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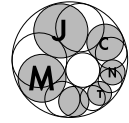
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# Simultaneous approximation to three numbers

Wolfgang M. Schmidt (Boulder),  
Leonhard Summerer (Wien)

## Abstract:

This paper deals with simultaneous approximation to three numbers  $\xi_1, \xi_2, \xi_3$  which together with 1 are linearly independent over  $\mathbb{Q}$ . The main goal is to establish lower bounds for the quotients  $\omega/\widehat{\omega}$  (resp.  $\omega^*/\widehat{\omega}^*$ ) in terms of  $\widehat{\omega}$  (resp.  $\widehat{\omega}^*$ ). The achieved estimates turn out to be best possible. For  $\omega/\widehat{\omega}$  the result was already proved by N. Moshchevitin in [3] using different techniques. The methods in this paper allow to treat the dual problem as well and lead to a conjecture for the simultaneous approximation of  $n$  numbers.

**Keywords:** Simultaneous approximation, approximation constants, successive minima, minimal points

**AMS Subject classification:** 11H06, 11J13

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## 1. Introduction

Let  $\xi_1, \xi_2, \xi_3$  be real numbers with  $1, \xi_1, \xi_2, \xi_3$  linearly independent over  $\mathbb{Q}$ . As usual, the quantities  $\omega$  (resp.  $\widehat{\omega}$ ) are defined as the supremum of the numbers  $\eta$  such that there are arbitrarily large values (resp. such that for every large value) of  $X$  for which the system of inequalities

$$|y| \leq X$$

$$|\xi_i y - y_i| \leq X^{-\eta} \text{ for } i = 1, 2, 3$$

has a non-trivial solution  $\mathbf{y} = (y, y_1, y_2, y_3) \in \mathbb{Z}^4$ .

On the other hand  $\omega^*$  (resp.  $\widehat{\omega}^*$ ) is the supremum of the numbers  $\eta$  such that there are arbitrarily large values (resp. such that for every large value) of  $X$  for which the system

$$|y + \xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3| \leq X^{-\eta}$$

$$|y_i| \leq X \text{ for } i = 1, 2, 3$$

has a non-trivial solution in  $\mathbb{Z}^4$ . We have the following estimates:

$$1/3 \leq \widehat{\omega} \leq \omega, \widehat{\omega} \leq 1, \text{ and } 3 \leq \widehat{\omega}^* \leq \omega^*,$$

where the lower bounds for  $\widehat{\omega}, \widehat{\omega}^*$  follow from Dirichlet's Theorem.

The aim of this paper is to establish lower bounds for the quotient  $\omega/\widehat{\omega}$  (resp.  $\omega^*/\widehat{\omega}^*$ ) in terms of  $\widehat{\omega}$  (resp.  $\widehat{\omega}^*$ ). We will prove

THEOREM 1. *When  $\widehat{\omega} < 1$ , then*

$$\frac{\omega}{\widehat{\omega}} \geq \frac{\widehat{\omega} + \sqrt{4\widehat{\omega} - 3\widehat{\omega}^2}}{2(1 - \widehat{\omega})}, \quad (1.1)$$

and if  $\widehat{\omega} = 1$ , then  $\omega = \infty$ .

*On the other hand, for the dual problem we have*

$$\frac{\omega^*}{\widehat{\omega}^*} \geq \frac{\sqrt{4\widehat{\omega}^* - 3} - 1}{2}. \quad (1.2)$$

The relation (1.1) was first established by Moshchevitin [3]. The relation (1.2) is dual to (1.1) in the sense that it can be obtained from (1.1) by changing  $\omega, \omega^*, \widehat{\omega}, \widehat{\omega}^*$  into  $1/\omega^*, 1/\omega, 1/\widehat{\omega}^*, 1/\widehat{\omega}$  respectively and reversing the inequality. Note that also Khintchine's inequalities get interchanged by this procedure: in the present case of three numbers  $\xi_1, \xi_2, \xi_3$ , these inequalities are

$$\omega \geq \frac{\omega^*}{2\omega^* + 3} \text{ and } \omega^* \geq 3\omega + 2.$$

By recent work of Damien Roy, the inequalities of the Theorem are best possible. See our remarks at the end of Section 3.

We will employ concepts and notations of our earlier papers [5], [6]. But only the first two sections of [6] will be needed. The quantities  $\omega, \omega^*, \widehat{\omega}, \widehat{\omega}^*$  are defined in terms of four integers  $y, y_1, y_2, y_3$ , and hence the case  $n = 4$  of [5] comes into play. Quantities  $\underline{\varphi}_i, \overline{\varphi}_i, \underline{\varphi}_i^*, \overline{\varphi}_i^*$ , ( $1 \leq i \leq 4$ ) were introduced in [5], and will again be defined in Section 3. As pointed out in [5],

$$(1 + \omega)(3 + \underline{\varphi}_1) = (1 + \widehat{\omega})(3 + \overline{\varphi}_1) = 4, \quad (1.3)$$

$$(1 + \omega^*)(1 - \overline{\varphi}_4) = (1 + \widehat{\omega}^*)(1 - \underline{\varphi}_4) = 4. \quad (1.4)$$

Moreover,

$$\underline{\varphi}_4 = -\overline{\varphi}_1^* \text{ and } \overline{\varphi}_4 = -\underline{\varphi}_1^*, \quad (1.5)$$

so that (1.4) may be rephrased as

$$(1 + \omega^*)(1 + \underline{\varphi}_1^*) = (1 + \widehat{\omega}^*)(1 + \overline{\varphi}_1^*) = 4. \quad (1.6)$$

The estimates

$$-3 \leq \underline{\varphi}_1 \leq \overline{\varphi}_1 \leq 0, \quad -1 \leq \overline{\varphi}_1, \text{ and } -1 \leq \underline{\varphi}_1^* \leq \overline{\varphi}_1^* \leq 0,$$

with the additional property that  $\underline{\varphi}_1 = -3$  if and only if  $\overline{\varphi}_1 = -1$  resp.  $\underline{\varphi}_1^* = -1$  if and only if  $\overline{\varphi}_1^* = -1$  can be found in [5] as well.

We will introduce a parameter  $\rho \geq 0$  and define two functions

$$F(\rho) := -\frac{3\rho^3 - \rho^2 - \rho - 1}{\rho^3 + \rho^2 + \rho + 1}, \quad (1.7)$$

as well as

$$G(\rho) := -\frac{2\rho^2 - \rho - 1}{2\rho^2 + \rho + 1}. \quad (1.8)$$

Both are decreasing, have  $F(0) = G(0) = 1$  and  $F(1) = G(1) = 0$ . As  $\rho$  tends to infinity,  $F$  tends to  $-3$  and  $G$  tends to  $-1$ . Hence their inverse functions  $\overline{F}$  resp.  $\overline{G}$  are defined on  $(-3, 1]$  resp.  $(-1, 1]$ .

Also set

$$F^*(\rho) := -F(1/\rho) = -\frac{\rho^3 + \rho^2 + \rho - 3}{\rho^3 + \rho^2 + \rho + 1}, \quad (1.9)$$

as well as

$$G^*(\rho) := -G(1/\rho) := -\frac{\rho^2 + \rho - 2}{\rho^2 + \rho + 2}. \quad (1.10)$$

THEOREM 2. Suppose  $\underline{\varphi}_1 > -3$ , so that  $\underline{\varphi}_1 = F(\rho)$  for some  $\rho \geq 1$ . Then  $\overline{\varphi}_1 \geq G(\rho)$ . In short,

$$\underline{\varphi}_1 = F(\rho) \Rightarrow \overline{\varphi}_1 \geq G(\rho). \quad (1.11)$$

Thus if we define  $H(u) = G(\overline{F}(u))$ , then  $\overline{\varphi}_1 \geq H(\underline{\varphi}_1)$ .

Suppose  $-1 < \underline{\varphi}_1^*$ , so that  $\underline{\varphi}_1^* = F^*(\rho)$  for some  $\rho \geq 1$ . Then  $\overline{\varphi}_1^* \geq G^*(\rho)$ .

