* $F \in \mathbb{Z}[x]$, $G \in \mathbb{Z}[y]$. *Then*

$$\int_0^1\int_0^1\int_0^1 F(x)G(y)\frac{dx\,dy\,dz}{(1-x(1-z))(1-yz)} = A - 2B\zeta(3),$$

where

$$B = -\frac{1}{2\pi\sqrt{-1}}\oint F(1-z)G\left(1-\frac{1}{z}\right)\frac{dz}{z} \in \mathbb{Z},$$

and $d_M^2 d_N A \in \mathbb{Z}$, *with* $M = \max\{\deg F, \deg G\}$ *and*

$$N = \min\{\max\{\deg F, \deg G - \mathrm{ord}_1 F\}, \max\{\deg G, \deg F - \mathrm{ord}_1 G\}\} =$$
$$= \max\{\deg F - \mathrm{ord}_1 G, \deg G - \mathrm{ord}_1 F, \min\{\deg F, \deg G\}\}.$$

In order to write the integer B in a more convenient way, we remark that for any $f, g \geqslant 0$ integers, if $F(x) = (1-x)^f$ and $G(y) = (1-y)^g$ then

$$B = -\frac{1}{2\pi\sqrt{-1}}\oint z^f z^{-g}\frac{dz}{z} = \begin{cases} -1 & \text{if} \quad f = g, \\ 0 & \text{if} \quad f \neq g. \end{cases}$$

Since

$$-\frac{1}{(2\pi\sqrt{-1})^3}\oint_{|z|=\rho_1}\oint_{|y-\frac{1}{z}|=\rho_2}\oint_{|x-\frac{1}{1-z}|=\rho_3}\frac{(1-x)^f(1-y)^g}{(1-x(1-z))(1-yz)}\,dx\,dy\,dz =$$

$$= \frac{1}{2\pi\sqrt{-1}} \oint\limits_{|z|=\rho_1} \frac{z^f}{(z-1)^{f+1}} \frac{(z-1)^g}{z^{g+1}}\, dz = \begin{cases} -1 & \text{if} \quad f = g, \\ 0 & \text{if} \quad f \neq g, \end{cases}$$

by linearity we get

$$B = -\frac{1}{(2\pi\sqrt{-1})^3} \oint\limits_{|z|=\rho_1} \oint\limits_{|x-\frac{1}{1-z}|=\rho_2} \oint\limits_{|y-\frac{1}{z}|=\rho_3} \frac{F(x)\, G(y)}{(1-x(1-z))(1-yz)}\, dx\, dy\, dz \qquad (13)$$

for all $F \in \mathbb{Z}[x]$ and $G \in \mathbb{Z}[y]$. We recall the definition of the first Appel hypergeometric series of two variables

$$F_1(\alpha; \beta, \beta'; \gamma; u, v) = \sum_{\mu,\nu \geqslant 0} \frac{(\alpha)_{\mu+\nu} (\beta)_\mu (\beta')_\nu}{(\gamma)_{\mu+\nu}\ \mu!\ \nu!} u^\mu v^\nu, \quad |u| < 1,\ |v| < 1$$

(see [2], vol.I, p.224 formula (6)). Here $(w)_0 = 1$ and $(w)_\tau = w(w+1)\cdots(w+\tau-1)$ if $\tau > 0$. The above series has the following representation as simple integral of Euler's type, valid if $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}(\gamma-\alpha) > 0$ ([2], vol.I, p.231, formula (5)):

$$F_1(\alpha; \beta, \beta'; \gamma; u, v) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int\limits_0^1 \frac{t^{\alpha-1}(1-t)^{\gamma-\alpha-1}}{(1-tu)^\beta (1-tv)^{\beta'}}\, dt.$$

Moreover, if $\beta + \beta' = \gamma$ the double series $F_1(\alpha; \beta, \beta'; \gamma; u, v)$ can be written as a simple hypergeometric series ([2], vol.I, p.238, formula (1)):

$$(1-v)^\alpha F_1(\alpha; \beta, \beta'; \beta+\beta'; u, v) = {}_2F_1\Big(\alpha, \beta, \beta+\beta'; \frac{u-v}{1-v}\Big).$$

