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Poly-Cauchy polynomials

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Abstract: We introduce the poly-Cauchy polynomials which generalize the classical Cauchy polynomials, and investigate their arithmetical and combinatorial properties. These polynomials are considered as analogues of the poly-Bernoulli polynomials that generalize the classical Bernoulli polynomials. Moreover, we investigate the zeta functions which interpolate the poly-Cauchy polynomials. The values of these functions at positive integers can be expressed by using the polylogarithm function or the truncated multiple zeta star values.

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1. Introduction

In 1997 Kaneko [8] introduced the poly-Bernoulli numbers $B_n^{(k)}$ for an integer k and a non-negative integer n by

$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!} ,$$

where

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}$$

denotes the formal power series (the k-th polylogarithm if $k \geqslant 1$; a rational function if $k \leqslant 0$). When $k = 1$, $B_n^{(1)} = B_n$ are the Bernoulli numbers (with $B_1 = 1/2$), defined by

$$\frac{xe^x}{e^x-1}=\sum_{n=0}^\infty B_n^{(1)}\frac{x^n}{n!}\,.$$

Kaneko [8, Theorem 1] proved that $B_n^{(k)}$ is explicitly given by

$$B_n^{(k)}=(-1)^n\sum_{m=0}^n\frac{(-1)^m m!}{(m+1)^k}\left\{ n \atop m \right\}\,,$$

where

$$\left\{ n \atop m \right\}=\frac{(-1)^m}{m!}\sum_{l=0}^m(-1)^l\binom{m}{l}l^m$$

are the Stirling numbers of the second kind.

In 2010 Coppo and Candelpergher [5] and in 2011 Bayad and Hamahata [2] introduced the poly-Bernoulli polynomials $B_n^{(k)}(z)$ given by

$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}}e^{-xz}=\sum_{n=0}^\infty B_n^{(k)}(z)\frac{x^n}{n!}\,. \tag{1}$$

Note that e^{-xz} on the left-hand side of (1) is replaced by e^{xz} in [2, (1.5)]. When $z=0$, $B_n^{(k)}(0)=B_n^{(k)}$ are the poly-Bernoulli numbers.

Recently, the second named author [9] introduced the poly-Cauchy numbers $c_n^{(k)}$ for an integer k and a non-negative integer n, given by

$$\mathrm{Lif}_k(\ln(1+x))=\sum_{n=0}^\infty c_n^{(k)}\frac{x^n}{n!}\,,$$

where $\mathrm{Lif}_k(z)$ is the polylogarithm factorial function defined by

$$\mathrm{Lif}_k(z)=\sum_{m=0}^\infty\frac{z^m}{m!(m+1)^k}\,.$$

If $k=-r$ is a non-positive integer, the polylogarithm factorial function can be expressed in terms of the Stirling numbers of the second kind:

$$\mathrm{Lif}_{-r}(x)=e^x\sum_{j=0}^r\left\{ r+1 \atop j+1 \right\}x^j\quad(r=0,1,2,\dots)\,.$$

When $k=1$, $c_n^{(1)}=c_n$ are the Cauchy numbers (e. g. [4]) defined by

$$c_n=n!\int_0^1\binom{x}{n}dx\,.$$

In this paper we introduce the poly-Cauchy polynomials $c_n^{(k)}(z)$ in the spirit of the poly-Bernoulli polynomials. We consider the formal power series $\mathrm{Lif}_k(z)$ instead of the polylogarithm $\mathrm{Li}_k(z)$. When $z=0$, $c_n^{(k)}(0)=c_n^{(k)}$ are the poly-Cauchy numbers.

2. Poly-Cauchy polynomials of the first kind

For an integer $k\geqslant 1$ we define the poly-Cauchy polynomials of the first kind $c_n^{(k)}(z)$ as

$$c_n^{(k)}(z)=n!\underbrace{\int_0^1\cdots\int_0^1}_{k}\binom{x_1x_2\cdots x_k+z}{n}dx_1dx_2\cdots dx_k\,.$$

The first several polynomials are

$$\begin{aligned}
c_0^{(k)}(z)&=1,\\
c_1^{(k)}(z)&=\frac{1}{2^k}+z,\\
c_2^{(k)}(z)&=-\frac{1}{2^k}+\frac{1}{3^k}+\left(-1+\frac{2}{2^k}\right)z+z^2,\\
c_3^{(k)}(z)&=\frac{2}{2^k}-\frac{3}{3^k}+\frac{1}{4^k}+\left(2-\frac{6}{2^k}+\frac{3}{3^k}\right)z+\left(-3+\frac{3}{2^k}\right)z^2+z^3,\\
c_4^{(k)}(z)&=-\frac{6}{2^k}+\frac{11}{3^k}-\frac{6}{4^k}+\frac{1}{5^k}+\left(-6+\frac{22}{2^k}-\frac{18}{3^k}+\frac{4}{4^k}\right)z+\\
&\quad+\left(11-\frac{18}{2^k}+\frac{6}{3^k}\right)z^2+\left(-6+\frac{4}{2^k}\right)z^3+z^4.
\end{aligned}$$

As stated in Section 1, Coppo and Candelpergher [5] defined the poly-Bernoulli polynomials by (1) and Bayad and Hamahata [2] defined them by replacing z by

$-z$ in (1). Therefore we may define the poly-Cauchy numbers in alternative ways, simply by replacing z by $-z$. We define the polynomials $c_n^{(k)}(z)$ such that $c_n^{(1)}(z)$ coincides with the Bernoulli polynomials of the second kind $b_n(z)$ (see Theorem 2 and Corollary 1 below).

Theorem 1.

$$c_n^{(k)}(z)=\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}(-1)^{n-m}\sum_{i=0}^{m}\binom{m}{i}\frac{z^i}{(m-i+1)^k}\,, \tag{2}$$

where $\left[\begin{smallmatrix}n\\m\end{smallmatrix}\right]$ *are the Stirling numbers of the first kind.*

Proof. By the identity

$$\binom{x}{n}=\frac{1}{n!}\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}(-1)^{n-m}x^m \tag{3}$$

(e. g. [7, (6.13)]), we have

$$c_n^{(k)}(z)=n!\underbrace{\int_0^1\cdots\int_0^1}_{k}\binom{x_1x_2\cdots x_k+z}{n}dx_1dx_2\cdots dx_k=$$

$$=\underbrace{\int_0^1\cdots\int_0^1}_{k}\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}(-1)^{n-m}(x_1x_2\cdots x_k+z)^m dx_1dx_2\cdots dx_k=$$

$$=\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}(-1)^{n-m}\sum_{i=0}^{m}\binom{m}{i}\frac{z^i}{(m-i+1)^k}\,.$$

□

Remark. (i) By using the formula (2), the poly-Cauchy numbers $c_n^{(k)}$ for $k\leqslant 0$ can be defined.

