ISSN 2220-5438

Moscow Journal
of Combinatorics and Number Theory

URSS

Vol. 1 • Issue 1

2011

Moscow Journal of Combinatorics and Number Theory. 2011. Vol. 1. Iss. 1. 80 p.

The journal was founded in 2010

The aim of this journal is to publish original, high-quality research articles from a broad range of interests within combinatorics, number theory and allied areas. One volume of four issues is published annually.

Published by the Moscow Institute of Physics and Technology
with the support of Yandex and Microsoft.

Website
`http://mjcnt.phystech.edu`

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`mjcnt@phystech.edu`

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Российская Федерация,
141700

URSS Publishers
56, Nakhimovsky Prospekt,
Moscow,
Russia,
117335

Издательство «УРСС»
Нахимовский пр-т, 56
Москва,
Российская Федерация,
117335

Журнал зарегистрирован в Федеральной службе по надзору в сфере массовых коммуникаций, связи и охраны культурного наследия 3 сентября 2010 г. Свидетельство ПИ № ФС77-41900.

Формат 70 × 100/16. Печ. л. 5. Зак. № 4714.
Отпечатано в ООО «ЛЕНАНД».
117312, Москва, пр-т Шестидесятилетия Октября, 11А, стр. 11.

ISSN 2220-5438

10015 ID 123882
9 785453 000173

Moscow Journal
of Combinatorics
and Number Theory
2011. Vol. 1. Iss. 1.
PP. 67–78

Estimates for the exponent of irrationality for certain values of hypergeometric functions

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Abstract: We modify Nesterenko's integral construction to obtain bounds for irrationality measures for numbers of the form $\sqrt{2k+1}\ln\frac{\sqrt{2k+1}-1}{\sqrt{2k+1}+1}$, $k\in\mathbb{N}$. In particular we improve the previous result for the number $\sqrt{5}\ln\frac{\sqrt{5}-1}{2}$.

Keywords: irrationaliry measure, integral representation of linear forms, values of logarithms

AMS Subject classification: 11J82

Received: 27.12.2010; **revised:** 24.03.2011

1. Introduction

For any irrational α we can consider the exponent $\mu(\alpha)$ which characterizes the rate of convergence of rational approximations to α. This exponent is known as the irrationality measure for α. It is defined as the supremum of all real κ such that the inequality $\left|\alpha-\frac{p}{q}\right|<q^{-\kappa}$ has infinitely many solutions in $p,q\in\mathbb{Z}$.

One of the methods of finding upper bounds for $\mu(\alpha)$ is based on integral constructions of linear forms and investigating asymptotics of their coefficients. M. Hata [1], 2000 was the first to consider for this purpose a construction with a double complex integral. A similar integral construction was recently applied by R. Marcovecchio [2], 2009 for getting bounds for irrationality measures for the quantities $\ln\frac{a}{a+1}$, $a\in\mathbb{N}$, as well as for the quadratic irrationality measure for the same numbers (the quadratic irrationality measure μ_2 describes the order of approximation of a number by roots of quardatic equations with rational coefficients). Marcovecchio's method leads to many interesting results. In particular it gives the bound $\mu(\ln 2)\leqslant 3.5745\ldots$. This bound improves the result $\mu(\ln 2)\leqslant 3.8913\ldots$, which was obtained by E.A. Rukhadze [3], 1989. But Marcovecchio's construction was difficult enough. A little bit later Yu.V. Nesterenko [4] gave a different proof of Marcovecchio's result by means of a simpler inregral construction with a single complex integral.

A certain modification of Marcovecchio's integral and the corresponting integral by Nesterenko enables to apply the method to estimate irrationality measures for logarithms of certain quadratic irrationalities, namely for the values of Gauss hypergeometric functions $F\left(1, \frac{1}{2}, \frac{3}{2}; \frac{1}{2k+1}\right) = \frac{1}{2\sqrt{2k+1}} \ln \frac{\sqrt{2k+1}+1}{\sqrt{2k+1}-1}$, $k \in \mathbb{N}$, with half-integer parameters. Irrationality measures for such numbers were obtained earlier by different authors. Some general methods are due to M. Huttner [5], K. Väänänen, T. Matala-Aho and A. Heimonen [6]. Particular results are due to G. Rhin [7], M. Hata [8], E.S. Salnikova [9] and others. The method of modified Nesterenko's integral construction enables to improve some of the previous results from the paper cited above.