Since ${}_2F_1(\alpha, \beta, \beta+\beta'; w) = {}_2F_1(\beta, \alpha, \beta+\beta'; w)$ and

$$(1-v)^\beta F_1(\beta; \alpha, \beta+\beta'-\alpha; \beta+\beta'; u, v) = {}_2F_1\Big(\beta, \alpha, \beta+\beta'; \frac{u-v}{1-v}\Big)$$

we have

$$F_1(\alpha; \beta, \beta'; \beta+\beta'; u, v) = (1-v)^{\beta-\alpha} F_1(\beta; \alpha, \beta+\beta'-\alpha; \beta+\beta'; u, v)$$

whence the following integral transformation holds (if $\operatorname{Re}\alpha$, $\operatorname{Re}\beta$, $\operatorname{Re}\beta'$, $\operatorname{Re}(\beta+\beta'-\alpha)>0$; $u,v\in\mathbb{C}\setminus[1,+\infty[$):

$$\int_0^1 \frac{t^{\alpha-1}(1-t)^{\beta+\beta'-\alpha-1}}{(1-tu)^{\beta}(1-tv)^{\beta'}}\,dt = $$
$$= (1-v)^{\beta-\alpha}\frac{\Gamma(\alpha)\Gamma(\beta+\beta'-\alpha)}{\Gamma(\beta)\Gamma(\beta')}\int_0^1 \frac{t^{\beta-1}(1-t)^{\beta'-1}}{(1-tu)^{\alpha}(1-tv)^{\beta+\beta'-\alpha}}\,dt. \quad (14)$$

LEMMA 4. *Let* $a_1,a_2,b_1,b_2\geqslant 0$ *be integers satisfying*

$$a_1+a_2=b_1+b_2.$$

Then for any $x,y\in\mathbb{C}\setminus[1,+\infty[$

$$\int_0^1 \frac{z^{a_1}(1-z)^{a_2}}{(1-x(1-z))^{b_2+1}(1-yz)^{b_1+1}}\,dz = $$
$$= \frac{a_1!a_2!}{b_1!b_2!}\int_0^1 \frac{z^{b_1}(1-z)^{b_2}}{(1-x(1-z))^{a_2+1}(1-yz)^{a_1+1}}\,dz \quad (15)$$

and

$$\int_0^1 \frac{z^{a_1}(1-z)^{a_2}}{(1-x(1-z))^{b_1+1}(1-yz)^{b_2+1}}\,dz = $$
$$= \left(\frac{1-y}{1-x}\right)^{b_1-a_1}\frac{a_1!a_2!}{b_1!b_2!}\int_0^1 \frac{z^{b_1}(1-z)^{b_2}}{(1-x(1-z))^{a_1+1}(1-yz)^{a_2+1}}\,dz. \quad (16)$$

PROOF. By applying (14) with $\alpha=a_1+1$, $\beta=b_1+1$, $\beta'=b_2+1$, $t=z$, $u=y$ and $v=x/(x-1)$ we obtain (15).

By applying (14) with $\alpha=a_1+1$, $\beta=b_1+1$, $\beta'=b_2+1$, $t=z$, $u=x/(x-1)$ and $v=y$ we obtain (16). $\square$

Let $h, j, k, l, m, q, r, s \geqslant 0$ be integers such that

$$h + m = k + r \quad \text{and} \quad j + q = l + s,$$

and that $q + h - r, r + l - q, j + m - k, k + s - j \geqslant 0$. Let

$$\alpha = \max\{0, q + h - r - l\}, \quad \beta = \{0, j - k, j - l + \max\{0, q - r\} - \alpha\},$$

$$\gamma = \max\{0, j + m - k - s\} = r + l - q - h + \alpha,$$

$$\delta = \max\{0, q - r, q - s + \max\{0, j - k\} - \gamma\}, \quad \varepsilon = q + k - j - m + \beta + \gamma - \delta,$$

$$\eta = j + r - q - h + \alpha + \delta - \beta. \quad (17)$$

We remark that $\alpha\gamma = 0$. With the notations (17) we define

$$I(h, j, k, l, m, q, r, s) := (-1)^{j+q+\beta+\delta} \int_0^1 \int_0^1 \int_0^1 x^\alpha (1 - x)^\delta D_{q+h-r}(x^l (1 - x)^h) \times$$

$$\times\, y^\gamma (1 - y)^\beta D_{j+m-k}(y^s (1 - y)^m) z^\varepsilon (1 - z)^\eta \, \frac{dx\, dy\, dz}{(1 - x(1 - z))(1 - yz)} \, .$$