(ii) When $z=0$, the numbers

$$c_n^{(k)}(0)=(-1)^n\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}\frac{(-1)^m}{(m+1)^k}$$

are the poly-Cauchy numbers of the first kind defined in [9, Theorem 1].

THEOREM 2. *The generating function of the poly-Cauchy polynomials of the first kind* $c_n^{(k)}(z)$ *is given by*

$$(1+x)^z \mathrm{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty} c_n^{(k)}(z) \frac{x^n}{n!} .$$

PROOF. By Theorem 1 and the identity

$$\frac{(\ln(1+x))^m}{m!} = (-1)^m \sum_{n=m}^{\infty} \left[{n \atop m} \right] \frac{(-x)^n}{n!} \tag{4}$$

(e. g. [7, (7.50)]), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} c_n^{(k)}(z) \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[{n \atop m} \right] (-1)^{n-m} \sum_{i=0}^{m} \binom{m}{i} \frac{z^i}{(m-i+1)^k} \frac{x^n}{n!} = \\
&= \sum_{m=0}^{\infty} (-1)^m \sum_{i=0}^{m} \binom{m}{i} \frac{z^i}{(m-i+1)^k} \sum_{n=m}^{\infty} \left[{n \atop m} \right] \frac{(-x)^n}{n!} = \\
&= \sum_{m=0}^{\infty} \frac{(\ln(1+x))^m}{m!} \sum_{i=0}^{m} \binom{m}{i} \frac{z^i}{(m-i+1)^k} = \\
&= \sum_{i=0}^{\infty} \frac{z^i}{i!} \sum_{m=i}^{\infty} \frac{(\ln(1+x))^m}{(m-i)!(m-i+1)^k} = \\
&= \sum_{i=0}^{\infty} \frac{(z\ln(1+x))^i}{i!} \sum_{\nu=0}^{\infty} \frac{(\ln(1+x))^\nu}{\nu!(\nu+1)^k} = \\
&= (1+x)^z \mathrm{Lif}_k(\ln(1+x)) . \qquad \square
\end{aligned}$$

By this theorem, the generating function of $c_n^{(k)} = c_n^{(k)}(0)$ is given by

$$\mathrm{Lif}_k(\ln(1+x)) = \sum_{n=0}^{\infty} c_n^{(k)} \frac{x^n}{n!}$$

(see [9, Theorem 2]).

We can write the generating function of the poly-Cauchy polynomials in Theorem 2 in the form of iterated integrals. When $k=1$ in the following corollary, we have

$$\frac{x(1+x)^z}{\ln(1+x)}=\sum_{n=0}^{\infty}c_n^{(1)}(z)\frac{x^n}{n!}$$

and $c_n^{(1)}(z)$ coincides with the Bernoulli polynomial of the second kind $b_n(z)$ (cf. [6], [13, § 4.3.2]).

Corollary 1. *For $k \geqslant 1$ we have*

$$\frac{(1+x)^z}{\ln(1+x)}\underbrace{\int_0^x\frac{1}{(1+x)\ln(1+x)}\cdots\int_0^x\frac{1}{(1+x)\ln(1+x)}}_{k-1}\times x\ \underbrace{dxdx\cdots dx}_{k-1}=$$

$$=\sum_{n=0}^{\infty}c_n^{(k)}(z)\frac{x^n}{n!}\,.$$

Proof. For $k \geqslant 1$ we have

$$\begin{aligned}\mathrm{Lif}_k(z)&=\sum_{m=0}^{\infty}\frac{z^m}{(m+1)^{k-1}(m+1)!}=\\&=\frac{1}{z}\int_0^z\sum_{m=0}^{\infty}\frac{z^m}{(m+1)^{k-2}(m+1)!}dz=\\&=\frac{1}{z}\int_0^z\mathrm{Lif}_{k-1}(z)dz.\end{aligned}$$

By repeating this procedure, we obtain that

$$\mathrm{Lif}_k(z)=\underbrace{\frac{1}{z}\int_0^z\frac{1}{z}\int_0^z\cdots\frac{1}{z}\int_0^z}_{k-1}\mathrm{Lif}_1(z)\,\underbrace{dzdz\cdots dz}_{k-1}=$$

$$= \underbrace{\frac{1}{z}\int\limits_0^z \frac{1}{z}\int\limits_0^z \cdots \frac{1}{z}\int\limits_0^z}_{k-1} \frac{e^z-1}{z} \underbrace{dzdz\cdots dz}_{k-1}.$$

Here we use

$$\begin{aligned}\mathrm{Lif}_1(z) &= \frac{1}{z}\sum_{m=0}^{\infty}\frac{z^{m+1}}{(m+1)!} = \\ &= \frac{e^z-1}{z}.\end{aligned}$$

Putting $z=\ln(1+x)$ together with Theorem 2, we get the result. □

THEOREM 3.

$$\sum_{m=0}^{n}\left\{ {n \atop m} \right\} c_m^{(k)}(z) = \sum_{i=0}^{n}\binom{n}{i}\frac{z^i}{(n-i+1)^k}\,. \tag{5}$$

PROOF. We recall the identity

$$\sum_{m=0}^{\max\{l,n\}}(-1)^{m-n}\left[{m \atop l} \right]\left\{ {n \atop m} \right\} = \delta_{ln}$$

(e. g. [7, (6.16)]), where δ_{ln} is the Kronecker delta defined by

$$\delta_{ln} = \begin{cases} 1 & (l=n), \\ 0 & (l\neq n). \end{cases}$$

By this identity and Theorem 1, we have

$$\begin{aligned}\sum_{m=0}^{n}\left\{ {n \atop m} \right\} c_m^{(k)}(z) &= \sum_{m=0}^{n}\left\{ {n \atop m} \right\}\sum_{l=0}^{m}\left[{m \atop l} \right](-1)^{m-l}\sum_{i=0}^{l}\binom{l}{i}\frac{z^i}{(l-i+1)^k} = \\ &= \sum_{i=0}^{n} z^i \sum_{l=i}^{n}\binom{l}{i}\frac{(-1)^l}{(l-i+1)^k}\sum_{m=l}^{n}(-1)^m\left[{m \atop l} \right]\left\{ {n \atop m} \right\} =\end{aligned}$$

$$= (-1)^n \sum_{i=0}^{n} z^i \sum_{l=i}^{n} \binom{l}{i} \frac{(-1)^l}{(l-i+1)^k} \delta_{ln} =$$

$$= \sum_{i=0}^{n} \binom{n}{i} \frac{z^i}{(n-i+1)^k} .$$

□

Remark. When $z=0$, then the identity (5) is the same as that in [9, Theorem 3].

