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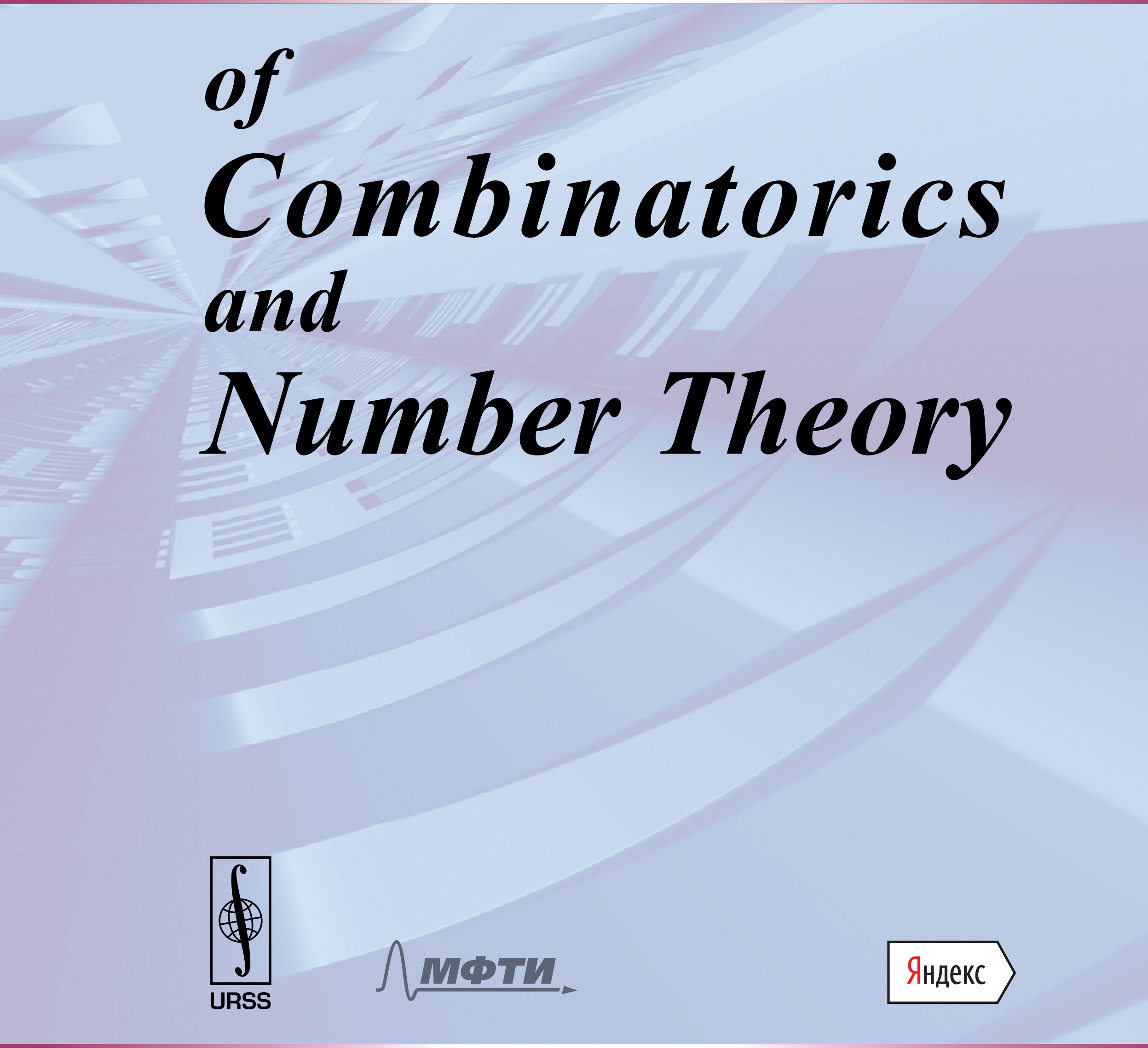

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New limiting distributions for the Möbius function

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Abstract: We establish some new limiting theorems for a family of signed distributions supported on the set of square-free numbers. These distributions are derived from the study of the Möbius function and they give nontrivial generalizations of the Dickman—de Bruijn distribution to the signed valued case. A special feature is the appearance of several singularities in the limiting distribution. This leads to a surprising change of limiting behavior depending on the smoothness of the test functions.