It is easy to see that $\varepsilon, \eta \geqslant 0$ and $\varepsilon + \eta = \alpha + \gamma$. We also remark that

$$\operatorname{ord}_0\Big(x^\alpha (1 - x)^\delta D_{q+h-r}(x^l (1 - x)^h)\Big) = \alpha + \max\{0, r + l - q - h\} = \max\{\alpha, \gamma\},$$

$$\operatorname{ord}_0\Big(y^\gamma (1 - y)^\beta D_{j+m-k}(y^s (1 - y)^m)\Big) = \gamma + \max\{0, k + s - j - m\} = \max\{\alpha, \gamma\}.$$

Let

$$\mathcal{S} = \{h, j, k, l, m, q, r, s,$$
$$h + l - j, j + m - k, k + q - l, l + r - m, m + s - q, q + h - r, r + j - s, s + k - h\},$$

and let $H = \max \mathcal{S}$, $K = \max' \mathcal{S}$, $L = \max'' \mathcal{S}$. We are going to prove that

$$I(h, j, k, l, m, q, r, s) = A - 2\zeta(3)B, \quad d_H d_K d_L A \in \mathbb{Z}, \ B \in \mathbb{Z} \qquad (18)$$

under the assumption

$$H \in \mathcal{S} \setminus \{h+l-j, l+r-m, m+s-q, s+k-h\}. \tag{19}$$

We distinguish two cases:

First case: $\gamma = 0$. Hence α, β, δ, ε and η in (17) may be rewritten as follows:

$$\alpha = q+h-r-l, \quad \beta = \max\{0, j-k, j-h, j+r-q-h\},$$

$$\delta = \max\{0, q-r, q-s, l-k\}, \quad \varepsilon = q+k-j-m+\beta-\delta, \quad \eta = j-l+\delta-\beta.$$

Since $0 \leqslant \eta \leqslant \alpha = \varepsilon + \eta$, by writing

$$\frac{x^\alpha(1-z)^\eta}{1-x(1-z)} = \frac{x^\varepsilon}{1-x(1-z)} - x^\varepsilon\big(1 + x(1-z) + \cdots + x^{\eta-1}(1-z)^{\eta-1}\big)$$

we decompose the integral into $\eta+1$ parts: $I(h, j, k, l, m, q, r, s) = J_0 + J_1 + \ldots + J_\eta$, where

$$J_0 = (-1)^{j+k+\beta+\delta} \int_0^1 \int_0^1 \int_0^1 x^\varepsilon (1-x)^\delta D_{q+h-r}(x^l(1-x)^h)$$

$$\times (1-y)^\beta D_{j+m-k}(y^s(1-y)^m) z^\varepsilon \, \frac{dx\, dy\, dz}{(1-x(1-z))(1-yz)},$$

and for $\theta = 1, \ldots, \eta$

$$J_\theta = (-1)^{j+k+\beta+\delta+1} \int_0^1 x^{\varepsilon+\theta-1}(1-x)^\delta D_{q+h-r}(x^l(1-x)^h)\; dx$$

$$\times \int_0^1 \int_0^1 (1-y)^\beta D_{j+m-k}(y^s(1-y)^m) z^\varepsilon (1-z)^{\theta-1} \, \frac{dy\, dz}{1-yz}.$$

Similarly, by writing

$$\frac{y^\varepsilon z^\varepsilon}{1-yz} = \frac{1}{1-yz} - (1 + yz + \cdots + y^{\varepsilon-1}z^{\varepsilon-1})$$

we decompose the integral J_0 into $\varepsilon+1$ parts: $J_0 = K_0 + K_1 + \cdots + K_\varepsilon$, where

$$K_0 = (-1)^{j+k+\beta+\delta} \int_0^1\int_0^1\int_0^1 x^\varepsilon (1-x)^\delta D_{q+h-r}(x^l(1-x)^h) \times$$

$$\times\, y^{-\varepsilon}(1-y)^\beta D_{j+m-k}(y^s(1-y)^m)\ \frac{dx\,dy\,dz}{(1-x(1-z))(1-yz)},$$

and, for $\lambda = 1, \ldots, \varepsilon$

$$K_\lambda = (-1)^{j+k+\beta+\delta+1} \int_0^1\int_0^1 x^\varepsilon (1-x)^\delta D_{q+h-r}(x^l(1-x)^h) z^{\lambda-1} \frac{dx\,dz}{(1-x(1-z))} \times$$

$$\times \int_0^1 y^{\lambda-\varepsilon-1}(1-y)^\beta D_{j+m-k}(y^s(1-y)^m)\ dy.$$