So for $k = 1$ we prove the inequality $\mu(\sqrt{3}\ln(2-\sqrt{3})) \leqslant 11.9185\ldots$, which is better than the previous bounds due to G. Rhin [7], and E.S. Salnikova [9], as well as the previous result by the author $\mu(\sqrt{3}\ln(2-\sqrt{3})) \leqslant 12.4518\ldots$, which was obtained in [10], by a different method. The best bound for the logarithm of the "golden ratio" $\mu(\sqrt{5}\ln\frac{\sqrt{5}-1}{2}) \leqslant 4.4562\ldots$, (case $k = 2$) was due to E.S. Salnikova [9]. She had a slight improvement of Hata's result from [8]. In the present paper we prove a new bound $\mu(\sqrt{5}\ln\frac{\sqrt{5}-1}{2}) \leqslant 3.7133\ldots$. Here we should note that for $k = 4$ we can deduce Marcovecchio's result for $\ln 2$.

2. Integral construction and its properties

Consider the integral

$$\widehat{Y}(x) = \frac{x^{-\frac{bn+1}{2}}}{2\pi i} \int\limits_{l} H(\varsigma)\left(\frac{\pi}{\sin\pi\varsigma}\right)^{3}(-x)^{-\varsigma}d\varsigma, \tag{1}$$

where

$$H(\varsigma) = \binom{\varsigma+(b-2a)n}{(b-4a)n}\binom{\varsigma+(b-a)n}{(b-2a)n}\binom{\varsigma+bn}{bn} =$$

$$= \frac{(\varsigma+2an+1)\ldots(\varsigma+(b-2a)n)}{((b-4a)n)!}\frac{(\varsigma+an+1)\ldots(\varsigma+(b-a)n)}{((b-2a)n)!}\frac{(\varsigma+1)\ldots(\varsigma+bn)}{(bn)!},$$

$a, b \in \mathbb{N}, b > 4a$, $bn+1$ is even, and $x \in \mathbb{C}, x \notin \{0, 1\}$, l is a vertical line of the form $\operatorname{Re}\varsigma = C$, where $-(b-2a)n < C < -2an-1$. We suppose that the line is parametrized from the bottom to the top. Suppose that $(-x)^{-\varsigma} = e^{-\varsigma\ln(-x)}$. We take the branch of the logarithm $\ln(-x) = \ln|x| + i\arg x + i\pi$ such that $-\pi \leqslant \arg x < \pi$. A similar construction was considered in [4]. Here we give a modification which adds a symmetry of the integral with respect to the substitution of x by $\frac{1}{x}$.

The integral (1) admits the following properties.

Proposition 1. *For any $x \in \mathbb{R}$, under the condition $x > 0$, $x \neq 1$ the following representation is valid*:

$$\widehat{Y}(x) = -\frac{1}{2}P(x)\ln^2 x + Q(x)\ln x - \frac{1}{2}R(x) - i\pi(P(x)\ln x - Q(x)), \qquad (2)$$

where the functions $P(x)$, $Q(x)$, $R(x)$ are defined as

$$P(x) = x^{-\frac{bn+1}{2}} \sum_{j=bn}^{3(b-2a)n} a_j \left(\frac{x}{x-1}\right)^{j+1},$$

$$a_j = \sum_{k=bn+1}^{j+1} (-1)^{k-1} H(-k) \binom{j}{k-1},$$

$$Q(x) = x^{-\frac{bn+1}{2}} \left(\sum_{k=1}^{an} H'(-k)x^k + \sum_{j=0}^{3(b-2a)n-1} b_j \left(\frac{x}{x-1}\right)^{j+1} \right),$$

$$b_j = \sum_{k=1}^{j+1} (-1)^{k-1} H'(-k) \binom{j}{k-1},$$

$$R(x) = x^{-\frac{bn+1}{2}} \left(\sum_{k=1}^{2an} H''(-k)x^k + \sum_{j=0}^{3(b-2a)n-2} c_j \left(\frac{x}{x-1}\right)^{j+1} \right),$$

$$c_j = \sum_{k=1}^{j+1} (-1)^{k-1} H''(-k) \binom{j}{k-1}.$$

The proof of this Proposition is quite similar to the proof of Proposition 1 from [4]. A more detailed argument can be found in [11].