3. Poly-Cauchy polynomials of the second kind

For $k \geqslant 1$ we define the poly-Cauchy polynomials of the second kind $\widehat{c}_n^{(k)}(z)$ as

$$\widehat{c}_n^{(k)}(z) = n! \underbrace{\int_0^1 \cdots \int_0^1}_{k} \binom{-x_1 x_2 \cdots x_k - z}{n} dx_1 dx_2 \cdots dx_k .$$

The first several polynomials are

$$\widehat{c}_0^{(k)}(z) = 1,$$

$$\widehat{c}_1^{(k)}(z) = -\frac{1}{2^k} - z,$$

$$\widehat{c}_2^{(k)}(z) = \frac{1}{2^k} + \frac{1}{3^k} + \left(1 + \frac{2}{2^k}\right) z + z^2,$$

$$\widehat{c}_3^{(k)}(z) = -\frac{2}{2^k} - \frac{3}{3^k} - \frac{1}{4^k} - \left(2 + \frac{6}{2^k} + \frac{3}{3^k}\right) z - \left(3 + \frac{3}{2^k}\right) z^2 - z^3,$$

$$\widehat{c}_4^{(k)}(z) = \frac{6}{2^k} + \frac{11}{3^k} + \frac{6}{4^k} + \frac{1}{5^k} + \left(6 + \frac{22}{2^k} + \frac{18}{3^k} + \frac{4}{4^k}\right) z +$$

$$+ \left(11 + \frac{18}{2^k} + \frac{6}{3^k}\right) z^2 + \left(6 + \frac{4}{2^k}\right) z^3 + z^4.$$

The numbers $\widehat{c}_n^{(1)}(0)$ are the Cauchy numbers of the second kind (e. g. [9], [11]). The polynomials $\widehat{c}_n^{(1)}(z)$ are essentially the same as the Cauchy polynomials of the second type $\Phi_n^{(2)}(z)$ given in [3, p. 2579]. In fact, it holds that $\widehat{c}_n^{(1)}(z) = (-1)^n \Phi_n^{(2)}(-z)$.

By using the identity (3) again, we have

$$\widehat{c}_n^{(k)}(z) = n! \underbrace{\int_0^1 \cdots \int_0^1}_{k} \binom{-x_1 x_2 \cdots x_k - z}{n} dx_1 dx_2 \cdots dx_k =$$

$$= (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \underbrace{\int_0^1 \cdots \int_0^1}_{k} (x_1 x_2 \cdots x_k + z)^m \, dx_1 dx_2 \cdots dx_k =$$

$$= \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-1)^n \sum_{i=0}^{m} \binom{m}{i} \frac{z^i}{(m-i+1)^k} .$$

Hence we obtain the following theorem, which is an analogue of Theorem 1.

Theorem 4.

$$\widehat{c}_n^{(k)}(z) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-1)^n \sum_{i=0}^{m} \binom{m}{i} \frac{z^i}{(m-i+1)^k} . \tag{6}$$

When $z = 0$, the numbers

$$\widehat{c}_n^{(k)}(0) = (-1)^n \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{1}{(m+1)^k}$$

are the poly-Cauchy numbers of the second kind ([9, Theorem 4]). We note that $\widehat{c}_n^{(k)}(z)$ for $k \leqslant 0$ can be also defined by using (6).

Theorem 5. *The generating function of* $\widehat{c}_n^{(k)}(z)$ *is given by*

$$\frac{\mathrm{Lif}_k(-\ln(1+x))}{(1+x)^z} = \sum_{n=0}^{\infty} \widehat{c}_n^{(k)}(z) \frac{x^n}{n!} .$$

Proof. Similarly to the proof of Theorem 2, by Theorem 4, we have

$$\sum_{n=0}^{\infty} \widehat{c}_n^{(k)}(z) \frac{x^n}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} (-1)^n \sum_{i=0}^{m} \binom{m}{i} \frac{z^i}{(m-i+1)^k} \frac{x^n}{n!} =$$

$$= \sum_{m=0}^{\infty} \sum_{i=0}^{m} \binom{m}{i} \frac{z^i}{(m-i+1)^k} \sum_{n=m}^{\infty} \left[{n \atop m} \right] \frac{(-x)^n}{n!} =$$

$$= \sum_{m=0}^{\infty} \frac{(-\ln(1+x))^m}{m!} \sum_{i=0}^{m} \binom{m}{i} \frac{z^i}{(m-i+1)^k} =$$

$$= \sum_{i=0}^{\infty} \frac{z^i}{i!} \sum_{m=i}^{\infty} \frac{(-\ln(1+x))^m}{(m-i)!(m-i+1)^k} =$$

$$= \sum_{i=0}^{\infty} \frac{(-z\ln(1+x))^i}{i!} \sum_{\nu=0}^{\infty} \frac{(-\ln(1+x))^\nu}{\nu!(\nu+1)^k} =$$

$$= \frac{\mathrm{Lif}_k(-\ln(1+x))}{(1+x)^z} . \qquad \square$$

By this theorem, the generating function of $\widehat{c}_n^{(k)} = c_n^{(k)}(0)$ is given by

$$\mathrm{Lif}_k(-\ln(1+x)) = \sum_{n=0}^{\infty} \widehat{c}_n^{(k)} \frac{x^n}{n!}$$

([9, Theorem 5]). In particular, the generating function of the Cauchy numbers of the second kind $\widehat{c}_n = \widehat{c}_n^{(1)}$ is given by

$$\frac{x}{(1+x)\ln(1+x)} = \sum_{n=0}^{\infty} \widehat{c}_n^{(1)} \frac{x^n}{n!}$$

([4, Chapter VII], [11, p. 1910]).

The generating function of the poly-Cauchy polynomials of the second kind can be also written in the form of iterated integrals by putting $z = -\ln(1+x)$ in

$$\mathrm{Lif}_k(z) = \underbrace{\frac{1}{z}\int_0^z \frac{1}{z}\int_0^z \cdots \frac{1}{z}\int_0^z}_{k-1} \mathrm{Lif}_1(z) \underbrace{dzdz\dots dz}_{k-1} .$$

Corollary 2. *For $k \geqslant 1$ we have*

$$\frac{1}{(1+x)^z \ln(1+x)} \underbrace{\int_0^x \frac{1}{(1+x)\ln(1+x)} \cdots \int_0^x \frac{1}{(1+x)\ln(1+x)}}_{k-1} \cdot \frac{x}{1+x} \underbrace{dx dx \dots dx}_{k-1} =$$

$$= \sum_{n=0}^{\infty} \widehat{c}_n^{(k)} \frac{x^n}{n!}.$$

Similarly to Theorem 3, by Theorem 4 we have an analogous result about the poly-Cauchy polynomials of the second kind.