We now apply our Lemma 3 to the integral K_0, with

$$F(x) = x^\varepsilon(1-x)^\delta D_{q+h-r}(x^l(1-x)^h),$$

$$G(y) = y^{-\varepsilon}(1-y)^\beta D_{j+m-k}(y^s(1-y)^m),$$

so that

$$\deg F = h+l-j+\beta, \quad \deg G = m+s-q+\delta,$$

$$\mathrm{ord}_1 F = \delta + \max\{0, r-q\}, \quad \deg G - \mathrm{ord}_1 F = \min\{m+s-q, s+m-r\},$$

$$\mathrm{ord}_1 G = \beta + \max\{0, k-j\}, \quad \deg F - \mathrm{ord}_1 G = \min\{h+l-j, l+h-k\}.$$

Hence $d_M d_N d_Q K_0 \in \mathbb{Z} + 2\zeta(3)\mathbb{Z}$, with

$$M = Q = \max\{h+l-j+\beta, m+s-q+\delta\},$$

$$N = \max\Big\{\min\{m+s-q, s+m-r\}, \min\{h+l-j, l+h-k\},$$
$$\min\{h+l-j+\beta, m+s-q+\delta\}\Big\}.$$

Since

$$h+l-j+\beta \leqslant h+l-j+\beta+\eta = h+\delta = \max\{h, q+h-r, h+q-s, l+h-k\}$$

and

$$m+s-q+\delta \leqslant m+s-q+\delta+\varepsilon = k+s-j+\beta = \max\{k+s-j, s, s+k-h, m+s-q\}$$

we have $d_H d_K d_L K_0 \in \mathbb{Z} + 2\zeta(3)\mathbb{Z}$.

As to the integrals J_θ with $\theta = 1, \ldots, \eta$, we apply our Lemma 1 with

$$F(y) = (1-y)^\beta D_{j+m-k}(y^s(1-y)^m), \quad G(z) = z^\varepsilon(1-z)^{\theta-1},$$

so that

$$\deg F = k+s-j+\beta, \quad \deg G = \varepsilon+\theta-1 < \alpha = \mathrm{ord}_0 F,$$

$$\mathrm{ord}_0 G = \varepsilon, \quad \deg F - \mathrm{ord}_0 G = m+s-q+\delta.$$

Since moreover

$$\deg\left(x^{\varepsilon+\theta-1}(1-x)^\delta D_{q+h-r}(x^l(1-x)^h)\right) < h+l-j+\beta+\eta = h+\delta,$$

we have $d_{M'} d_{N'} d_{Q'} J_\theta \in \mathbb{Z} + 2\zeta(3)\mathbb{Z}$ for $\theta = 1, \ldots, \eta$, where

$$M' = k+s-j+\beta = m+s-q+\delta+\varepsilon = \max\{k+s-j, s, s+k-h, m+s-q\},$$

$$N' = \max\{q+h-r-l, m+s-q+\delta\},$$

with

$$q+h-r-l \leqslant \min\{h, q+h-r, s, k+s-j\}$$

and

$$m+s-q+\delta = \max\{m+s-q, s+m-r, m, j+m-k\}$$

(note that $s+m-r=s+k-h$),

$$Q' = h+l-j+\beta+\eta = h+\delta = \max\{h, q+h-r, h+q-s, l+m-k\}.$$

Again this implies $d_H d_K d_L J_\theta \in \mathbb{Z} + 2\zeta(3)\mathbb{Z}$ because $H \notin \{m+s-q, s+m-r\}$.