Proposition 2. *The functions $P(x)$, $Q(x)$, $R(x)$ defined in* 1 *satisfy the equalities*

$$P(x) = P\left(\frac{1}{x}\right), \quad Q(x) = -Q\left(\frac{1}{x}\right), \quad R(x) = R\left(\frac{1}{x}\right).$$

Proof. From [4], [11] we know that

$$P(x) = -x^{-\frac{bn+1}{2}} \sum_{k=bn+1}^{\infty} x^k H(-k),$$

$$Q(x) = -x^{-\frac{bn+1}{2}} \sum_{k=(b-a)n+1}^{\infty} x^k H'(-k),$$

$$R(x) = -x^{-\frac{bn+1}{2}} \sum_{k=(b-2a)n+1}^{\infty} x^k H''(-k).$$

Consider the integral $\widehat{Y}\left(\frac{1}{x}\right)$.

$$\widehat{Y}\left(\frac{1}{x}\right)=\frac{x^{\frac{bn+1}{2}}}{2\pi i}\int\limits_{l} H(\varsigma)\left(\frac{\pi}{\sin\pi\varsigma}\right)^{3}\left(-\frac{1}{x}\right)^{-\varsigma}d\varsigma=\begin{bmatrix}\varsigma=-\xi-bn-1\\ d\varsigma=-d\xi\end{bmatrix}=$$

$$=\frac{x^{\frac{bn+1}{2}}}{2\pi i}\int\limits_{l_1} H(-\xi-bn-1)\left(\frac{\pi}{\sin\pi(-\xi-bn-1)}\right)^{3}\left(-\frac{1}{x}\right)^{\xi+bn+1}(-d\xi)=$$

$$=\frac{x^{-\frac{bn+1}{2}}}{2\pi i}\int\limits_{l_1} H(-\xi-bn-1)\left(\frac{\pi}{\sin\pi(-\xi-bn-1)}\right)^{3}\left(-\frac{1}{x}\right)^{\xi}(-d\xi).$$

Here l_1 is the vertical line $\operatorname{Re}\xi=-C-bn-1$,parametrized from the top to the bottom. As $2an+1<-C<(b-2a)n$ we have $(-b+2a)n<\operatorname{Re}\xi<-2an-1$. So we may suppose that l_1 coincides with l but is parametrized in the opposite direction. Moreover,

$$\begin{aligned}&H(-\xi-bn-1)=\\ &=\frac{(-\xi-(b-2a)n)\dots(-\xi-2an-1)}{((b-4a)n)!}\frac{(-\xi-bn)\dots(-\xi-1)}{(bn)!}\times\\ &\quad\times\frac{(-\xi-(b-a)n)\dots(-\xi-an-1)}{((b-2a)n)!}=\\ &=(-1)^{(3b-6a)n}H(\xi)=-H(\xi).\end{aligned}$$

As $\sin\pi(k+x)=(-1)^k\sin(\pi x)$ for $k\in\mathbb{Z}$, we see that

$$\frac{\pi}{\sin(\pi(-\xi-bn-1))}=-(-1)^{bn+1}\frac{\pi}{\sin\pi\xi}=-\frac{\pi}{\sin\pi\xi}.$$

By changing the direction of the line l_1 we get

$$\widehat{Y}\left(\frac{1}{x}\right)=\frac{x^{-\frac{bn+1}{2}}}{2\pi i}\int\limits_{l} H(\varsigma)\left(\frac{\pi}{\sin\pi\varsigma}\right)^{3}\left(-\frac{1}{x}\right)^{\varsigma}d\varsigma.$$

One can easily see that

$$\widehat{Y}\left(\frac{1}{x}\right)=x^{-\frac{bn+1}{2}}\sum_{k=(b-2a)n+1}^{\infty}Res_{\varsigma=-k}\left(H(\varsigma)\left(\frac{\pi}{\sin\pi\varsigma}\right)^{3}\left(-\frac{1}{x}\right)^{\varsigma}\right).$$

Let us find the residues in the points$\varsigma=-k$.

The following equalities are valid in certain neighborhoods of these points:

$$\left(\frac{\pi}{\sin\pi\varsigma}\right)^{3}=(-1)^{k}\left(\frac{1}{(\varsigma+k)^{3}}+\frac{\pi^{2}}{2(\varsigma+k)}+O(1)\right),$$

$$H(\varsigma) = H(-k) + H'(-k)(\varsigma + k) + \frac{H''(-k)}{2}(\varsigma + k)^2 + O((\varsigma + k)^3),$$

$$\left(-\frac{1}{x}\right)^{\varsigma} = e^{(\varsigma+k)(-\ln x+\pi i)}(-1)^k x^k =$$

$$= (-1)^k x^k (1 + (-\ln x + \pi i)(\varsigma + k) + \frac{1}{2}(-\ln x + \pi i)^2(\varsigma + k)^2 + O((\varsigma + k)^3)).$$