Theorem 6.

$$\sum_{m=0}^{n} \left\{ {n \atop m} \right\} \widehat{c}_m^{(k)}(z) = (-1)^n \sum_{i=0}^{n} \binom{n}{i} \frac{z^i}{(n-i+1)^k}.$$

4. Relations between two kinds of poly-Cauchy polynomials

There are some relations between the poly-Cauchy polynomials of the first kind and those of the second kind. We first show the following lemma.

Lemma 1. *For $n \geqslant 1$ and $l \geqslant 1$, we have*

$$\frac{(-1)^l}{n!} \left[{n \atop l} \right] = \sum_{m=1}^{n} \frac{(-1)^m}{m!} \left[{m \atop l} \right] \binom{n-1}{m-1}.$$

Proof.

By the identity (4), we have

$$\sum_{n=1}^{\infty} \frac{(-1)^l}{n!} \left[{n \atop l} \right] x^n = \frac{1}{l!} (\ln(1-x))^l.$$

On the other hand, by the identity

$$\sum_{n=1}^{\infty} \binom{n-1}{m-1} x^n = \frac{x^m}{(1-x)^m} \quad (m \geqslant 1)$$

(e. g. [7, § 7.2]), we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\begin{bmatrix}m\\l\end{bmatrix}\binom{n-1}{m-1}x^n=\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\begin{bmatrix}m\\l\end{bmatrix}\frac{x^m}{(1-x)^m}=$$
$$=\sum_{m=1}^{\infty}\frac{1}{m!}\begin{bmatrix}m\\l\end{bmatrix}\left(\frac{x}{x-1}\right)^m=$$
$$=\frac{1}{l!}\left(-\log\left(1-\frac{x}{x-1}\right)\right)^l=$$
$$=\frac{1}{l!}\left(-\log\left(\frac{1}{1-x}\right)\right)^l=$$
$$=\frac{1}{l!}\left(\log\left(1-x\right)\right)^l .$$

Hence

$$\sum_{n=1}^{\infty}\frac{(-1)^l}{n!}\begin{bmatrix}n\\l\end{bmatrix}x^n=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\begin{bmatrix}m\\l\end{bmatrix}\binom{n-1}{m-1}x^n$$

and we obtain the result by comparing the coefficients of x^n. □

THEOREM 7. *For $n \geqslant 1$ we have*

$$(-1)^n\frac{c_n^{(k)}(z)}{n!}=\sum_{m=1}^{n}\binom{n-1}{m-1}\frac{\widehat{c}_m^{(k)}(z)}{m!},$$
$$(-1)^n\frac{\widehat{c}_n^{(k)}(z)}{n!}=\sum_{m=1}^{n}\binom{n-1}{m-1}\frac{c_m^{(k)}(z)}{m!}.$$

PROOF. We shall prove the first identity. The second one is proved similarly. By Theorem 1, Theorem 4 and Lemma 1, we have

$$\sum_{m=1}^{n}\binom{n-1}{m-1}\frac{\widehat{c}_m^{(k)}(z)}{m!}=\sum_{m=1}^{n}\binom{n-1}{m-1}\frac{(-1)^m}{m!}\sum_{l=0}^{m}\begin{bmatrix}m\\l\end{bmatrix}\sum_{i=0}^{l}\binom{l}{i}\frac{z^i}{(l-i+1)^k}=$$
$$=\sum_{l=1}^{n}\sum_{m=l}^{n}\frac{(-1)^m}{m!}\binom{n-1}{m-1}\begin{bmatrix}m\\l\end{bmatrix}\sum_{i=0}^{l}\binom{l}{i}\frac{z^i}{(l-i+1)^k}=$$

$$= \sum_{l=1}^{n} \frac{(-1)^l}{n!} \begin{bmatrix} n \\ l \end{bmatrix} \sum_{i=0}^{l} \binom{l}{i} \frac{z^i}{(l-i+1)^k} =$$

$$= (-1)^n \frac{c_n^{(k)}(z)}{n!} . \qquad \square$$

5. Some expressions of poly-Cauchy polynomials with negative indices

The poly-Bernoulli numbers satisfy the duality formula $B_n^{(-k)} = B_k^{(-n)}$ for $n, k \geqslant 0$ ([8, Theorem 2]), because of the symmetric formula

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{(-k)} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}} .$$

However, the corresponding duality formula does not hold for the poly-Cauchy polynomials for any z by the following results.

Proposition 1. *We have*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_n^{(-k)}(z) \frac{x^n}{n!} \frac{y^k}{k!} = e^y (1+x)^{e^y+z} ,$$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \widehat{c}_n^{(-k)}(z) \frac{x^n}{n!} \frac{y^k}{k!} = \frac{e^y}{(1+x)^{e^y+z}} .$$

Proof. We shall prove the first identity. The second one can be proved similarly. By Theorem 2, we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_n^{(-k)}(z) \frac{x^n}{n!} \frac{y^k}{k!} = \sum_{k=0}^{\infty} (1+x)^z \mathrm{Lif}_{-k}\big(\ln(1+x)\big) \frac{y^k}{k!} =$$

$$= (1+x)^z \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{\big(\ln(1+x)\big)^m}{m!} (m+1)^k \frac{y^k}{k!} =$$

$$= (1+x)^z \sum_{m=0}^{\infty} \frac{\big(\ln(1+x)\big)^m}{m!} \sum_{k=0}^{\infty} \frac{\big((m+1)y\big)^k}{k!} =$$

$$= (1+x)^z \sum_{m=0}^{\infty} \frac{\bigl(\ln(1+x)\bigr)^m}{m!} e^{(m+1)y} =$$
$$= e^y (1+x)^z \sum_{m=0}^{\infty} \frac{\bigl(e^y \ln(1+x)\bigr)^m}{m!} =$$
$$= e^y (1+x)^{e^y+z} . \qquad \square$$

Though the symmetric formula does not hold for the poly-Cauchy polynomials, we have explicit expressions of them with negative indices.