We next apply our Lemma 1 to the integrals K_λ (where z is changed into $1-z$) with

$$F(x) = x^\varepsilon (1-x)^\delta D_{q+h-r}(x^l(1-x)^h), \quad G(z) = (1-z)^{\lambda-1}.$$

Since

$$\deg\left(y^{\lambda-\varepsilon-1}(1-y)^\beta D_{j+m-k}(y^s(1-y)^m)\right) < \beta+k+s-j$$

we get $d_{M''} d_{N''} d_{Q''} K_\lambda \in \mathbb{Z} + 2\zeta(3)\mathbb{Z}$, where

$$\begin{aligned} M'' = N'' = \deg F = h+l-j+\beta &= \max\{h+l-j, l+h-k, l, r+l-q\} \\ &\leqslant h+\delta = \max\{h, q+h-r, h+q-s, l+h-k\} \end{aligned}$$

(note that $h+l-j=h+q-s$),

$$Q'' = k+s-j+\beta = m+s-q+\delta+\varepsilon = \max\{k+s-j, s, s+k-h, m+s-q\},$$

whence $d_H d_K d_L K_\lambda \in \mathbb{Z} + 2\zeta(3)\mathbb{Z}$ because $H \notin \{h+l-j, l+h-k\}$.

Second case: $\alpha = 0$. By exchanging x and y, and changing z into $1-z$ in the integration, we see that $I(h,j,k,l,m,q,r,s) = I(m,q,r,s,h,j,k,l)$. In other words, the permutation $(h\, m)(j\, q)(k\, r)(l\, s)$ leaves the integral $I(h,j,k,l,m,q,r,s)$ unchanged, and by (17) exchanges α with γ, β with δ and ε with η. Hence (18) follows from the previously considered case $\gamma = 0$.

From now on we make the assumption that all the sixteen integers in the finite sequence $\mathcal{S}$ are non-negative, but we do not assume (19) anymore. We shall prove that $I(h,j,k,l,m,q,r,s)$ is invariant under the action of a permutation group isomorphic to $\mathfrak{D}_8$, as a special case of the invariance of the quotient

$$\frac{I(h,j,k,l,m,q,r,s)}{h!j!k!l!m!q!r!s!}$$

under the action of a permutation group of order 1920. Then (18) will follow without assuming (19). By applying (7) with $a_1 = l,\ a_2 = h,\ b_1 = q + h - r,\ b_2 = r + l - q$ we have

$$D_{q+h-r}(x^l(1-x)^h) = (-x)^{r+l-q-h}\frac{l!h!}{(q+h-r)!(r+l-q)!}D_l(x^{q+h-r}(1-x)^{r+l-q}).$$

By applying (7) with $a_1 = s,\ a_2 = m,\ b_1 = j + m - k,\ b_2 = k + s - j$ we get

$$D_{j+m-k}(y^s(1-y)^m) = (-y)^{k+s-j-m}\frac{m!s!}{(j+m-k)!(k+s-j)!}D_s(y^{j+m-k}(1-y)^{k+s-j}).$$

These together yield

$$\frac{I(h,j,k,l,m,q,r,s)}{h!l!m!s!} = \frac{I(r+l-q,j,k,q+h-r,k+s-j,q,r,j+m-k)}{(q+h-r)!(r+l-q)!(j+m-k)!(k+s-j)!}.$$

We remark that to apply (7) separately to the polynomial in x and in y would yield to an integral similar to $I(h, j, k, l, m, q, r, s)$ but the conditions $h + m = k + r$ and $j + q = l + s$ would not be preserved.

By applying (8) with $a_1 = h,\ a_2 = l,\ b_1 = q + h - r$ and $b_2 = r + l - q$ we have

$$D_{q+h-r}(x^l(1-x)^h) = (1-x)^{r-q}\frac{l!h!}{(q+h-r)!(r+l-q)!}D_h(x^{r+l-q}(1-x)^{q+h-r}),$$

whence

$$\frac{I(h,j,k,l,m,q,r,s)}{h!l!} = \frac{I(q+h-r,j,k,r+l-q,m,r,q,s)}{(q+h-r)!(r+l-q)!}.$$