Note that the first assertion could be reformulated to say that when  $\overline{\varphi}_1 = G(\rho)$ , then  $\underline{\varphi}_1 \leq F(\rho)$ . The second assertion is equivalent to the statement that when  $\overline{\varphi}_1^* = G^*(\rho)$ , then  $\underline{\varphi}_1^* \leq F^*(\rho)$ . The assertions are dual in some sense: it is easily seen that, since  $\overline{\varphi}_4 \geq 0$ , we have  $\overline{\varphi}_4 = F(\sigma)$  for some  $\sigma \leq 1$ , so in view of (1.5) we have  $\underline{\varphi}_1^* = -F(\sigma) = F^*(1/\sigma) = F^*(\rho)$  with  $\rho = 1/\sigma$ . The second assertion of the Theorem yields  $\overline{\varphi}_1^* \geq G^*(\rho) = -G(1/\rho) = -G(\sigma)$  and therefore  $\underline{\varphi}_4 = -\overline{\varphi}_1^* \leq G(\sigma)$ . In short,

$$\overline{\varphi}_4 = F(\sigma) \implies \underline{\varphi}_4 \leq G(\sigma). \quad (1.12)$$

This fits into the general duality philosophy that many relations remain true if  $\underline{\varphi}_i, \overline{\varphi}_i$  are replaced by  $\overline{\varphi}_{5-i}, \underline{\varphi}_{5-i}$  and inequalities are reversed.

Theorem 2 also is best possible.

We are grateful to the referee for his valuable comments.

## 2. Deduction of Theorem 1 from Theorem 2

Before we proceed to the proof of Theorem 2, we will explain how Theorem 1 can be deduced from it. The relations (1.3) yield

$$\omega = \frac{1 - \underline{\varphi}_1}{3 + \underline{\varphi}_1} \text{ and } \widehat{\omega} = \frac{1 - \overline{\varphi}_1}{3 + \overline{\varphi}_1}, \quad (2.1)$$

and hence

$$\frac{\omega}{\widehat{\omega}} = \frac{1 - \underline{\varphi}_1}{3 + \underline{\varphi}_1} \frac{3 + \overline{\varphi}_1}{1 - \overline{\varphi}_1}. \quad (2.2)$$

From (1.6) it follows that

$$\omega^* = \frac{3 - \underline{\varphi}_1^*}{1 + \underline{\varphi}_1^*} \text{ and } \widehat{\omega}^* = \frac{3 - \overline{\varphi}_1^*}{1 + \overline{\varphi}_1^*}, \quad (2.3)$$

and hence

$$\frac{\omega^*}{\widehat{\omega}^*} = \frac{3 - \underline{\varphi}_1^*}{1 + \underline{\varphi}_1^*} \frac{1 + \overline{\varphi}_1^*}{3 - \overline{\varphi}_1^*}. \quad (2.4)$$

On the other hand it follows easily from our definitions (1.7), (1.8) that

$$\frac{1 - F(\rho)}{3 + F(\rho)} = \frac{\rho^3}{\rho^2 + \rho + 1}, \quad \frac{1 - G(\rho)}{3 + G(\rho)} = \frac{\rho^2}{\rho^2 + \rho + 1}, \quad (2.5)$$

so that

$$\frac{1 - F(\rho)}{3 + F(\rho)} \frac{3 + G(\rho)}{1 - G(\rho)} = \rho, \quad \frac{1 - G(\rho)}{1 + G(\rho)} = \frac{2\rho^2}{\rho + 1}. \quad (2.6)$$

Moreover,

$$\frac{1 + F^*(\rho)}{3 - F^*(\rho)} = \frac{1}{\rho^3 + \rho^2 + \rho}, \quad \frac{1 + G^*(\rho)}{3 - G^*(\rho)} = \frac{1}{\rho^2 + \rho + 1}, \quad (2.7)$$

implies that

$$\frac{3 - F^*(\rho)}{1 + F^*(\rho)} \frac{1 + G^*(\rho)}{3 - G^*(\rho)} = \rho, \quad \frac{1 + G^*(\rho)}{1 - G^*(\rho)} = \frac{2}{\rho^2 + \rho}. \quad (2.8)$$

Therefore when  $\overline{\varphi}_1 = G(\rho)$ , hence  $\underline{\varphi}_1 \leq F(\rho)$ , the equations (2.2), (2.6) yield

$$\frac{\omega}{\widehat{\omega}} \geq \rho.$$

By (2.1), (2.5),

$$\widehat{\omega} = \frac{1 - G(\rho)}{3 + G(\rho)} = \frac{\rho^2}{\rho^2 + \rho + 1},$$

so that

$$\frac{\omega}{\widehat{\omega}} \geq \rho = \frac{\widehat{\omega} + \sqrt{4\widehat{\omega} - 3\widehat{\omega}^2}}{2(1 - \widehat{\omega})},$$

and (1.1) follows. The implication  $\widehat{\omega} = 1 \Rightarrow \omega = \infty$  is classical; see, e. g., [1], Corollary 2.

When  $\overline{\varphi}_1^* = G^*(\rho)$ , hence  $\underline{\varphi}_1^* \leq F^*(\rho)$ , then by (2.4), (2.8),  $\omega^*/\widehat{\omega}^* \geq \rho$ . By (2.3), (2.7),  $\widehat{\omega}^* = \rho^2 + \rho + 1$ , so that

$$\frac{\omega^*}{\widehat{\omega}^*} \geq \rho = \frac{\sqrt{4\widehat{\omega}^* - 3} - 1}{2},$$

and (1.2), hence Theorem 1 is established.

### 3. Some geometry of numbers

We will work in  $\mathbb{R}^4$ , so that in the context of our earlier work [5],[6] we have  $n = 4$  and the lattice  $\Lambda = \Lambda(\xi_1, \xi_2, \xi_3)$  will be the lattice of points  $(y, \xi_1 y - y_1, \xi_2 y - y_2, \xi_3 y - y_3)$  where  $(y, y_1, y_2, y_3)$  runs through  $\mathbb{Z}^4$ . Let  $\mathcal{B}(q) \subseteq \mathbb{R}^4$  be the box of points  $\mathbf{x} = (x_1, x_2, x_3, x_4)$  with

$$|x_1| \leq e^{3q}, \quad |x_i| \leq e^{-q} \text{ for } i = 2, 3, 4.$$

Thus if  $\mathbf{a}_1 = \mathbf{a}_1(q) = e^{-3q}\mathbf{e}_1$  and  $\mathbf{a}_i = \mathbf{a}_i(q) = e^q\mathbf{e}_i$  for  $i = 2, 3, 4$ , where  $\mathbf{e}_1, \dots, \mathbf{e}_4$  are the basic unit vectors in  $\mathbb{R}^4$ , then  $\mathcal{B}(q)$  consists of points  $\mathbf{x}$  whose inner products  $\mathbf{a}_i\mathbf{x}$  have  $|\mathbf{a}_i\mathbf{x}| \leq 1$  for  $i = 1, 2, 3, 4$ .