So

$$Res_{\varsigma=-k}\left(x^{-\frac{bn+1}{2}} H(\varsigma)\left(\frac{\pi}{\sin\pi\varsigma}\right)^3\left(-\frac{1}{x}\right)^{\varsigma}\right) =$$

$$= x^{k-\frac{bn+1}{2}}\left(H'(-k)(-\ln x+\pi i)+\frac{\pi^2}{2}H(-k)+H(-k)\frac{1}{2}(-\ln x+\pi i)^2+\frac{H''(-k)}{2}\right).$$

We take into account the properties of $H(\varsigma)$ and its derivatives to obtain

$$\widehat{Y}\left(\frac{1}{x}\right) = x^{-\frac{bn+1}{2}}\left(\frac{1}{2}\ln^2 x\cdot\sum_{k=bn+1}^{\infty} x^k H(-k) - \ln x\cdot\sum_{k=(b-a)n+1}^{\infty} x^k H'(-k)+\right.$$

$$\left.+\sum_{k=(b-2a)n+1}^{\infty} x^k\frac{H''(-k)}{2}+\pi i\left(-\ln x\cdot\sum_{k=bn+1}^{\infty} x^k H(-k)+\sum_{k=(b-a)n+1}^{\infty} x^k H'(-k)\right)\right) =$$

$$= -\frac{1}{2}P(x)\ln^2 x + Q(x)\ln x - \frac{1}{2}R(x) + \pi i(P(x)\ln x - Q(x)).$$

On the other hand, from (2) we have

$$\widehat{Y}\left(\frac{1}{x}\right) = -\frac{1}{2}P\left(\frac{1}{x}\right)\ln^2 x - Q\left(\frac{1}{x}\right)\ln x - \frac{1}{2}R\left(\frac{1}{x}\right) - \pi i\left(-P\left(\frac{1}{x}\right)\ln x - Q\left(\frac{1}{x}\right)\right).$$

As $\ln x$ is a transcentental function we see that

$$P(x) = P\left(\frac{1}{x}\right), \quad Q(x) = -Q\left(\frac{1}{x}\right), \quad R(x) = R\left(\frac{1}{x}\right).$$

This is just what we need. □

As we can see from this statement the integral (1) satisfies the equality

$$\widehat{Y}\left(\frac{1}{x}\right) = \overline{\widehat{Y}(x)}.$$

For more detailed properties of the functions under consideration we need the following simple lemma. This lemma was used in [10].

LEMMA 1. *Suppose that* $S(x) \in \mathbb{Q}(x)$.
If $S(x) \equiv S\left(\frac{1}{x}\right)$, *then* $S(x) = \widehat{S}(x + \frac{1}{x})$, *where* $\widehat{S}(t) \in \mathbb{Q}(t)$.
If $S(x) \equiv -S\left(\frac{1}{x}\right)$, *then* $S(x) = (x - \frac{1}{x})\widehat{S}(x + \frac{1}{x})$, *where* $\widehat{S}(t) \in \mathbb{Q}(t)$.

Now from Proposition 2 and Lemma 1, one can see that

$$P(x)=\widehat{P}\left(x+\frac{1}{x}\right),\quad Q(x)=\left(x-\frac{1}{x}\right)\widehat{Q}\left(x+\frac{1}{x}\right),\quad R(x)=\widehat{R}\left(x+\frac{1}{x}\right),$$

where $\widehat{P}(t),\widehat{Q}(t),\widehat{R}(t)\in\mathbb{Q}(t)$. So if we consider a parameter x such that $x+\dfrac{1}{x}\in\mathbb{Q}$, then $P(x)$, $\dfrac{Q(x)}{x-1/x}$, $R(x)\in\mathbb{Q}$. We also consider the quantities

$$x=\frac{k+1-\sqrt{2k+1}}{k},\quad k\in\mathbb{N}. \tag{3}$$

For these quantities we have

$$\frac{1}{x}=\frac{k+1+\sqrt{2k+1}}{k},\quad x+\frac{1}{x}=\frac{2k+2}{k}\in\mathbb{Q},\quad x-\frac{1}{x}=-\frac{2\sqrt{2k+1}}{k},$$