By applying (8) with $a_1 = m,\ a_2 = s,\ b_1 = j + m - k,\ b_2 = k + s - j$ we obtain

$$D_{j+m-k}(y^s(1-y)^m) = (1-y)^{k-j}\frac{m!s!}{(j+m-k)!(k+s-j)!}D_m(y^{k+s-j}(1-y)^{j+m-k}),$$

hence

$$\frac{I(h,j,k,l,m,q,r,s)}{m!s!} = \frac{I(h,k,j,l,j+m-k,q,r,k+s-j)}{(j+m-k)!(k+s-j)!}.$$

In the triple integral $I(h,j,k,l,m,q,r,s)$ we perform a $(q+h-r)$-fold partial integration with respect to x and a $(j+m-k)$-partial integration with respect to y:

$$I(h,j,k,l,m,q,r,s) = (-1)^{\beta+\delta}\int_0^1\int_0^1\int_0^1 x^l(1-x)^h D_{q+h-r}\frac{x^\alpha(1-x)^\delta}{1-x(1-z)} \times$$
$$\times\, y^s(1-y)^m D_{j+m-k}\frac{y^\gamma(1-y)^\beta}{1-yz} z^\varepsilon(1-z)^\eta\, dx\, dy\, dz.$$

Since $x^\alpha(1-x)^\delta$ is a polynomial in x of degree not exceeding $q+h-r$ we have

$$D_{q+h-r}\frac{x^\alpha(1-x)^\delta}{1-x(1-z)} = \frac{\dfrac{1}{(1-z)^\alpha}\left(1-\dfrac{1}{1-z}\right)^\delta(1-z)^{q+h-r}}{(1-x(1-z))^{q+h-r+1}},$$

and since $y^\gamma(1-y)^\beta$ is a polynomial in y of degree not exceeding $j+m-k$ we have

$$D_{j+m-k}\frac{y^\gamma(1-y)^\beta}{1-yz} = \frac{\dfrac{1}{z^\gamma}\left(1-\dfrac{1}{z}\right)^\beta z^{j+m-k}}{(1-yz)^{j+m-k+1}}.$$

It follows that

$$I(h,j,k,l,m,q,r,s) =$$
$$= \int_0^1\int_0^1\int_0^1 \frac{x^l(1-x)^h y^s(1-y)^m z^q(1-z)^j}{(1-x(1-z))^{q+h-r+1}(1-yz)^{j+m-k+1}}\, dx\, dy\, dz. \quad (20)$$

We may now apply (15) with $a_1 = q$, $a_2 = j$, $b_1 = j+m-k$, $b_2 = q+h-r$, to obtain

$$\int_0^1 \frac{z^q(1-z)^j}{(1-x(1-z))^{q+h-r+1}(1-yz)^{j+m-k+1}}\, dz =$$
$$= \frac{q!j!}{(q+h-r)!(j+m-k)!}\int_0^1 \frac{z^{j+m-k}(1-z)^{q+h-r}}{(1-x(1-z))^{j+1}(1-yz)^{q+1}}\, dz,$$

whence

$$\frac{I(h,j,k,l,m,q,r,s)}{j!q!} = \frac{I(h,q+h-r,k,l,m,j+m-k,r,s)}{(j+m-k)!(q+h-r)!}.$$

Similarly by applying (16) with $a_1 = q,\ a_2 = j,\ b_1 = q+h-r,\ b_2 = j+m-k$ we get

$$\int_0^1 \frac{z^q(1-z)^j}{(1-x(1-z))^{q+h-r+1}(1-yz)^{j+m-k+1}}\,dz =$$

$$= \left(\frac{1-y}{1-x}\right)^{h-r}\frac{q!j!}{(q+h-r)!(j+m-k)!}\int_0^1 \frac{z^{q+h-r}(1-z)^{j+m-k}}{(1-x(1-z))^{q+1}(1-yz)^{j+1}}\,dz,$$

hence

$$\frac{I(h,j,k,l,m,q,r,s)}{j!q!} = \frac{I(r,j+m-k,m,l,k,q+h-r,h,s)}{(j+m-k)!(q+h-r)!}.$$