For a nonzero lattice point  $\mathbf{x} \in \Lambda$ , set

$$\lambda_{\mathbf{x}}(q) := \max_{1 \leq i \leq 4} |\mathbf{a}_i(q)\mathbf{x}|. \quad (3.1)$$

Further let  $\lambda_1(q), \dots, \lambda_4(q)$  be the successive minima of  $\mathcal{B}(q)$  with respect to  $\Lambda$ . Then

$$\lambda_1(q) = \min_{\mathbf{x} \in \Lambda \setminus \{\mathbf{0}\}} \lambda_{\mathbf{x}}(q).$$

Setting  $L_i(q) = \log \lambda_i(q)$ ,  $L_{\mathbf{x}}(q) = \log \lambda_{\mathbf{x}}(q)$  we obtain

$$L_1(q) = \min_{\mathbf{x} \in \Lambda \setminus \{\mathbf{0}\}} L_{\mathbf{x}}(q). \quad (3.2)$$

As in [5] we define for  $1 \leq i \leq 4$ ,

$$\underline{\varphi}_i := \liminf L_i(q)/q, \quad \overline{\varphi}_i := \limsup L_i(q)/q. \quad (3.3)$$

Observe that

$$L_{\mathbf{x}}(q) = \max(\log |x_1| - 3q, \max_{1 < i \leq 4} \log |x_i| + q),$$

so that each function  $L_{\mathbf{x}}$  decreases with slope  $-3$  up to some  $q_{\mathbf{x}}$ , and then increases with slope  $1$ . Therefore there will be numbers  $q_1 < p_1 < q_2 < p_2 \dots$ , where  $q_1 \geq 0$ , such that there are "minimal points"  $\mathbf{x}_t \in \Lambda \setminus \{\mathbf{0}\}$  having

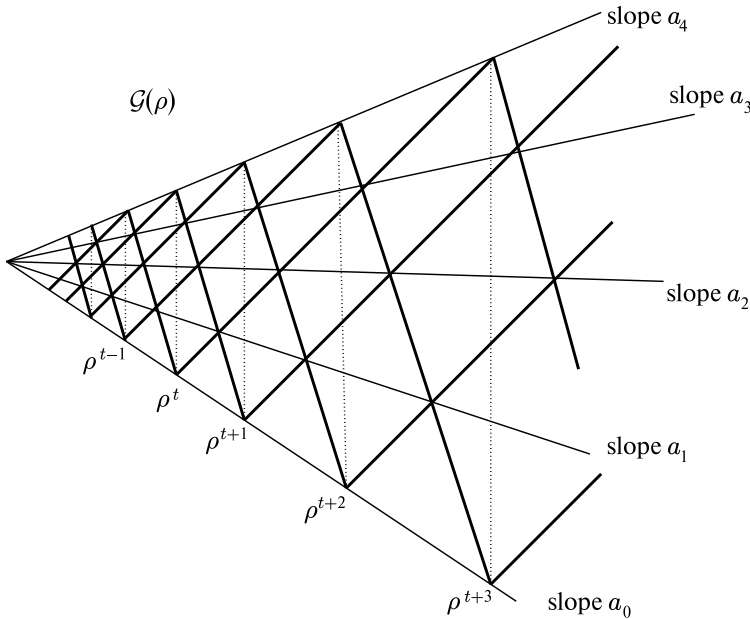
$$L_1(q) = L_{\mathbf{x}_t}(q) \text{ for } p_t \leq q \leq p_{t+1}, \quad (3.4)$$

and  $L_1$  has a local minimum at  $q_1, q_2, \dots$ , a local maximum at  $p_1, p_2, \dots$ . The sequence  $\mathbf{x}_1, \mathbf{x}_2, \dots$  of minimal points is infinite, for since  $1, \xi_1, \xi_2, \xi_3$  are linearly independent over  $\mathbb{Q}$ , no point  $\mathbf{x} \in \Lambda \setminus \{\mathbf{0}\}$  can be in  $\mathcal{B}(q)$  for arbitrarily large  $q$ . By (3.4) and since two successive minimal points are linearly independent,

$$L_1(p_t) = L_{\mathbf{x}_t}(p_t) = L_{\mathbf{x}_{t+1}}(p_t) = L_2(p_t). \tag{3.5}$$

The union of the graphs of  $L_1, L_2, L_3, L_4$  is called the *combined graph*. Note that by Minkowski's Theorem,  $|L_1(q) + \dots + L_4(q)|$  is bounded.

We now interrupt our main argument to present the *regular graph*  $\mathcal{G}(\rho)$ , defined for  $\rho > 1$ .  $\mathcal{G}(\rho)$  is contained in the set of points  $(q, L) \in \mathbb{R}^2$  with  $q \geq 0$ ,  $a_0q \leq L \leq a_4q$ , where  $a_0 = a_0(\rho), \dots, a_4 = a_4(\rho)$  will be defined below.



$\mathcal{G}(\rho)$  is the union of line segments of slope  $-3$  or  $1$  with end points on the lines  $L = a_0q$  and  $L = a_4q$ . In fact we have segments  $S_t(-3)$  of slope  $-3$  with end points  $(\rho^t, a_4\rho^t)$  and  $(\rho^{t+1}, a_0\rho^{t+1})$ , and segments  $S_t(1)$  of slope  $1$  and end points  $(\rho^t, a_0\rho^t), (\rho^{t+3}, a_4\rho^{t+3})$ . Clearly  $\mathcal{G}(\rho)$  is invariant under dilation by the factor  $\rho$ .

Considering the segment  $S_t(-3)$  we see that

$$a_0\rho^{t+1} = a_4\rho^t - 3(\rho^{t+1} - \rho^t),$$

and looking at the segment  $S_t(1)$  we obtain

$$a_4\rho^{t+3} = a_0\rho^t + (\rho^{t+3} - \rho^t).$$

After division by  $\rho^t$  we obtain

$$a_0\rho - a_4 = 3 - 3\rho \text{ and } a_0 - a_4\rho^3 = 1 - \rho^3,$$

and in view of (1.7)

$$a_0 = -\frac{3\rho^3 - \rho^2 - \rho - 1}{\rho^3 + \rho^2 + \rho + 1} = F(\rho), \quad a_4 = \frac{\rho^3 + \rho^2 + \rho - 3}{\rho^3 + \rho^2 + \rho + 1} = F(1/\rho).$$

For  $0 < i < 4$ , the segments  $S_t(1)$  and  $S_{t+i-1}(-3)$  intersect in a point  $(p_{it}, p_{it}a_i)$ , where  $a_i$  is independent of  $t$  by the invariance of  $\mathcal{G}(\rho)$  under dilation by  $\rho$ . We have

$$a_i p_{it} = a_0 \rho^t + p_{it} - \rho^t = a_0 \rho^{t+i} + 3(\rho^{t+i} - p_{it}).$$

Therefore  $(a_0 - 1)\rho^t + 4p_{it} = (a_0 + 3)\rho^{t+i}$  which we rewrite as

$$4p_{it} = ((a_0 + 3)\rho^i - (a_0 - 1))\rho^t.$$

On the other hand

$$4a_i p_{it} = 4(a_0 - 1)\rho^t + 4p_{it} = ((a_0 + 3)\rho^i + 3(a_0 - 1))\rho^t.$$

As a consequence, as  $a_0 = F(\rho)$ , (2.5) implies

$$a_i = \frac{(a_0 + 3)\rho^i + 3(a_0 - 1)}{(a_0 + 3)\rho^i - (a_0 - 1)} = \frac{\rho^i + 3(a_0 - 1)/(a_0 + 3)}{\rho^i - (a_0 - 1)/(a_0 + 3)} = \frac{\rho^i - 3\rho^3/(\rho^2 + \rho + 1)}{\rho^i + \rho^3/(\rho^2 + \rho + 1)}.$$

From this we may conclude that

$$a_i = \frac{\rho^2 + \rho + 1 - 3\rho^{3-i}}{\rho^2 + \rho + 1 + \rho^{3-i}}, \quad (3.6)$$

and in fact, this equality holds for  $0 \leq i \leq 4$ . Observe further that

$$a_i(\rho) = a_{4-i}(1/\rho) \text{ and } a_1 = -\frac{2\rho^2 - \rho - 1}{2\rho^2 + \rho + 1} = G(\rho).$$

The regular graph has all the properties of a combined graph of functions  $L_i(q)$  ( $1 \leq i \leq 4$ ).

The dual graph  $\mathcal{G}^*(\rho)$  is obtained from  $\mathcal{G}(\rho)$  by the map  $(q, L) \mapsto (q, -L)$ , i. e. reflection on the  $q$ -axis. It is the union of line segments of slope 3 and  $-1$ . These line segments intersect at lines  $L = a_i^*q$ , ( $0 \leq i \leq 4$ ) where  $a_i^* = -a_{5-i} = -a_i(1/\rho)$ . In particular,  $a_0^* = -F(1/\rho) = F^*(\rho)$  and  $a_1^* = -G(1/\rho) = G^*(\rho)$ .