and

$$\ln x=\ln\frac{\sqrt{2k+1}-1}{\sqrt{2k+1}+1}.$$

The equality (2) turns into

$$\begin{aligned}\widehat{Y}(x)=-\widetilde{p}_n\ln^2\frac{\sqrt{2k+1}-1}{\sqrt{2k+1}+1}+\widetilde{q}_n\sqrt{2k+1}\ln\frac{\sqrt{2k+1}-1}{\sqrt{2k+1}+1}-\\-\widetilde{r}_n-i\pi\left(\widetilde{p}_n\ln\frac{\sqrt{2k+1}-1}{\sqrt{2k+1}+1}-\sqrt{2k+1}\widetilde{q}_n\right),\end{aligned} \tag{4}$$

where $\widetilde{p}_n,\widetilde{q}_n,\widetilde{r}_n\in\mathbb{Q}$.

PROPOSITION 3. *Suppose that* x *is of the form* (3). *Then the following statements hold.*

1. *If* $k=2l$, *then* $2l^Nx^N, 2l^N\dfrac{1}{x^N}\in\mathbb{Z}[\sqrt{2k+1}]$ *and* $2\cdot\left(\dfrac{x}{x-1}\right)^N\in\mathbb{Z}[\sqrt{2k+1}]$;
2. *If* $k=2l+1$, *then* $k^Nx^N, \dfrac{k^N}{x^N}\in\mathbb{Z}[\sqrt{2k+1}]$ *and* $2^{[\frac{N}{2}]+1}\cdot\left(\dfrac{x}{x-1}\right)^N\in\mathbb{Z}[\sqrt{2k+1}]$.

PROOF. For x of the form (3) we have $k^Nx^N, \dfrac{k^N}{x^N}\in\mathbb{Z}[\sqrt{2k+1}]$ for all k. For $k=2l$ we have $x=\dfrac{1}{l}\left(\dfrac{\sqrt{4l+1}-1}{2}\right)^2=\dfrac{1}{l}\widetilde{x}^2$. The number $\widetilde{x}$ is a root of the equation $t^2+t-l=0$, so $\widetilde{x}^N=a_N\widetilde{x}+b_N, a_N,b_N\in\mathbb{Z}$ and $2\widetilde{x}^N\in\mathbb{Z}[\sqrt{4l+1}]$, that is $2 \cdot l^N x^N \in \mathbb{Z}[\sqrt{2k+1}]$. For the number $\frac{1}{x}$ we have a similar statement as $\frac{1}{x} = \frac{1}{l}\left(\frac{\sqrt{2k+1}+1}{2}\right)^2$.

We have similar statements for $\frac{x}{x-1} = \frac{1-\sqrt{2k+1}}{2} = t$ also. These numbers are roots of the equations $t^2 - t - \frac{k}{2} = 0$. So for $k = 2l$ we have $2t^N \in \mathbb{Z}[\sqrt{2k+1}]$. For $k = 2l+1$ we have

$$\left(\frac{1-\sqrt{2k+1}}{2}\right)^2 = \frac{k+1-\sqrt{2k+1}}{2} = \frac{2(l+1)-\sqrt{4l+3}}{2}.$$

So for $M \in \mathbb{N}$ we see that

$$2^M \cdot \left(\frac{1-\sqrt{2k+1}}{2}\right)^{2M} = (k+1-\sqrt{2k+1})^M \in \mathbb{Z}[\sqrt{2k+1}]$$

and

$$2^{M+1}\left(\frac{1-\sqrt{2k+1}}{2}\right)^{2M+1} = (1-\sqrt{2k+1}) \cdot 2^M \left(\frac{k+1-\sqrt{2k+1}}{2}\right)^M \in \mathbb{Z}[\sqrt{2k+1}].$$

This is just what we need. □

Do deduce the bounds for irrationality measures we need the following lemma due to M. Hata (see [12], Proposition 2.1).

LEMMA 2. *Suppose that $n \in \mathbb{N}, \gamma \in \mathbb{R}$, and γ is irrational. Suppose that $l_n = q_n\gamma + p_n$ where $q_n, p_n \in \mathbb{Z}$ and*

$$\lim_{n\to\infty} \frac{1}{n} \ln |q_n| = \sigma, \quad \limsup_{n\to\infty} \frac{1}{n} \ln |l_n| \leqslant -\tau, \quad \sigma, \tau > 0. \tag{5}$$

Then $\mu(\gamma) \leqslant 1 + \frac{\sigma}{\tau}$.