We can rephrase all the previous identities like the last one in the following way. The quotient

$$\frac{I(h,j,k,l,m,q,r,s)}{h!j!k!l!m!q!r!s!}$$

is invariant under the following transformations:

$$\phi_1 : (h,j,k,l,m,q,r,s) \to (r+l-q,j,k,q+h-r,k+s-j,q,r,j+m-k);$$

$$\phi_2 : (h,j,k,l,m,q,r,s) \to (q+h-r,j,k,r+l-q,m,r,q,s);$$

$$\phi_3 : (h,j,k,l,m,q,r,s) \to (h,k,j,l,j+m-k,q,r,k+s-j);$$

$$\phi_4 : (h,j,k,l,m,q,r,s) \to (h,q+h-r,k,l,m,j+m-k,r,s);$$

$$\phi_5 : (h,j,k,l,m,q,r,s) \to (r,j+m-k,m,l,k,q+h-r,h,s).$$

We now follow closely [8], pp.282–284 in the study of the structure of the group generated by $\phi_1, \ldots, \phi_5$. We introduce ten auxiliary integers:

$$u_0 = k + r, \quad u_1 = h + l, \quad u_2 = j + m, \quad u_3 = k + q, \quad u_4 = l + r,$$

$$u_5 = m + s, \quad u_6 = q + h, \quad u_7 = r + j, \quad u_8 = s + k, \quad u_9 = j + q.$$

With the above notations the transformations $\phi_1, \ldots, \phi_5$ correspond to the five permutations

$$\varphi_1 = (u_2\ u_8)(u_4\ u_6), \quad \varphi_2 = (u_0\ u_3)(u_7\ u_9), \quad \varphi_3 = (u_0\ u_7)(u_3\ u_9),$$

$$\varphi_4 = (u_2\ u_3)(u_6\ u_7), \quad \varphi_5 = (u_1\ u_4)(u_5\ u_8).$$

It is easy to see that the permutation group generated by $\varphi_1, \ldots, \varphi_5$ is contained in the alternating group $\mathfrak{A}_{10}$ and is isomorphic to the group generated by $\phi_1, \ldots, \phi_5$. Let us introduce the set of pairs

$$\mathcal{P} = \{\{u_0, u_9\}, \{u_1, u_5\}, \{u_2, u_6\}, \{u_3, u_7\}, \{u_4, u_8\}\},$$

and let $\mathfrak{H}$ denote the subgroup of $\mathfrak{A}_{10}$ consisting of the $\binom{5}{0} + \binom{5}{2} + \binom{5}{4} = 16$ permutations of $u_0, \ldots, u_9$ that interchange the elements in an even number of pairs lying in $\mathcal{P}$ and acts identically on the elements of the remaining pairs. The permutations $\varphi_1, \ldots, \varphi_5$ induce five permutations $\varphi_1^*, \ldots, \varphi_5^*$ on the pairs of $\mathcal{P}$:

$$\varphi_1^* = (\{u_2, u_6\}\ \{u_4, u_8\}), \quad \varphi_2^* = \varphi_3^* = (\{u_0, u_9\}\ \{u_3, u_7\}),$$

$$\varphi_4^* = (\{u_2, u_6\}\ \{u_3, u_7\}), \quad \varphi_5^* = (\{u_1, u_5\}\quad \{u_4, u_8\}).$$

The mapping $\varphi_1 \to \varphi_1^*, \ldots, \varphi_5 \to \varphi_5^*$ extends to a homomorphism $\Phi \xrightarrow{*} \mathfrak{G}_5$ of the permutation group generated by $\varphi_1, \ldots, \varphi_5$ into the symmetric group $\mathfrak{G}_5$ of all permutations of $\mathcal{P}$. The homomorphism * is plainly surjective. Its kernel is the subgroup $\mathfrak{H}$ described above and generated, for example, by

$$\chi_1 = (u_0\ u_9)(u_3\ u_7) = \varphi_2\varphi_3,$$

$$\chi_2 = (u_0\ u_9)(u_2\ u_6) = \varphi_4\chi_1\varphi_4,$$

$$\chi_3 = (u_0\ u_9)(u_4\ u_8) = \varphi_1\chi_2\varphi_1,$$

$$\chi_4 = (u_0\ u_9)(u_1\ u_5) = \varphi_5\chi_3\varphi_5.$$

Hence Φ is the subgroup of all permutations φ of $\mathfrak{A}_{10}$ such that

$$\mathcal{P}=\{\{\varphi(u_0),\varphi(u_9)\},\{\varphi(u_1),\varphi(u_5)\},\{\varphi(u_2),\varphi(u_6)\},\{\varphi(u_3),\varphi(u_7)\},\{\varphi(u_4),\varphi(u_8)\}\}$$

and that

$$\frac{(\varphi(u_0)-\varphi(u_9))(\varphi(u_1)-\varphi(u_5))(\varphi(u_2)-\varphi(u_6))(\varphi(u_3)-\varphi(u_7))(\varphi(u_4)-\varphi(u_8))}{(u_0-u_9)(u_1-u_5)(u_2-u_6)(u_3-u_7)(u_4-u_8)}=1.$$