THEOREM 8. *For non-negative integers n and k, we have*

$$c_n^{(-k)}(z) = \sum_{j=0}^{k} (-1)^{n+j} j! \left\{ {k+1 \atop j+1} \right\} \sum_{l=0}^{n} \binom{n}{l} \left[{n-l \atop j} \right] (-z-1)_l , \tag{7}$$
$$\widehat{c}_n^{(-k)}(z) = \sum_{j=0}^{k} (-1)^{n} j! \left\{ {k+1 \atop j+1} \right\} \sum_{l=0}^{n} \binom{n}{l} \left[{n-l \atop j} \right] (z+1)_l , \tag{8}$$

where $(x)_l$ is the Pochhammer symbol defined by

$$(x)_l = \begin{cases} 1 & (l=0), \\ x(x+1)\cdots(x+l-1) & (l \geqslant 1). \end{cases} \tag{9}$$

PROOF. By Proposition 1 with

$$\frac{(e^y-1)^j}{j!} = \sum_{k=j}^{\infty} \left\{ {k \atop j} \right\} \frac{y^k}{k!} \quad \text{and} \quad \frac{\bigl(-\ln(1+x)\bigr)^j}{j!} = \sum_{n=j}^{\infty} \left[{n \atop j} \right] \frac{(-x)^n}{n!}$$

([7, (7.49) (7.50)]), we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_n^{(-k)}(z) \frac{x^n}{n!} \frac{y^k}{k!} = \exp\bigl((e^y-1)\ln(1+x)\bigr)(1+x)^{z+1} e^y =$$
$$= \sum_{j=0}^{\infty} j! \frac{(e^y-1)^j}{j!} \frac{\bigl(\ln(1+x)\bigr)^j}{j!} (1+x)^{z+1} e^y =$$

$$= \sum_{j=0}^{\infty}(-1)^j j! e^y \sum_{k=j}^{\infty} \left\{ {k \atop j} \right\} \frac{y^k}{k!}(1+x)^{z+1} \sum_{n=j}^{\infty} \left[{n \atop j} \right] \frac{(-x)^n}{n!} \,.$$

Note that

$$\begin{aligned} e^y \sum_{k=j}^{\infty} \left\{ {k \atop j} \right\} \frac{y^k}{k!} &= \sum_{l=0}^{\infty} \frac{y^l}{l!} \sum_{k=j}^{\infty} \left\{ {k \atop j} \right\} \frac{y^k}{k!} = \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \binom{k}{i} \left\{ {i \atop j} \right\} \right) \frac{y^k}{k!} = \\ &= \sum_{k=0}^{\infty} \left\{ {k+1 \atop j+1} \right\} \frac{y^k}{k!} \end{aligned} \tag{10}$$

and

$$\begin{aligned} &(1+x)^{1+z} \sum_{n=j}^{\infty} \left[{n \atop j} \right] \frac{(-x)^n}{n!} = \\ &= \sum_{l=0}^{\infty} \binom{z+1}{l} x^l \sum_{n=j}^{\infty} \left[{n \atop j} \right] \frac{(-x)^n}{n!} = \\ &= \sum_{l=0}^{\infty} \frac{(z+1)z(z-1)\cdots(2+z-l)}{l!} x^l \sum_{n=0}^{\infty} \left[{n \atop j} \right] \frac{(-x)^n}{n!} = \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} \left[{n-l \atop j} \right] (-1)^{n+l}(-z-1)_l \frac{x^n}{n!} \,. \end{aligned}$$

Hence

$$\begin{aligned} &\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_n^{(-k)}(z) \frac{x^n}{n!} \frac{y^k}{k!} = \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{k} (-1)^{n+j} j! \left\{ {k+1 \atop j+1} \right\} \sum_{l=0}^{n} \binom{n}{l} \left[{n-l \atop j} \right] (-1)^l (-z-1)_l \frac{x^n}{n!} \frac{y^k}{k!} \end{aligned}$$

which proves (7). Similarly, by (10) and

$$(1+x)^{-z-1}\sum_{n=j}^{\infty}\begin{bmatrix} n \\ j \end{bmatrix}\frac{(-x)^n}{n!} =$$

$$= \sum_{l=0}^{\infty}\binom{-z-1}{l}x^l\sum_{n=j}^{\infty}\begin{bmatrix} n \\ j \end{bmatrix}\frac{(-x)^n}{n!} =$$

$$= \sum_{l=0}^{\infty}\frac{(-z-1)(-z-2)\cdots(-z-l)}{l!}x^l\sum_{n=0}^{\infty}\begin{bmatrix} n \\ j \end{bmatrix}\frac{(-x)^n}{n!} =$$

$$= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\begin{bmatrix} n-l \\ j \end{bmatrix}(-1)^n(z+1)_l\frac{x^n}{n!},$$

we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\widehat{c}_n^{(-k)}(z)\frac{x^n}{n!}\frac{y^k}{k!} =$$

$$= \exp\Big((e^y-1)\ln(1+x)^{-1}\Big)(1+x)^{-z-1}e^y =$$

$$= \sum_{j=0}^{\infty}j!\frac{(e^y-1)^j}{j!}\frac{\big(-\ln(1+x)\big)^j}{j!}(1+x)^{-z-1}e^y =$$

$$= \sum_{j=0}^{\infty}j!e^y\sum_{k=j}^{\infty}\begin{Bmatrix} k \\ j \end{Bmatrix}\frac{y^k}{k!}(1+x)^{-z-1}\sum_{n=j}^{\infty}\begin{bmatrix} n \\ j \end{bmatrix}\frac{(-x)^n}{n!} =$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{k}(-1)^n j!\begin{Bmatrix} k+1 \\ j+1 \end{Bmatrix}\sum_{l=0}^{n}\binom{n}{l}\begin{bmatrix} n-l \\ j \end{bmatrix}(z+1)_l\frac{x^n}{n!}\frac{y^k}{k!}$$

and this proves (8). □

Remark. When $z=0$, by

$$(-1)_l = \begin{cases} 1 & (l=0), \\ -1 & (l=1), \\ 0 & (l\geqslant 2), \end{cases}$$

the first identity (7) is reduced to

$$c_n^{(-k)} = c_n^{(-k)}(0) = \sum_{j=0}^{k}(-1)^{n+j}j!\left(\begin{bmatrix} n \\ j \end{bmatrix} - n\begin{bmatrix} n-1 \\ j \end{bmatrix}\right)\begin{Bmatrix} k+1 \\ j+1 \end{Bmatrix}.$$

By the identity

$$\sum_{l=0}^{n}\binom{n}{l}\begin{bmatrix} n-l \\ j \end{bmatrix} l! = \begin{bmatrix} n+1 \\ j+1 \end{bmatrix}$$

(e. g. [7, (6.23)]), the second identity (8) is reduced to

$$\widehat{c}_n^{(-k)} = \widehat{c}_n^{(-k)}(0) = \sum_{j=0}^{k}(-1)^{n}j!\begin{bmatrix} n+1 \\ j+1 \end{bmatrix}\begin{Bmatrix} k+1 \\ j+1 \end{Bmatrix}.$$

6. Zeta functions interpolating the Cauchy polynomials

In this and the next section, we study complex-variable functions $Z_k(s,z)$ and $\widehat{Z}_k(s,z)$ which interpolate the poly-Cauchy polynomials $c_n^{(k)}(z)$ and $\widehat{c}_n^{(k)}(z)$. We show that their values at positive integers can be expressed by using the polylogarithm function or the truncated multiple zeta star values. As a corollary, we give a duality relation between $Z_k(n,z)$ (or $\widehat{Z}_k(n,z)$) and the generalized Arakawa-Kaneko zeta function $\xi_k(s,z)$.