Recently, in February 2013, Damien Roy showed us, for given  $\rho > 1$ , a construction of numbers  $\xi_1, \xi_2, \xi_3$  whose combined graph differs from the graph  $\mathcal{G}(\rho)$  by a bounded amount. More precisely, if  $L_i(q)$  are the functions defined in terms of  $\xi_1, \xi_2, \xi_3$ , and  $\widehat{L}_i(q)$  the functions implied by  $\mathcal{G}(\rho)$ , then  $|L_i(q) - \widehat{L}_i(q)|$  is bounded for  $1 \leq i \leq 4$ . For this graph,  $\underline{\varphi}_1 = a_0(\rho) = F(\rho)$ ,  $\overline{\varphi}_1 = \underline{\varphi}_2 = a_1(\rho) = G(\rho)$ , so that the first assertion of Theorem 2, hence also of Theorem 1, holds with equality. A dual argument applies to the second assertion.

A regular graph can be defined for any  $n > 1$ . Already at a conference in 2011, Damien Roy showed us a regular graph for  $n = 3$ , stemming from simultaneous approximation to two explicit numbers  $\xi_1, \xi_2$  related to [4]. His recent construction can be done for any  $n > 1$ . There is a conjectured version of Theorem 2 for any such  $n$ , which holds with equality for the regular graph.

## 4. Geometry of numbers, continued

We now continue the arguments of Section 3, interrupted by the introduction of the regular graph.

As in [6], let  $\mathbf{A}_1, \dots, \mathbf{A}_6$  in  $\mathbb{R}^6$  be the exterior products  $\mathbf{a}_i \wedge \mathbf{a}_j$ , ( $1 \leq i < j \leq 4$ ). After suitable ordering and up to sign,  $\mathbf{A}_k = e^{2q}\mathbf{E}_k$  for  $k = 1, 2, 3$  and  $\mathbf{A}_k = e^{-2q}\mathbf{E}_k$  for  $k = 4, 5, 6$ , where  $\mathbf{E}_1, \dots, \mathbf{E}_6$  are the basic unit vectors in  $\mathbb{R}^6$ . The second pseudocompound  $\mathcal{B}^{(2)}(q)$  of  $\mathcal{B}(q)$  consists of the points  $\mathbf{X} \in \mathbb{R}^6$  whose inner products  $\mathbf{A}_k\mathbf{X}$  have  $|\mathbf{A}_k\mathbf{X}| \leq 1$  for  $k = 1, \dots, 6$ .

For nonzero  $\mathbf{X} \in \Lambda^{(2)}$ , which is the lattice generated by the products  $\mathbf{x}_i \wedge \mathbf{x}_j$  with  $\mathbf{x}_i, \mathbf{x}_j \in \Lambda$ , set

$$\mu_{\mathbf{X}}(q) = \max_{1 \leq k \leq 6} |\mathbf{A}_k(q)\mathbf{X}|. \quad (4.1)$$

When  $\mu_1(q), \dots, \mu_6(q)$  are the successive minima of  $\mathcal{B}^{(2)}(q)$  with respect to  $\Lambda^{(2)}$ , then

$$\mu_1(q) = \min_{\mathbf{X} \in \Lambda^{(2)} \setminus \{0\}} \mu_{\mathbf{X}}(q).$$

Setting  $M_k(q) := \log \mu_k(q)$  for  $k = 1, \dots, 6$  and  $M_X(q) := \log \mu_X(q)$ , we obtain

$$M_1(q) = \min_{\mathbf{x} \in \Lambda^{(2)} \setminus \{\mathbf{0}\}} M_X(q). \quad (4.2)$$

We have

$$M_X(q) = \max(\max_{1 \leq k \leq 3} (\log |X_k| - 2q), \max_{4 \leq k \leq 6} (\log |X_k| + 2q)),$$

and therefore  $M_X(q)$  decreases with slope  $-2$  up to some  $q_X$ , then increases with slope  $2$ .

Laplace's equation gives for  $1 \leq i < j \leq 4$ :

$$(\mathbf{a}_i \wedge \mathbf{a}_j) \cdot (\mathbf{x} \wedge \mathbf{x}') = \begin{vmatrix} \mathbf{a}_i \mathbf{x} & \mathbf{a}_i \mathbf{x}' \\ \mathbf{a}_j \mathbf{x} & \mathbf{a}_j \mathbf{x}' \end{vmatrix}.$$

Therefore by (3.1), (4.1), independent points  $\mathbf{x}, \mathbf{x}'$  in  $\Lambda \setminus \{\mathbf{0}\}$  have

$$\mu_Y(q) \leq 2\lambda_x(q)\lambda_{x'}(q)$$

where  $\mathbf{Y} := \mathbf{x} \wedge \mathbf{x}'$ .

Let  $S$  be a two-dimensional space spanned by points  $\mathbf{x}, \mathbf{x}'$  in  $\Lambda$ . Let  $\mathbf{X} \in \mathbb{R}^6$  be a primitive Grassmann coordinate vector of  $S$ , i. e. a Grassmann vector  $\mathbf{X} \in \Lambda^{(2)} \setminus \{\mathbf{0}\}$  with no point of  $\Lambda^{(2)}$  between  $\mathbf{0}$  and  $\mathbf{X}$ . Then  $\mathbf{Y} = \mathbf{x} \wedge \mathbf{x}'$  is an integer multiple of  $\mathbf{X}$  and

$$\mu_X(q) \leq \mu_Y(q) \leq 2\lambda_x(q)\lambda_{x'}(q).$$

Taking logarithms we obtain

$$M_X(q) \leq L_x(q) + L_{x'}(q) + \log 2 = L_x(q) + L_{x'}(q) + O(1). \quad (4.3)$$

If moreover  $\lambda_x(q) = \lambda_1(q)$  and  $\lambda_{x'}(q) = \lambda_2(q)$ , then

$$\mu_1(q) \leq \mu_X(q) \leq 2\lambda_1(q)\lambda_2(q).$$

But according to Mahler, see [2],  $\mu_1(q) \geq c_1\lambda_1(q)\lambda_2(q)$  with an absolute constant  $c_1 > 0$ . Hence, taking logarithms again, we have

$$M_X(q) - L_1(q) - L_2(q) = O(1).$$

From (3.5) we see that, when  $\mathbf{x}_t, \mathbf{x}_{t+1}$  are in  $S$ , then

$$M_X(p_t) - 2L_1(p_t) = O(1). \quad (4.4)$$

## 5. The sequence of minimal points

Suppose  $r < s$  and the minimal points

$$\mathbf{x}_{r-1}, \mathbf{x}_r, \dots, \mathbf{x}_s, \mathbf{x}_{s+1}$$

span  $\mathbb{R}^4$ , whereas  $\mathbf{x}_r, \dots, \mathbf{x}_s$  span a two-dimensional space  $S$ . Moshchevitin has observed in [3] that these conditions, to which we will refer later as  $(\star)$ , are fulfilled for certain arbitrarily large  $r, s$ . Then (4.4) can be applied for every  $t$  with  $r \leq t < s$ , and therefore

$$L_1(p_t) - L_1(p_r) = (1/2)(M_X(p_t) - M_X(p_r)) + O(1) \text{ for } r \leq t < s. \quad (5.1)$$

Recall that  $M_X(q)$  has slope  $-2$  up to  $q_X$ , then slope  $2$ . Set  $\tilde{q} = \max(p_r, q_X)$ , so that  $p_r \leq \tilde{q}$ , and let  $l \in \{r, r+1, \dots, s-1\}$  be largest with  $p_l \leq \tilde{q}$ . Observe that when  $s = r+1$ , so that by our hypothesis  $\mathbf{x}_{r-1}, \mathbf{x}_r, \mathbf{x}_s, \mathbf{x}_{s+1}$  are linearly independent, then  $l = r$ .