In particular $|\Phi| = 2^4 \cdot 5! = 1920$, and the permutations

$$\vartheta = (u_0\ u_9)(u_1\ u_2\ u_3\ u_4\ u_5\ u_6\ u_7\ u_8), \quad \varsigma = (u_1\ u_8)(u_2\ u_7)(u_3\ u_6)(u_4\ u_5)$$

belong to Φ. These permutations correspond to the transformations

$$\theta : (h, j, k, l, m, q, r, s) \to (j, k, l, m, q, r, s, h)$$

and

$$\sigma : (h, j, k, l, m, q, r, s) \to (k, j, h, s, r, q, m, l).$$

By applying the change of variables

$$X = 1 - x, \quad Y = 1 - y, \quad Z = \frac{1-z}{1-yz},$$

the integral (20) is changed into

$$I(h, j, k, l, m, q, r, s) = \int_0^1\int_0^1\int_0^1 \frac{x^h(1-x)^l y^k(1-y)^s z^j(1-z)^q}{(1-(1-xy)z)^{q+h-r+1}}\, dx\, dy\, dz. \quad (21)$$

In [8] the transformation θ arises from a highly nontrivial change of variables in (21). The group generated by θ and σ is isomorphic to $\mathfrak{D}_8$, and every transformation in this group leaves the value of $I(h, j, k, l, m, q, r, s)$ unchanged. Since $\mathfrak{D}_8$ is transitive on the eight integers

$$h + l - j, j + m - k, k + q - l, l + r - m, m + s - q, q + h - r, r + j - s, s + k - h,$$

up to applying a power of θ we may assume that (19) holds, whence we have (18).

Now let $h = 16n,\ j = 17n,\ k = 19n,\ l = 15n,\ m = 12n,\ q = 11n,\ r = 9n,$ $s = 13n$ where $n \geqslant 0$ is an integer, and let

$$I(16n, 17n, 19n, 15n, 12n, 11n, 9n, 13n) = A_n - 2B_n\zeta(3).$$

By [8], section 4, p.284–287 and section 5 there exists a sequence of integers Δ_n such that Δ_n divides B_n and $d_{19n}d_{18n}d_{17n}A_n$, and

$$\lim_{n\to\infty} \frac{1}{n} \log \Delta_n = 18.04470204... + 6.14298325... = 24.1876853.... $$

By standard computations using (20)

$$\lim_{n\to\infty} \frac{1}{n} \log I(16n, 17n, 19n, 15n, 12n, 11n, 9n, 13n) = -47.15472079.... $$

By a straightforward application of Cauchy's integral formula in (13), with the polynomials $F(x)$ and $G(y)$ given by

$$F(x) = x^{\alpha}(1 - x)^{\delta} D_{q+h-r}(x^{l}(1 - x)^{h})$$

and

$$G(y) = y^{\gamma}(1 - y)^{\beta} D_{j+m-k}(y^{s}(1 - y)^{m})$$

respectively, we also have the following explicit expression for the coefficient of $\zeta(3)$ in (18):

$$B = -\frac{1}{(2\pi\sqrt{-1})^3} \oint \oint \oint \frac{x^{l}(1 - x)^{h} y^{s}(1 - y)^{m} z^{q}(1 - z)^{j}}{(1 - x(1 - z))^{q+h-r+1}(1 - yz)^{j+m-k+1}}\, dx\, dy\, dz.$$

Here the integration contours are the same as in (13). It easily follows that

$$\lim_{n\to\infty} \frac{1}{n} \log |B_n| \leqslant 48.46940964.... $$

By a standard lemma one obtains $\mu(\zeta(3)) < 5.513890...$.

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