The generalized Arakawa-Kaneko zeta function is defined as

$$\xi_k(s,z) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}e^{-zt}t^{s-1}\,dt \quad (\mathrm{Re}(s)>0 \text{ and } z>0)$$

(see [2], [5] and [14]). It is easy to see that $\xi_1(s,z) = s\zeta(s+1,z)$ where $\zeta(s,z)$ is the Hurwitz zeta function defined by

$$\zeta(s,z) = \sum_{n=0}^{\infty}\frac{1}{(n+z)^s} \quad (\mathrm{Re}(s)>1,\ z>0).$$

It is known that the function $\xi_k(s,z)$ can be analytically continued to the whole complex s-plane and its values at non-positive integers are given by

$$\xi_k(-n,z) = (-1)^n B_n^{(k)}(z) \quad (n \geqslant 0) \tag{11}$$

([5, Theorem 2]).

When $z = 1$, the function $\xi_k(s, 1) = \xi_k(s)$ is the original Arakawa-Kaneko zeta function, first defined in [1]. By (11), the values of the Arakawa-Kaneko zeta function at negative integers are expressed by poly-Bernoulli numbers:

$$\xi_k(-n) = B_n^{(k)}(1) = \sum_{l=0}^{n} (-1)^l \binom{n}{l} B_l^{(k)} \quad (n \geqslant 0)$$

([1, Theorem 6]). On the other hand, Ohno [12, Theorem 2] proved that the values of $\xi_k(s)$ at positive integers are expressed by multiple zeta star values:

$$\xi_k(n) = \zeta^\star(k+1, \underbrace{1, \dots, 1}_{n-1}) \quad (n \geqslant 1), \tag{12}$$

where $\zeta^\star(k_1, \dots, k_n)$ are multiple zeta star values defined by

$$\zeta^\star(k_1, \dots, k_n) = \sum_{m_1 \geqslant \cdots \geqslant m_k \geqslant 1} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}.$$

Ohno proved the identity (12) by using the iterated integral expression of multiple zeta values, and later Kuba [10, Theorem 1] gave a direct proof.

Now we give the asymptotic behavior of the function $\mathrm{Lif}_k(z)$, which is needed to define analogues of the generalized Arakawa-Kaneko zeta function.

LEMMA 2. *For* $k \geqslant 1$, *we have*

$$\mathrm{Lif}_k(-z) \sim \frac{1}{(k-1)!} \frac{(\ln z)^{k-1}}{z} \quad (z \to \infty). \tag{13}$$

Here $f(z) \sim g(z)$ $(z \to \infty)$ *means that* f *and* g *are asymptotically equal*, *i.e.*,

$$\lim_{z \to \infty} \frac{f(z)}{g(z)} = 1.$$

PROOF. We prove this lemma by induction on k. We have

$$\mathrm{Lif}_1(-z) = \frac{e^{-z} - 1}{-z} \sim \frac{1}{z},$$

hence (13) holds for $k = 1$. We assume that (13) holds for some $k \geqslant 1$. Then

$$\mathrm{Lif}_{k+1}(-z) = \frac{1}{-z}\int_0^{-z} \mathrm{Lif}_k(u)\,du = $$
$$= \frac{1}{z}\int_0^{z} \mathrm{Lif}_k(-u)\,du.$$

Let α be an arbitrary constant. By the inductive assumption, we have

$$\mathrm{Lif}_{k+1}(-z) \sim \frac{1}{z}\int_\alpha^{z} \frac{1}{(k-1)!}\frac{(\ln u)^{k-1}}{u}\,du =$$
$$= \frac{1}{z}\left[\frac{(\ln u)^k}{k!}\right]_\alpha^z \sim$$
$$\sim \frac{1}{k!}\frac{(\ln z)^k}{z}.$$

This means (13) holds for $k+1$ and the lemma is proved. □

Let k be a positive integer. We define the function $Z_k(s,z)$ for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ and $z > -1$ as

$$Z_k(s,z) := \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}(1-t)^z \mathrm{Lif}_k(\ln(1-t))\,dt. \tag{14}$$

By the change of the variables $t = 1 - e^{-u}$, this can be written as

$$Z_k(s,z) = \frac{1}{\Gamma(s)}\int_0^\infty (1-e^{-u})^{s-1}\mathrm{Lif}_k(-u)e^{-(z+1)u}\,du. \tag{15}$$

By Lemma 2, the integral in (15) is convergent for $\mathrm{Re}(s) > 0$ and $z > -1$.

The following proposition indicates that the function $Z_k(s,z)$ is an analogue of the generalized Arakawa-Kaneko zeta function $\xi_k(s,z)$.

Proposition 2. *The function $Z_k(s, z)$ can be extended to an entire function, and its values at non-positive integers are given as follows*:

$$Z_k(-n, z) = c_n^{(k)}(z) \quad (n = 0, 1, 2, \ldots).$$

Proof. Let q be an arbitrary non-negative integer and γ an arbitrary real number with $0 < \gamma < 1$. Then we have

$$\int_0^1 t^{s-1}(1-t)^z \mathrm{Lif}_k(\ln(1-t))\, dt =$$

$$= \int_0^\gamma t^{s-1}(1-t)^z \mathrm{Lif}_k(\ln(1-t))\, dt + \int_\gamma^1 t^{s-1}(1-t)^z \mathrm{Lif}_k(\ln(1-t))\, dt =$$

$$= \int_0^\gamma t^{s-1} \sum_{m=0}^{q} \frac{(-1)^m c_m^{(k)}(z)}{m!} t^m\, dt + \int_0^\gamma t^{s-1} \sum_{m=q+1}^{\infty} \frac{(-1)^m c_m^{(k)}(z)}{m!} t^m\, dt +$$

$$+ \int_\gamma^1 t^{s-1}(1-t)^z \mathrm{Lif}_k(\ln(1-t))\, dt.$$

The first term is equal to

$$\sum_{m=0}^{q} \frac{(-1)^m c_m^{(k)}(z)}{m!} \left[\frac{t^{s+m}}{s+m}\right]_0^\gamma = \sum_{m=0}^{q} \frac{(-1)^m c_m^{(k)}(z)}{m!} \frac{\gamma^{s+m}}{(s+m)}.$$