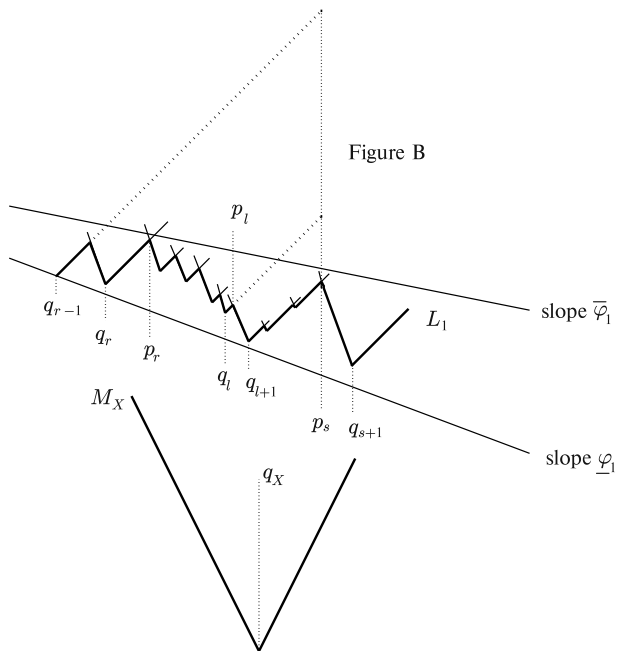
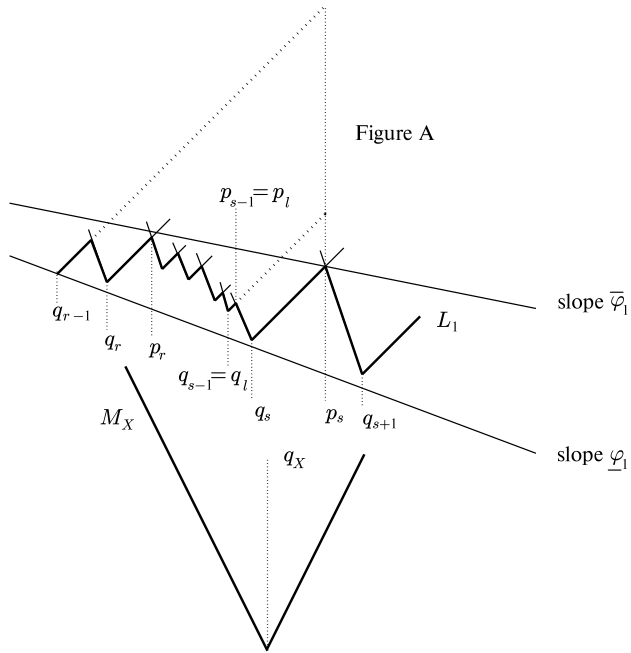
If  $l > r$ , then  $M_X(q)$  has slope  $-2$  in the interval  $p_r \leq q \leq p_l$ , so that by (5.1),

$$L_1(p_l) - L_1(p_r) = -(p_l - p_r) + O(1). \quad (5.2)$$

We will distinguish two cases  $A$  and  $B$ . Case  $A$  is when  $l = s-1$ , and is illustrated in the following Figure A.

The graph of  $L_1$  is indicated by the thick zig-zag line. Just little pieces of the graph of  $L_2$ , as it emanates from points  $(p_t, L_1(p_t))$ , are shown. The lines of slope 1 indicated by dots belong to the graph of  $L_{x_{r-1}}$  resp.  $L_{x_{s-1}}$ . The bottom of the figure shows part of the graph of  $M_X$ .

In case  $B$ ,  $l < s-1$ , so that  $s > r+1$ . It is illustrated in Figure B below. Up to the point  $(q_l, L_1(q_l))$  the picture looks as in case A. The difference then results from the fact that in case B not all maxima of  $L_1$  at numbers smaller than  $p_s$  are attained at values of  $q$  for which  $M_X$  is still decreasing.



The point  $\mathbf{x}_{l+1}$ , and also  $\mathbf{x}_{l+2}, \dots, \mathbf{x}_s$  lie in  $S$ . By (4.4) with  $t = l + 1$ ,

$$M_{\mathbf{X}}(p_{l+1}) = 2L_1(p_{l+1}) + O(1).$$

Now by our choice of  $l$ ,  $M_{\mathbf{X}}(q)$  has slope 2 for  $q \geq p_{l+1} > q_{\mathbf{X}}$ , so that

$$M_{\mathbf{X}}(p_s) = M_{\mathbf{X}}(p_{l+1}) + 2(p_s - p_{l+1}),$$

and therefore

$$M_{\mathbf{X}}(p_s) = 2L_1(p_{l+1}) + 2(p_s - p_{l+1}) + O(1). \quad (5.3)$$

Since both  $\mathbf{x}_{l+1}$  and  $\mathbf{x}_s$  are in  $S$ , (4.3) gives

$$\begin{aligned} M_{\mathbf{X}}(p_s) &\leq L_{\mathbf{x}_{l+1}}(p_s) + L_{\mathbf{x}_s}(p_s) + O(1) = \\ &= L_{\mathbf{x}_{l+1}}(p_{l+1}) + L_{\mathbf{x}_s}(p_s) + (p_s - p_{l+1}) + O(1) = \\ &= L_1(p_{l+1}) + L_1(p_s) + (p_s - p_{l+1}) + O(1). \end{aligned}$$

In conjunction with (5.3) this yields

$$L_1(p_s) \geq L_1(p_{l+1}) + p_s - p_{l+1} + O(1),$$

and since  $L_1(p_{l+1}) = L_1(q_{l+1}) + p_{l+1} - q_{l+1}$ ,

$$L_1(p_s) \geq L_1(q_{l+1}) + p_s - q_{l+1} + O(1).$$

But on the other hand,

$$L_1(p_s) \leq L_{\mathbf{x}_{l+1}}(p_s) = L_{\mathbf{x}_{l+1}}(q_{l+1}) + p_s - q_{l+1},$$

so that

$$L_1(p_s) = L_1(q_{l+1}) + p_s - q_{l+1} + O(1). \quad (5.4)$$

Observe that in case A we had  $s = l + 1$ , and therefore (5.4) holds in case A as well as in case B.

## 6. Proof of Theorem 1.2, first part

We start with some preliminaries. For convenience we set

$$f := \underline{\varphi}_1, \quad g := \overline{\varphi}_1.$$

We have to show that when  $-3 < f < 0$  and  $f = F(\rho)$ , then  $g \geq G(\rho)$ . Suppose to the contrary that

$$f = F(\rho) \text{ and } g < G(\rho). \quad (6.1)$$

$L_1(q)$  has local minima at numbers  $q_1 < q_2 < \dots$  and local maxima at numbers  $p_1 < p_2 < \dots$ , where  $q_1 < p_1 < q_2 < p_2 < \dots$ . Set

$$f_t := L_1(q_t)/q_t \text{ and } g_t := L_1(p_t)/p_t,$$

so that, as easily seen and shown in [6] (with the notation  $\pi_1(q) := P_1(q)/q$ ),

$$f = \liminf_{t \rightarrow \infty} f_t, \quad g = \limsup_{t \rightarrow \infty} g_t.$$

Suppose a quantity  $\bar{f} \in (-3, f)$  has  $\bar{f} = F(\bar{\rho})$ . For large  $t$  we have  $f_t > \bar{f}$ . Since the function  $H(f) = G(\overline{F}(f))$  is continuous,  $\bar{g} := G(\bar{\rho})$  will be close to  $G(\rho)$  when  $\bar{f}$  is close to  $f$ , so that by (6.1),  $g < G(\bar{\rho})$ . Here  $G(\bar{\rho}) > -1$ , so we may pick  $g'$  with  $\max(-1, g) < g' < G(\bar{\rho})$ , and then  $g_t < g'$  for large  $t$ . Altogether

$$f_t > \bar{f} = F(\bar{\rho}), \quad \max(-1, g_t) < g' < G(\bar{\rho}).$$

Changing our notation by writing  $\rho, f, g$  for  $\bar{\rho}, \bar{f}, \bar{g}$ , we have

$$f = F(\rho), \quad g = G(\rho) \quad (6.2)$$

and

$$f_t > f, \quad \max(-1, g_t) < g' < g \quad (6.3)$$

for large  $t$ . This will lead to the desired contradiction.