The equality (2) for every x gives us linear forms

$$\frac{\operatorname{Im}(\widehat{Y}(x))}{\pi} = -P(x) \ln x + Q(x), \tag{6}$$

$$\operatorname{Re}(\widehat{Y}(x)) - \frac{\operatorname{Im}(\widehat{Y}(x))}{\pi} \ln x = \frac{1}{2}P(x) \ln^2 x - \frac{1}{2}R(x). \tag{7}$$

The simultaneous consideration of these linear forms leads to bounds for irrationality measures for $\ln x$, as it was done in [2], [11]. But for our detailed consideration of irrationality measures we only need the first linear form.
For x from (3) we get

$$\frac{\sqrt{2k+1}\operatorname{Im}(\widehat{Y}(x))}{\pi} = \widehat{p}_n\sqrt{2k+1} \ln \frac{\sqrt{2k+1}-1}{\sqrt{2k+1}+1} + \widehat{q}_n, \quad \widehat{p}_n, \widehat{q}_n \in \mathbb{Q}.$$

In order to apply Lemma 2 we need to transform this linear form into a form with integral coefficients, We shall deduce an estimate for the denominators of the coefficients using the explicit representation for the functions $P(x), Q(x), R(x)$ from Proposition 1.

We use notation $[x]$ and $\{x\}$ for the integer part and for the fractional part of x, respectively. Let d_n be the least common multiple of the numbers $1, 2, \dots, n$.

Lemma 3. *Suppose that* $m \in \mathbb{N}$, $A_m(x) = \dfrac{(x+1)\dots(x+m)}{m!}$. *Then for any* $k \in \mathbb{Z}$ *one has*

$$d_m A_m'(k) \in \mathbb{Z}.$$

A proof of this lemma one can find in [4] (Lemma 3).

Corollary 1. *For any* $k \in \mathbb{Z}$ *one has* $d_{bn} H'(k) \in \mathbb{Z}$.

By Ω we denote the set of y from the interval $0 \leqslant y < 1$ such that for any x the inequality

$$\begin{gathered}([x-2ay]-[x-(b-2a)y]-[(b-4a)y])+ \\ +([x-ay]-[x-(b-a)y]-[(b-2a)y])+([x]-[x-by]-[by]) \geqslant 1 \quad (8)\end{gathered}$$

is valid. The set Ω is the union of certain segments, half-intervals and points. For given values of parameters a, b we find this set explicitly, by a computer program. Define Δ to be the product of all prime numbers $p > \sqrt{3(b-2a)n}$ such that $\left\{\dfrac{n}{p}\right\} \in \Omega$.

Lemma 4. *For any* $k \in \mathbb{N}$, $k \leqslant 3(b-2a)n$, *one has* $\dfrac{d_{bn}}{\Delta} H'(k) \in \mathbb{Z}$.

Now from Propositions 1,3 and Lemma 4 as a simple corollary we deduce the following

Proposition 4. *Suppose that* x *is of the form* (3). *Then*

1. *If* $k = 2l$, $l \in \mathbb{N}$, *then* $2 \cdot l^{\frac{bn+1}{2}} P(x)$, $\quad 2 \cdot l^{\frac{bn+1}{2}} \cdot \dfrac{d_{bn}}{\Delta} \sqrt{4l+1}\, l\, Q(x) \in \mathbb{Z}$.

2. *If* $k = 2l+1$, $l \in \mathbb{N}_0$, *then* $2^{[\frac{3(b-2a)n+1}{2}]} \cdot (2l+1)^{\frac{bn+1}{2}} P(x)$, $2^{[\frac{3(b-2a)n+1}{2}]} \cdot (2l+1)^{\frac{bn+1}{2}} \cdot \dfrac{d_{bn}}{\Delta} \sqrt{4l+3}(2l+1) Q(x) \in \mathbb{Z}$.

Here we should note that for odd $k > 1$ the denominators are large enough, and this does not allow obtaining any bound for the corresponding quantities.

Now we get asymptotics for the coefficients of the linear form (6). As we have $\left|\operatorname{Im}(\widehat{Y}(x))\right| \leqslant \left|\widehat{Y}(x)\right|$, it is enough to calculate the values $\lim\limits_{n\to\infty} \dfrac{1}{n} \ln |P(x)|$ and $\limsup\limits_{n\to\infty} \dfrac{1}{n} \ln \left|\widehat{Y}(x)\right|$. We use the method from [4]. Here we formulate the main conclusions from this approach.