It is easy to see that

$$\frac{1}{\Gamma(s)} \frac{1}{(s+m)} = \frac{(s)_m}{\Gamma(s+m+1)},$$

where $(s)_m$ is the Pochhammer symbol defined by (9). Therefore we have the following expression for $\mathrm{Re}(s) > 0$:

$$\begin{aligned}Z_k(s,z)=&\sum_{m=0}^{q}\frac{(s)_m}{m!\Gamma(s+m+1)}c_m^{(k)}(z)(-1)^m\gamma^{s+m}+\\&+\frac{1}{\Gamma(s)}\Bigg(\int_0^\gamma t^{s-1}\left((1-t)^z\mathrm{Lif}_k(\ln(1-t))-\sum_{m=0}^{q}\frac{(-1)^m c_m^{(k)}(z)}{m!}t^m\right)dt+\\&+\int_\gamma^1 t^{s-1}(1-t)^z\mathrm{Lif}_k(\ln(1-t))dt\Bigg).\end{aligned} \tag{16}$$

The first integration converges if $\mathrm{Re}(s) > -1-q$ and the second integration converges for an arbitrary $s \in \mathbb{C}$. Hence the right-hand side of (16) defines holomorphic function for $\mathrm{Re}(s) > -1-q$. Since q is arbitrary, we obtain the holomorphic continuation of $Z_k(s, z)$ to the whole s-plane. Finally, when $s = -n$, only the term for $m = n$ in the first part remains and $Z_k(-n, z) = c_n^{(k)}(z)$ is obtained. □

The previous proposition gives the values of $Z_k(s, z)$ at negative integers. The values at positive integers are expressed by using values of polylogarithm function.

PROPOSITION 3. *Let k and n be positive integers. For $z \geqslant 0$, we have*

$$Z_k(n, z) = \frac{1}{(n-1)!}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^{l+1}\mathrm{Li}_k\left(-\frac{1}{l+1+z}\right).$$

PROOF. By the expression (15), we have

$$Z_k(n, z) = \frac{1}{(n-1)!}\int_0^\infty\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l e^{-u(l+1+z)}\sum_{m=0}^{\infty}\frac{(-u)^m}{m!(m+1)^k}\,du.$$

By changing the variables $u = v/(l+1+z)$, we have

$$Z_k(n, z) = \frac{1}{(n-1)!}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l\int_0^\infty e^{-v}\sum_{m=0}^{\infty}\frac{(-1)^m v^m}{m!(m+1)^k(l+1+z)^{m+1}}dv.$$

Since $m! = \int_0^\infty e^{-v} v^m dv$, we obtain that

$$Z_k(n,z) = \frac{1}{(n-1)!} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)^k (l+1+z)^{m+1}} =$$

$$= \frac{1}{(n-1)!} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^{l+1} \mathrm{Li}_k \left(-\frac{1}{l+1+z} \right). \qquad \square$$

LEMMA 3. E. G. [10, (2)]. *For any $r \geqslant 0$, we have*

$$\sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^{l-1}}{l^r} = \zeta_n^\star(\{1\}_r),$$

where $\zeta_n^\star(\{1\}_r) := \begin{cases} \displaystyle\sum_{n \geqslant k_1 \geqslant \cdots \geqslant k_r \geqslant 1} \frac{1}{k_1 \cdots k_r} & (r \geqslant 1), \\ 0 & (r = 0). \end{cases}$

Remark. Lemma 3 implies that

$$\lim_{r \to \infty} \zeta_n^\star(\{1\}_r) = n \quad (n = 1, 2, \ldots).$$

THEOREM 9. *For $k \geqslant 1$, $n \geqslant 1$ and $0 \leqslant z < 1$, we have*

$$Z_k(n,z) = \frac{1}{n!} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^k} \sum_{j=0}^{\infty} \binom{m+j-1}{m-1} \zeta_n^\star(\{1\}_{m+j-1})(-z)^j. \tag{17}$$

In particular, when $z = 0$, we have

$$Z_k(n,0) = \frac{1}{n!} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^k} \zeta_n^\star(\{1\}_{m-1}).$$

PROOF. By Proposition 3, we have

$$Z_k(n,z) = \frac{1}{(n-1)!} \sum_{l=0}^{n-1} \binom{n-1}{l} \sum_{m=1}^{\infty} \frac{1}{m^k} \frac{(-1)^{l+m+1}}{(l+1+z)^m} =$$

$$= \frac{1}{(n-1)!} \sum_{m=1}^{\infty} \frac{1}{m^k} \sum_{l=0}^{n-1} \binom{n-1}{l} \sum_{j=0}^{\infty} \binom{m+j-1}{m-1} \frac{(-1)^{l+m+1}(-z)^j}{(l+1)^{m+j}} =$$

$$= \frac{1}{n!} \sum_{m=1}^{\infty} \frac{1}{m^k} \sum_{j=0}^{\infty} \binom{m+j-1}{m-1} \sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^{l+m}}{l^{m+j-1}} (-z)^j.$$

Here we use the identity

$$\frac{1}{(x+z)^m} = \frac{1}{x^m} \sum_{j=0}^{\infty} \binom{m+j-1}{m-1} \left(-\frac{z}{x}\right)^j$$

for $|z/x| < 1$. By Lemma 3, equation (17) is obtained. □

We put $T_k(s,z) := \Gamma(s) Z_k(s,z)$. This means that the gamma factor in (14) is removed. Then the following duality formula holds.