From now on, we will always assume that  $t$  is so large that (6.3) holds. Implied constants in  $O(1)$  may depend on  $\rho, f, g, g'$ . We have

$$g_t p_t = L_1(p_t) = L_1(q_t) + p_t - q_t = f_t q_t + p_t - q_t,$$

hence  $(g_t - 1)p_t = (f_t - 1)q_t$ , or

$$\frac{p_t}{q_t} = \frac{1 - f_t}{1 - g_t}. \quad (6.4)$$

Observe that  $g_t < g < 0$ . Also

$$g_t p_t = L_1(p_t) = L_1(q_{t+1}) + 3(q_{t+1} - p_t) = f_{t+1} q_{t+1} + 3(q_{t+1} - p_t),$$

hence  $(g_t + 3)p_t = (f_{t+1} + 3)q_{t+1}$ , or

$$\frac{p_t}{q_{t+1}} = \frac{3 + f_{t+1}}{3 + g_t}. \quad (6.5)$$

The denominator on the right hand side here is positive, since  $f > -3$ , hence  $g_t > -3$  for large  $t$ .

Combining (6.4) with the version of (6.5) with  $t$  replaced by  $t - 1$  gives

$$\frac{p_t}{p_{t-1}} = \frac{1 - f_t}{3 + f_t} \frac{3 + g_{t-1}}{1 - g_t} =: \rho_t, \quad (6.6)$$

say. Since each  $f_t \geq f$ ,  $g_t < g' < g$ , and from (2.6), (6.2), we may conclude that  $\rho_t < \rho$ , so that

$$\frac{p_t}{p_{t-1}} < \rho. \quad (6.7)$$

Now suppose  $\mathbf{x}_{r-1}, \mathbf{x}_r, \dots, \mathbf{x}_s, \mathbf{x}_{s+1}$  are minimal points satisfying  $(\star)$  and  $l$  is defined as in Section 5. For convenience, set

$$\sigma := g_l.$$

By (5.2)

$$\sigma p_l - g_r p_r = L_1(p_l) - L_1(p_r) = p_r - p_l + O(1),$$

hence

$$(1 + \sigma)p_l = (1 + g_r)p_r + O(1) < (1 + g')p_r + O(1),$$

where  $1 + g' > 0$  by (6.3). Therefore

$$p_r > \frac{1 + \sigma}{1 + g'} p_l + O(1).$$

Since  $p_{r-1} > p_r/\rho$  by (6.7), we obtain

$$p_{r-1} > \frac{1+\sigma}{1+g'}\rho^{-1}p_l + O(1). \quad (6.8)$$

Now

$$\begin{aligned} L_{\mathbf{x}_{r-1}}(p_{r-1}) &= g_{r-1}p_{r-1} < g'p_{r-1}, \\ L_{\mathbf{x}_{r-1}}(p_s) &= L_{\mathbf{x}_{r-1}}(p_{r-1}) + p_s - p_{r-1} < p_s - (1-g')p_{r-1}, \end{aligned}$$

so that by (6.8)

$$L_{\mathbf{x}_{r-1}}(p_s) < p_s - \frac{1-g'}{1+g'} \frac{1+\sigma}{\rho} p_l + O(1). \quad (6.9)$$

On the other hand, in view of our definition of  $\sigma$ ,

$$L_{\mathbf{x}_l}(p_s) = L_{\mathbf{x}_l}(p_l) + p_s - p_l = p_s - (1-\sigma)p_l. \quad (6.10)$$

Also note that

$$L_1(p_s) = L_2(p_s) = g_s p_s < g' p_s \quad (6.11)$$

and  $L_1(p_s) = L_2(p_s) = L_{\mathbf{x}_s}(p_s) = L_{\mathbf{x}_{s+1}}(p_s)$ .

The points  $\mathbf{x}_l, \mathbf{x}_s$  span  $S$ , so that  $\mathbf{x}_{r-1}, \mathbf{x}_l, \mathbf{x}_s, \mathbf{x}_{s+1}$  span  $\mathbb{R}^4$ . Therefore

$$\begin{aligned} \Sigma(p_s) &:= L_1(p_s) + L_2(p_s) + L_3(p_s) + L_4(p_s) \leq \\ &\leq L_{\mathbf{x}_s}(p_s) + L_{\mathbf{x}_{s+1}}(p_s) + L_{\mathbf{x}_l}(p_s) + L_{\mathbf{x}_{r-1}}(p_s), \end{aligned}$$

and hence by (6.9), (6.10) and (6.11),

$$\Sigma(p_s) < 2g'p_s + 2p_s - \left(1 - \sigma + \frac{1-g'}{1+g'} \frac{1+\sigma}{\rho}\right) p_l + O(1). \quad (6.12)$$

In view of (5.4),

$$g_s p_s = f_{l+1} q_{l+1} + p_s - q_{l+1} + O(1),$$

hence  $(1-g_s)p_s = (1-f_{l+1})q_{l+1} + O(1)$ , so that

$$p_s = \frac{1-f_{l+1}}{1-g_s} q_{l+1} + O(1) < \frac{1-f}{1-g} q_{l+1} + O(1).$$

Now by (6.5)

$$q_{l+1} = \frac{3 + g_l}{3 + f_{l+1}} p_l < \frac{3 + \sigma}{3 + f} p_l,$$

so that

$$p_s < \frac{1 - f}{3 + f} \frac{3 + \sigma}{1 - g'} p_l + O(1).$$

In conjunction with (6.12) this gives

$$\Sigma(p_s) < \left( 2(1 + g') \frac{1 - f}{3 + f} \frac{3 + \sigma}{1 - g'} + \sigma - 1 - \frac{1 - g'}{1 + g'} \frac{1 + \sigma}{\rho} \right) p_l + O(1). \quad (6.13)$$

Since  $g' < g$ ,

$$\begin{aligned} \Sigma(p_s) &< \left( 2 \frac{1 + g}{1 - g} \frac{1 - f}{3 + f} (3 + \sigma) + \sigma - 1 - \frac{1 - g}{1 + g} \frac{1 + \sigma}{\rho} - c' \right) p_l + O(1) =: \\ &=: (A(\sigma) - c') p_l + O(1), \end{aligned}$$

say, where  $c' = c'(\rho, f, g, g') > 0$ . By (2.6), (6.2)

$$\begin{aligned} A(g) &= 2\rho(1 + g) - (1 - g)(1 + 1/\rho) = \\ &= 2\rho(1 - g) \left( \frac{1 + g}{1 - g} - \frac{\rho + 1}{2\rho^2} \right) = 0. \end{aligned}$$

On the other hand, for the derivative we have

$$A'(\sigma) = 2 \frac{1 + g}{1 - g} \frac{1 - f}{3 + f} + 1 - \frac{1 - g}{1 + g} \rho^{-1}.$$

Using (2.5), (2.6) and (6.2) we compute

$$\begin{aligned} 1 + 2 \frac{1 + g}{1 - g} \frac{1 - f}{3 + f} - \frac{1 - g}{1 + g} \rho^{-1} &= 1 + \frac{\rho + 1}{\rho^2} \frac{\rho^3}{\rho^2 + \rho + 1} - \frac{2\rho}{\rho + 1} = \\ &= 1 + \rho \frac{\rho + 1}{\rho^2 + \rho + 1} - \frac{2\rho}{\rho + 1} = \\ &= 1 - \frac{\rho}{\rho + 1} + \rho \left( \frac{\rho + 1}{\rho^2 + \rho + 1} - \frac{1}{\rho + 1} \right) > 0, \end{aligned}$$

so that  $A'(\sigma) > 0$ . Since  $\sigma \leq g$  and  $A(g) = 0$ , we see that  $A(\sigma) \leq 0$ , and

$$\Sigma(p_s) \leq -c'p_l + O(1) < -(1/2)c'p_l.$$

Since  $\Sigma(p_s) = O(1)$ , and since  $p_l \geq p_r$  will be arbitrarily large for large  $r$ , this yields a contradiction, and the first assertion of Theorem 2 is established.