Lemma 5. *Suppose that* $x \in \mathbb{R}, 0 < x < 1$. *Suppose that the equation*

$$\frac{t(t-a)(t-2a)}{(t-(b-2a))(t-(b-a))(t-b)} = \frac{1}{x} \tag{9}$$

has only one real root $x_0 > 0$. *Then*

$$\lim_{n\to\infty} \frac{1}{n} \ln |P(x)| = M =$$

$$= \ln \left(\frac{(x_0-(b-2a))^{b-2a}(x_0-(b-a))^{b-a}(x_0-b)^b}{(b-4a)^{b-4a}(b-2a)^{b-2a}b^b(x_0-a)^a(x_0-2a)^{2a}} \right) - \frac{b}{2} \ln x. \tag{10}$$

Lemma 6. *Denote*

$$m = \ln \frac{|z_0+(b-2a)|^{b-2a}|z_0+(b-a)|^{b-a}|z_0+b|^b}{(b-4a)^{b-4a}(b-2a)^{b-2a}b^b|z_0+a|^a|z_0+2a|^{2a}} - \frac{b}{2} \ln x,$$

where z_0 *is a complex root of the equation*

$$\frac{t(t+a)(t+2a)}{(t+(b-2a))(t+(b-2a))(t+b)} = \frac{1}{x}, \tag{11}$$

under the condition $\operatorname{Im} z_0 > 0$. *If* $m < 0$, *then one has*

$$\limsup_{n\longrightarrow\infty} \frac{1}{n} \ln |\widehat{Y}(x)| \leqslant m.$$

The following lemma dealing with the calculation of asymptotics of denominators is considered in the modern literature as a classical statement.

Lemma 7. *Suppose that real numbers* u, v *satisfy the inequalities* $0 < u < v < 1$. *Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{u \leqslant \{\frac{n}{p}\} < v} \ln p = \psi(v) - \psi(u), \tag{12}$$

where $\psi(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$ *is the logarithmic derivative of gamma-function and the summation is over all the primes* p *under the condition that the fractional part* $\left\{\dfrac{n}{p}\right\}$ *satisfies the inequality written under the sign of summation.*

One can find a proof of this lemma in [4] (Lemma 6).

3. Main results

We apply Lemma 2 to the linear form (6). For odd k we have a result for $k = 1$ only.

Theorem 1. *We have the following bound for the irrationality measure*:

$$\mu(\sqrt{3}\ln(2-\sqrt{3})) \leqslant 11.918524\ldots$$

PROOF. Take $k = 1$. Then $x = \beta = 2 - \sqrt{3}$, $x + \dfrac{1}{x} = 4 \in \mathbb{Z}$ and $\dfrac{x}{1-x} = \dfrac{\sqrt{3}-1}{2}$. Put $a = 1, b = 7$. The same parameters we used in Nesterenko's integral construction We use computer for optimal choice of a, b. The values above give the best result.

The representation (6), Lemma 4 and Proposition 4 lead to the equality

$$2\frac{\sqrt{3}\cdot 2^{[\frac{15n+1}{2}]}d_{7n}}{\Delta}\frac{\operatorname{Im}(\widehat{Y}(\beta))}{\pi} = p_n\sqrt{3}\ln(2-\sqrt{3}) + q_n,$$

where

$$p_n = -\frac{2\cdot 2^{[\frac{15n+1}{2}]}d_{7n}P(\beta)}{\Delta} \in \mathbb{Z}, \quad q_n = \frac{2\sqrt{3}\cdot 2^{[\frac{15n+1}{2}]}d_{7n}Q(\beta)}{\Delta} \in \mathbb{Z}.$$

Now we find the asymptopics for the integral and the function $P(x)$. The equation (9) turns into

$$-2.73205t^3 + 64.17691t^2 - 397.32944t + 783.73066 = 0.$$

The unique real root of this equation is $x_0 = 15.13248$. So by Lemma 5 we have

$$\lim_{n\to\infty}\frac{1}{n}\ln|P(\beta)| = M = 6.57772 - \frac{7}{2}\ln\beta = 11.36712.$$

The equation (11) can be derived analogously. The corresponding complex root is $z_0 = -4.17894 + 1.22201i$. Then in accordance with Lemma 6 we have

$$\limsup_{n\to\infty}\frac{1}{n}\ln|\widehat{Y}(\beta)| \leqslant -\frac{7}{2}\ln\beta - 13.51369 = -8.90433 = m.$$