COROLLARY 3. *Let $k \geqslant 2$ and $r \geqslant 2$ be integers. For $0 \leqslant z < 1$, we have*

$$\sum_{n=1}^{\infty} \frac{T_k(n,z)}{n^r} = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{\xi_r(m, 1+z)}{m^k}. \tag{18}$$

PROOF. By Theorem 9, the left-hand side of (18) is equal to

$$\sum_{j=0}^{\infty} (-z)^j \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^k} \binom{m+j-1}{m-1} \zeta^\star(r+1, \{1\}_{m+j-1}) =$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^k} \sum_{j=0}^{\infty} (-z)^j \binom{m+j-1}{m-1} \xi_r(m+j).$$

Now we have

$$\sum_{j=0}^{\infty} (-z)^j \binom{m+j-1}{m-1} \xi_r(m+j) = \sum_{j=0}^{\infty} \frac{(-z)^j}{(m-1)!\, j!} \int_0^\infty \frac{t^{m+j-1} \mathrm{Li}_r(1-e^{-t})}{e^t - 1}\, dt =$$

$$= \frac{1}{(m-1)!} \int_0^\infty \frac{e^{-zt} t^{m-1} \mathrm{Li}_r(1-e^{-t})}{e^t - 1}\, dt =$$

$$= \frac{1}{(m-1)!} \int_0^\infty \frac{e^{-(1+z)t} t^{m-1} \mathrm{Li}_r(1-e^{-t})}{1-e^{-t}} \, dt =$$

$$= \xi_r(m, 1+z)$$

and this completes the proof. □

We remark that Corollary 3 itself can be proved by calculating

$$\int_0^\infty \frac{e^{-uz} \mathrm{Li}_r(1-e^{-u}) \mathrm{Lif}_k(-u)}{e^u - 1} \, du \tag{19}$$

in two ways, without using Theorem 9. In fact, by expanding $\mathrm{Lif}_k(-u)$, equation (19) is

$$\int_0^\infty \frac{e^{-uz} \mathrm{Li}_r(1-e^{-u})}{e^u - 1} \sum_{m=1}^\infty \frac{(-u)^{m-1}}{(m-1)! m^k} \, du =$$

$$= \sum_{m=1}^\infty \frac{(-1)^{m+1}}{m^k} \frac{1}{\Gamma(m)} \int_0^\infty \frac{e^{-uz} u^{m-1} \mathrm{Li}_r(1-e^{-u})}{e^u - 1} \, du$$

and this equals the right-hand side of (18). On the other hand, by expanding $\mathrm{Li}_r(1-e^{-u})$, equation (19) is

$$\int_0^\infty e^{-u(z+1)} \mathrm{Lif}_k(-u) \sum_{n=1}^\infty \frac{(1-e^{-u})^{n-1}}{n^r} \, du =$$

$$= \sum_{n=1}^\infty \frac{1}{n^r} \int_0^\infty e^{-u(z+1)} (1-e^{-u})^{n-1} \mathrm{Lif}_k(-u) \, du$$

and this equals the left-hand side of (18).

7. The second kind case

The method similar to that in the previous section can be applied for the poly-Cauchy polynomials of the second kind $\widehat{c}_n^{(k)}(z)$.

Lemma 4. *For $k \geqslant 0$, we have*

$$\mathrm{Lif}_k(z) \sim \frac{e^z}{z^k} \ (z \to \infty). \tag{20}$$

Proof. We prove this lemma by induction on k. Since $\mathrm{Lif}_0(z) = e^z$, the evaluation (20) holds for $k = 0$. We assume that (20) holds for some $k \geqslant 0$. It is obvious that $\lim_{z\to\infty} \mathrm{Lif}_k(z) = \infty$ for any $k \geqslant 0$. By definition, we have

$$\frac{d}{dz} z\mathrm{Lif}_k(z) = \mathrm{Lif}_{k-1}(z).$$

Hence, by l'Hopital's rule and the inductive assumption, we have

$$\begin{aligned}\lim_{z\to\infty} \frac{\mathrm{Lif}_{k+1}(z)}{e^z z^{-k-1}} &= \lim_{z\to\infty} \frac{z\mathrm{Lif}_k(z)}{e^z z^{-k}} = \\ &= \lim_{z\to\infty} \frac{\mathrm{Lif}_{k-1}(z)}{e^z z^{-k}(1 - k/z)} = \\ &= 1\end{aligned}$$

and (20) holds for $k + 1$. □

Let $k \geqslant 1$ be an integer and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. We assume that

$$\begin{cases} z < 0 & \text{if } k = 1, \\ z \leqslant 0 & \text{if } k \geqslant 2. \end{cases} \tag{21}$$

Then we define

$$\widehat{Z}_k(s, z) := \frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}}{(1-t)^z} \mathrm{Lif}_k(-\ln(1-t))\, dt,$$

or equivalently,

$$\widehat{Z}_k(s, z) = \frac{1}{\Gamma(s)} \int_0^\infty (1 - e^{-u})^{s-1} \mathrm{Lif}_k(u) e^{(z-1)u}\, du. \tag{22}$$

By Lemma 4, the integral of the right-hand side of (22) is convergent for $\mathrm{Re}(s) > 0$ under the assumption (21). The function $\widehat{Z}_k(s, z)$ satisfies properties similar to those of $Z_k(s, z)$. Since they are proved in exactly the same way, we state here only the results and omit their proofs.

PROPOSITION 4. *The function $\widehat{Z}_k(s, z)$ can be continued to an entire function, and its values at non-positive integers are given as follows:*

$$\widehat{Z}_k(-n, z) = \widehat{c}_n^{(k)}(z) \quad (n = 0, 1, 2, \ldots).$$

THEOREM 10. *For positive integers k and n, we have*

$$\widehat{Z}_k(n, z) = \frac{1}{(n-1)!} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l \mathrm{Li}_k\left(\frac{1}{l+1-z}\right).$$

THEOREM 11. *Let k and n be positive integers. For $-1 < z < 0$ or $-1 < z \leqslant 0$ according to $k = 1$ or $k \geqslant 2$, we have*

$$\widehat{Z}_k(n, z) = \frac{1}{n!} \sum_{m=1}^{\infty} \frac{1}{m^k} \sum_{j=0}^{\infty} \binom{m+j-1}{m-1} \zeta_n^{\star}(\{1\}_{m+j-1})\, z^j.$$

In particular, when $z = 0$, we have

$$\widehat{Z}_k(n, 0) = \frac{1}{n!} \sum_{m=1}^{\infty} \frac{1}{m^k} \zeta_n^{\star}(\{1\}_{m-1}) \quad (k \geqslant 2).$$

We put $\widehat{T}_k(s, z) = \Gamma(s)\widehat{Z}_k(s, z)$ Then we have the following duality formula.

COROLLARY 4. *Let $k \geqslant 2$ and $r \geqslant 2$ be integers. For $-1 < z \leqslant 0$, we have*

$$\sum_{n=1}^{\infty} \frac{\widehat{T}_k(n, z)}{n^r} = \sum_{m=1}^{\infty} \frac{\xi_r(m, 1-z)}{m^k}.$$

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