## 7. The dual case

Instead of the lattice  $\Lambda$  introduced in Section 3, we now deal with the lattice  $\Lambda^*$  of points  $(y + \xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3, y_1, y_2, y_3)$  where  $(y, y_1, y_2, y_3)$  runs through  $\mathbb{Z}^4$ , and the box  $\mathcal{B}^*(q)$  of points  $\mathbf{x} = (x_1, x_2, x_3, x_4)$  with

$$|x_1| \leq e^{-3q}, \quad |x_i| \leq e^q \text{ for } i = 2, 3, 4.$$

In the same way as in Section 3, one can define  $\lambda_{\mathbf{x}}^*(q)$  and the successive minima  $\lambda_i^*(q)$  of  $\mathcal{B}^*(q)$  with respect to  $\Lambda^*$ , and the resulting functions  $L_{\mathbf{x}}^*(q)$  and  $L_j^*(q)$  for  $j = 1, 2, 3, 4$ . Furthermore  $\underline{\varphi}_i^*$  and  $\overline{\varphi}_i^*$  are respectively the lim inf and the lim sup of  $L_i^*(q)/q$ . We set  $f := \underline{\varphi}_1^*$  resp.  $g := \overline{\varphi}_1^*$ . In order to keep the notation simple, we will omit the superscript  $*$  in the remainder of this section.

The functions  $L_j$  will have slopes 3,  $-1$  in alternating intervals.  $L_1$  has local minima at  $q_1, q_2, \dots$  with  $q_t$  associated to the “minimal point”  $\mathbf{x}_t$  in the lattice for which  $L_1(q_t) = L_{\mathbf{x}_t}(q_t)$ .  $L_1$  will have its local maximum in  $(q_{t-1}, q_t)$  at some  $p_t$ , so that  $q_1 < p_2 < q_2 < p_3 < q_3 < \dots$ , which differs from the convention in Section 3 in order to better bring out the duality.

It may happen that for large  $k$ , the points  $x_k, x_{k+1}, \dots$  lie in a proper subspace of  $\mathbb{R}^4$ . In this case we are in effect dealing with a linear form in 3 (instead of 4) variables, and Jarnik’s inequality  $\omega^*/\widehat{\omega}^* \geq \widehat{\omega}^* - 1$  holds, which implies the inequality (1.2) of Theorem 1. We had seen that Theorem 2 implies Theorem 1, but it is easily seen that in fact these two Theorems are equivalent, so that also Theorem 2 holds.

We may therefore suppose that for any  $k$ , the points  $x_k, x_{k+1}, \dots$  span  $\mathbb{R}^4$ . Then there will be arbitrarily large integers  $s < r$  with  $\mathbf{x}_{s-1}, \mathbf{x}_s, \dots, \mathbf{x}_r, \mathbf{x}_{r+1}$  spanning  $\mathbb{R}^4$ , and  $\mathbf{x}_s, \dots, \mathbf{x}_r$  spanning a two-dimensional space  $S$ . Let  $\mathbf{X}$  be a primitive Grassmann coordinate vector of  $S$ . With  $M_{\mathbf{X}}$  defined as in Section 4 and considering that  $q_{t-1} < p_t < q_t$ , we now have (4.4) when both  $\mathbf{x}_{t-1}, \mathbf{x}_t$  are in  $S$ . Therefore (5.1)

holds for  $s < t \leq r$ . We will note this explicitly:

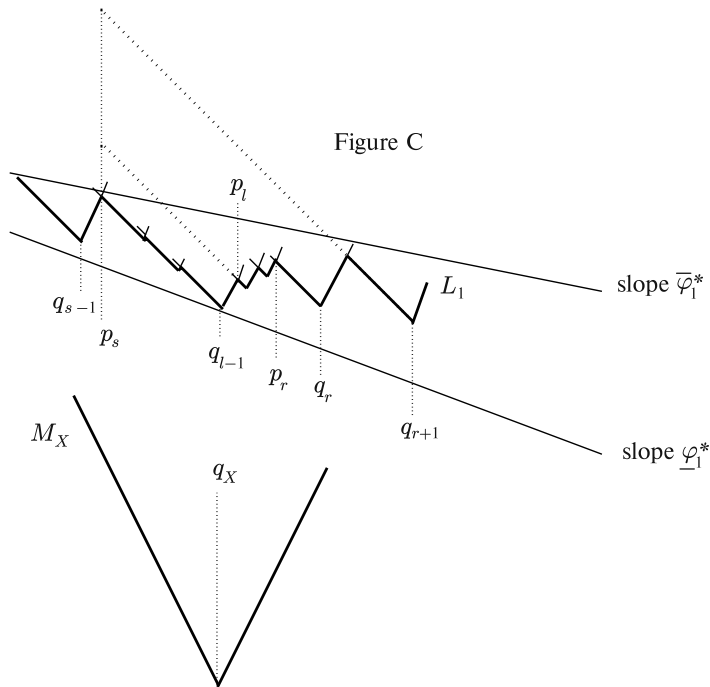
$$L_1(p_t) - L_1(p_r) = (1/2)(M_X(p_t) - M_X(p_r)) + O(1) \text{ for } s < t \leq r. \tag{7.1}$$

In general, the duals of formulas (5.x), (6.x) will be denoted by (7.x), (8.x) respectively.

Again,  $M_X$  has slope  $-2$  up to some  $q_X$ , then slope  $2$ . Set  $\tilde{q} = \min(p_r, q_X)$ , so that  $p_r \geq \tilde{q}$ , and let  $l \in \{s + 1, \dots, r - 1, r\}$  be smallest with  $p_l \geq \tilde{q}$ . In the interval  $p_l \leq q \leq p_r$ ,  $M_X(q)$  has slope  $2$ , so that by (7.1)

$$L_1(p_l) - L_1(p_r) = p_l - p_r + O(1). \tag{7.2}$$

For a sketch see Figure C below, which is essentially dual to Figure B.



We have  $l > s$ . In the case  $l > s + 1$  the points  $\mathbf{x}_s, \dots, \mathbf{x}_{l-2}$  lie in  $S$ . In view of our present notation with  $q_{t-1} < p_t < q_t$ , (4.4) now holds if  $\mathbf{x}_{t-1}, \mathbf{x}_t$  are in  $S$ . Applying this with  $t = l - 1$ , we see that

$$M_X(p_{l-1}) = 2L_1(p_{l-1}) + O(1).$$

Now by our choice of  $l$ ,  $M_X(q)$  has slope  $-2$  for  $q \leq p_{l-1} < q_X$ , so that

$$M_X(p_s) = M_X(p_{l-1}) - 2(p_s - p_{l-1}),$$

and therefore

$$M_X(p_s) = 2L_1(p_{l-1}) - 2(p_s - p_{l-1}) + O(1). \quad (7.3)$$

Since both  $x_{l-1}$  and  $x_s$  lie in  $S$ , (4.3) gives

$$\begin{aligned} M_X(p_s) &\leq L_{x_{l-1}}(p_s) + L_{x_s}(p_s) + O(1) = \\ &= L_{x_{l-1}}(p_{l-1}) + L_{x_s}(p_s) - (p_s - p_{l-1}) + O(1) = \\ &= L_1(p_{l-1}) + L_1(p_s) - (p_s - p_{l-1}) + O(1). \end{aligned}$$

In conjunction with (7.3) this yields

$$L_1(p_s) \geq L_1(p_{l-1}) - (p_s - p_{l-1}) + O(1),$$

and since  $L_1(p_{l-1}) = L_1(q_{l-1}) - (p_{l-1} - q_{l-1})$ ,

$$L_1(p_s) \geq L_1(q_{l-1}) - (p_s - q_{l-1}) + O(1).$$

But on the other hand,

$$L_1(p_s) \leq L_{x_{l-1}}(p_s) = L_{x_{l-1}}(q_{l-1}) - (p_s - q_{l-1}),$$

so that

$$L_1(p_s) = L_1(q_{l-1}) + q_{l-1} - p_s + O(1). \quad (7.4)$$

When  $l = s + 1$  this holds trivially.