The set Δ for the parameters a, b under consideration is of the form $\Delta = \left[\frac{1}{6}, \frac{3}{7}\right) \cup \left[\frac{1}{2}, \frac{5}{7}\right) \cup \left[\frac{3}{4}, \frac{6}{7}\right)$. So by Lemma 7 we have

$$\lim_{n\to\infty}\frac{1}{n}\ln\Delta = \Psi\left(\frac{3}{7}\right) - \Psi\left(\frac{1}{6}\right) + \Psi\left(\frac{5}{7}\right) - \Psi\left(\frac{1}{2}\right) + \Psi\left(\frac{6}{7}\right) - \Psi\left(\frac{3}{4}\right) = 4.995102\ldots$$

As from the prime number theorem one has

$$\lim_{n\to\infty}\frac{1}{n}\ln d_{An} = A, \quad A \in \mathbb{N},$$

we deduce that

$$\lim_{n\to\infty}\frac{1}{n}\ln d_{7n}\cdot 2^{[\frac{15n+1}{2}]} = 7 + 7.5\ln 2 = 12.19860\ldots$$

By Lemma 2 we get the announced result:

$$\mu(\sqrt{3}\ln(2-\sqrt{3})) \leqslant 1 - \frac{11.36712 + 12.1986 - 4.99510}{-8.90434 + 12.1986 - 4.99510} = 11.918524\ldots$$

□

It is possibe to obtain a generalization for even values of k.

THEOREM 2. *Suppose that* $a, b \in \mathbb{N}$ *satisfy* $b > 4a$. *Suppose that* x *is of the form* (3) *for* $k = 2l$. *Suppose that numbers* M, m *are defined as in Lemmas* 5, 6, *the set* Ω *is defined in* (8) *and* $M_0 = \lim\limits_{n\to\infty} \frac{1}{n} \ln \Delta$.

If $m + c_0 - M_0 < 0$, *where* $c_0 = b + \frac{b}{2} \ln l$, *then the following inequality is valid*:

$$\mu\left(\sqrt{2k+1} \ln \frac{\sqrt{2k+1}-1}{\sqrt{2k+1}+1}\right) \leqslant 1 - \frac{M + c_0 - M_0}{m + c_0 - M_0}. \tag{13}$$

The proof of Theorem 2 follows from Propositions 1, 4 and Lemma 2.

Here we mention several results obtained from formula (13). It turned out that the values of parameters $a = 1$, $b = 7$ are optimal for $\ln 2$ in Nesterenko's paper and in Theorem 1 above. So in the Table below we show not the optimal result only but the result for $a = 1, b = 7$ also.

k	x	a	b	the best result	the result with $a = 1$, $b = 7$
2	$\frac{3-\sqrt{5}}{2}$	1	7	3.71331..	–
6	$\frac{7-\sqrt{13}}{6}$	4	37	11.826..	12.139..
8	$\frac{9-\sqrt{17}}{8}$	4	39	18.937..	20.929..

The most important results occur for $k = 2$. They contain essential improvements of previous bounds. Here we should note that in this case the bound

$$\mu\left(\sqrt{5} \ln \frac{\sqrt{5}-1}{2}\right) \leqslant 3.71331\ldots$$

is also obtained with $a = 1, b = 7$.

The additional factor $x^{-\frac{bn+1}{2}}$ leads to worse asymptotics when k increases. So further results are not as good as the results known before. The method of estimating the irrationality measures for the numbers $F\left(1, \frac{1}{2}, \frac{3}{2}; \frac{1}{s}\right)$ from the paper [6] in particular gives the bound $\mu\left(\sqrt{13} \ln \frac{\sqrt{13}-1}{\sqrt{13}+1}\right) \leqslant 3.86\ldots$. When s increases the bounds of this method decrease. In our construction the situation is opposite: as k increases the bounds increase very rapidly.

The simultaneous consideration of the forms (6), (7) enables to deduce estimates for quadratic irrationality measure for $\ln x$. In this situation the only result obtained is that for $x = \dfrac{3-\sqrt{5}}{2}$ with no growth of denominator we have $\mu_2\left(\sqrt{5}\ln\dfrac{\sqrt{5}-1}{2}\right) \leqslant 33.009\ldots$. A detailed proof of this bound by the means of Marcovecchio's construction is given in [13].

Acknowledgements. This research is supported by Russian Foundation for Basic Researchs № 09-01-00743. The author is grateful to her superviser Professor V. Kh. Salikhov for the constant interest to her work, as well as to Yu. V. Nesterenko and N. G.Moshchevitin for useful discussion.

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