As compared with Section 4, our picture is essentially reversed, with left-right reversed. Of course, this is only a local picture, for  $q_{r-1} \leq q \leq q_{s+1}$  in Section 5 and  $q_{s-1} \leq q \leq q_{r+1}$  here. The lines of slope  $f = \underline{\varphi}_1^*$  and  $g = \overline{\varphi}_1^*$  are not reversed.

## 8. Proof of Theorem 2, second part

We set again

$$f_t := L_1(q_t)/q_t \text{ and } g_t := L_1(p_t)/p_t,$$

so that,

$$f = \liminf_{t \rightarrow \infty} f_t, \quad g = \limsup_{t \rightarrow \infty} g_t.$$

To prove the second assertion of Theorem 2, it suffices to show that

$$f = F^*(\rho) \text{ and } g < G^*(\rho) \tag{8.1}$$

is impossible. As in Section 6, this can be reduced to proving that

$$f = F^*(\rho), \quad g = G^*(\rho) \tag{8.2}$$

and

$$f_t > f, \quad g_t < g' < g \tag{8.3}$$

for large  $t$  leads to a contradiction.

Always recalling that now  $q_{t-1} < p_t < q_t$  and that  $L_1$  has slopes  $-1$  and  $3$ , we see that

$$g_t p_t = L_1(p_t) = L_1(q_t) + q_t - p_t = f_t q_t + q_t - p_t,$$

hence

$$\frac{p_t}{q_t} = \frac{1 + f_t}{1 + g_t}. \tag{8.4}$$

Moreover

$$g_t p_t = L_1(p_t) = L_1(q_{t-1}) + 3(p_t - q_{t-1}) = f_{t-1} q_{t-1} + 3(p_t - q_{t-1}),$$

hence

$$\frac{p_t}{q_{t-1}} = \frac{3 - f_{t-1}}{3 - g_t}. \tag{8.5}$$

From this we obtain

$$\frac{p_t}{p_{t-1}} = \frac{1 + g_{t-1}}{3 - g_t} \frac{3 - f_{t-1}}{1 + f_{t-1}} =: \rho_t, \tag{8.6}$$

say. Since  $f_t > f = F^*(\rho)$ ,  $g_t < g' < G^*(\rho)$ , and by (2.8), we have  $\rho_t < \rho$ , so that

$$\frac{p_t}{p_{t-1}} < \rho. \quad (8.7)$$

We set again  $\sigma := g_l$ . By (7.2)

$$\sigma p_l - g_r p_r = L_1(p_l) - L_1(p_r) = p_l - p_r + O(1),$$

hence

$$(1 - \sigma)p_l = (1 - g_r)p_r + O(1) > (1 - g')p_r + O(1),$$

and therefore

$$p_r < \frac{1 - \sigma}{1 - g'} p_l + O(1).$$

Since  $p_{r+1} < p_r \rho$  we obtain

$$p_{r+1} < \frac{1 - \sigma}{1 - g'} \rho p_l + O(1). \quad (8.8)$$

From

$$L_{\mathbf{x}_{r+1}}(p_{r+1}) = g_{r+1} p_{r+1} < g' p_{r+1}$$

we get

$$L_{\mathbf{x}_{r+1}}(p_s) = L_{\mathbf{x}_{r+1}}(p_{r+1}) + p_{r+1} - p_s < (1 + g')p_{r+1} - p_s,$$

and therefore

$$L_{\mathbf{x}_{r+1}}(p_s) < \frac{1 + g'}{1 - g'} (1 - \sigma) \rho p_l - p_s + O(1). \quad (8.9)$$

Furthermore

$$L_{\mathbf{x}_l}(p_s) = L_{\mathbf{x}_l}(p_l) + p_l - p_s = (1 + \sigma)p_l - p_s \quad (8.10)$$

and

$$L_1(p_s) = L_2(p_s) = L_{\mathbf{x}_s}(p_s) = L_{\mathbf{x}_{s-1}}(p_s) < g' p_s. \quad (8.11)$$

Since  $\mathbf{x}_{s-1}, \mathbf{x}_s, \mathbf{x}_l, \mathbf{x}_{r+1}$  are linearly independent

$$\begin{aligned} \Sigma(p_s) &:= L_1(p_s) + L_2(p_s) + L_3(p_s) + L_4(p_s) \\ &\leq L_{\mathbf{x}_{s-1}}(p_s) + L_{\mathbf{x}_s}(p_s) + L_{\mathbf{x}_l}(p_s) + L_{\mathbf{x}_{r+1}}(p_s), \end{aligned}$$

and by (8.9), (1.10), (1.11),

$$\Sigma(p_s) < 2(g' - 1)p_s + \left(1 + \sigma + \frac{1+g'}{1-g'}(1-\sigma)\rho\right) p_l + O(1). \quad (8.12)$$

In view of (7.4),

$$g_s p_s = f_{l-1} q_{l-1} + q_{l-1} - p_s + O(1),$$

hence  $(1+g')p_s > (1+g_s)p_s = (1+f_{l-1})q_{l-1} + O(1)$ , so that

$$p_s > \frac{1+f}{1+g'} q_{l-1} + O(1).$$

Now by (8.5)

$$q_{l-1} = \frac{3-g_l}{3-f_{l-1}} p_l > \frac{3-\sigma}{3-f} p_l,$$

so that

$$p_s > \frac{1+f}{3-f} \frac{3-\sigma}{1+g'} p_l + O(1).$$

Inserting this into (8.12) gives

$$\Sigma(p_s) < \left(-2(1-g') \frac{1+f}{3-f} \frac{3-\sigma}{1+g'} + \sigma + 1 + \frac{1+g'}{1-g'}(1-\sigma)\rho\right) p_l + O(1), \quad (8.13)$$

and since  $g' < g$ ,

$$\Sigma(p_s) < (B(\sigma) - c') p_l + O(1),$$

where

$$B(\sigma) = -2 \frac{1-g}{1+g} \frac{1+f}{3-f} (3-\sigma) + \sigma + 1 + \frac{1+g}{1-g} (1-\sigma)\rho$$

and  $c' > 0$ . Here

$$\begin{aligned} B(g) &= -2 \frac{3-g}{1+g} \frac{1+f}{3-f} (1-g) + (1+g)(1+\rho) \\ &= -2\rho^{-1}(1-g) + (1+g)(\rho+1) = 0 \end{aligned}$$

by (2.8), (8.2). Moreover

$$B'(\sigma) = 2 \frac{1-g}{1+g} \frac{1+f}{3-f} + 1 - \frac{1+g}{1-g} \rho > 1 - \frac{1+g}{1-g} \rho = 1 - \frac{2}{\rho+1} > 0.$$

Since  $\sigma = g_l < g$  we have  $B(\sigma) \leq B(g) = 0$ , and

$$\Sigma(p_s) \leq -c' p_l + O(1) < -(1/2)c' p_l.$$

When  $s$ , hence  $l$ , hence  $p_l$  is large, this contradicts  $\Sigma(p_s) = O(1)$ , and Theorem 2 is proved.

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WOLFGANG M. SCHMIDT

Department of Mathematics  
University of Colorado  
Boulder, CO 80309–0395, USA  
wolfgang.schmidt@colorado.edu

LEONHARD SUMMERER

Fakultät für Mathematik  
Universität Wien  
Nordbergstr. 15  
A-1090 Wien, AUSTRIA  
leonhard.summerer@univie.ac.at