Keywords: Moebius function, limit theorems, signed measures, generalized Dickman-de Bruijn distributions

AMS Subject classification: 11K65, 11N37, 28A33, 60F05

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1. Introduction

The Möbius function is defined on the set of natural numbers by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ different prime factors,} \\ 0 & \text{if } n \text{ is not square-free.} \end{cases} \tag{1.1}$$

In recent works [6–8], Cellarosi and Sinai studied several problems related to the distribution of $\mu^2(n)$, the binary correlations and some ergodic properties. For this purpose they consider a probabilistic model for square-free numbers, and provide limit theorems for several random variables defined on the ensemble. To fix the notations, we take any integer $N \geq 2$ and denote by $\Omega(N)$ the ensemble whose elements are $\xi = \{q_1, \dots, q_s\}$ with $1 < q_1 < \dots < q_s < N$, q_j are prime. There exists a natural bijection between $\Omega(N)$ and the set of square-free numbers $\mathcal{Q}$ whose prime factors are less than N, namely for any $Q = q_1 \cdot \dots \cdot q_s$ we identify it with $\xi = \{q_1, \dots, q_s\}$ and vice versa. For later purposes we also introduce the restricted ensemble defined by

$$\Omega_{q_0,N}(N) = \big\{\xi = \{q_1, \dots, q_s\} : 1 < q_1 < \dots < q_s < N, q_j \text{ are prime and } q_j \neq q_0\big\}. \quad (1.2)$$

Here q_0 is a prime number. In other words we just remove the prime q_0 from the ensemble.

Define a measure $\widetilde{u_N^1}$ (we use the superscript 1 here so as not to confuse with the notations later on) on $\Omega(N)$ so that

$$\widetilde{u_N^1}(\xi) = \frac{1}{q_1 \cdot \dots \cdot q_s}. \quad (1.3)$$

Equivalently, we can also write

$$\widetilde{u_N^1}(Q) = \frac{1}{Q}, \quad \mathrm{Q} = q_1 \cdot \dots \cdot q_s, \; q_i < N.$$

The total mass of the measure $\widetilde{u_N^1}$ has the form

$$Z_N = \sum_{Q \in \mathcal{Q}:\, q_i < N} \frac{1}{Q} = \prod_{q<N} \left(1 + \frac{1}{q}\right) = \frac{e^\gamma}{\zeta(2)} \log N + o(\log N), \quad (1.4)$$

where in the last equality γ is the Euler—Mascheroni constant and log denotes the natural logarithm. Here we have used the classical Mertens product formula [14]. Define the normalized measure u_N^1 by

$$u_N^1(\xi) = \frac{1}{Z_N} \widetilde{u_N^1}(\xi), \quad \xi \in \Omega(N).$$

For any random variable $f : \Omega(N) \to \mathbb{R}$, introduce the expectation

$$\mathbb{E}_N f = \sum_{\xi \in \Omega(N)} f(\xi) u_N^1(\xi)$$

and the characteristic function

$$\varphi_f(\lambda) = \mathbb{E}_N e^{i\lambda f}.$$

Consider random variables:

$$\eta_N(\xi) = \begin{cases} \displaystyle\sum_{j=1}^{s} \frac{\log q_j}{\log N}, & \xi = \{q_1, \ldots, q_s\}; \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently, we can define $\eta_N(Q) := \dfrac{\log Q}{\log N}$ for $Q \in \mathcal{Q}$ and 0 for not square-free Q.

Let $\mathbb{P}_N$ be the distribution on $\mathbb{R}$ corresponding to η_N on $\left(\Omega(N), u_N^1\right)$. We recall the following result proven in [6]:

THEOREM 1.1 (CELLAROSI—SINAI [6]). *As $N \to \infty$ the probability distribution $\mathbb{P}_N$ converges weakly to a probability distribution $\mathbb{P}$ whose characteristic function $\varphi(\lambda)$ has the form*

$$\varphi(\lambda) = \exp\left\{ \int_0^1 \frac{e^{i\lambda v} - 1}{v} dv \right\}.$$

The distribution in Theorem 1.1 is known as the Dickman—de Bruijn distribution and it first appeared in the theory of smooth numbers (cf. [2–4, 9]). For some computational aspects of the smooth numbers one can refer to the recent survey article by Granville [10] (see also Hildebrand and Tenenbaum [12]). In a series of papers, Tenenbaum and Wu [17–19] (see also Hanrot, Tenenbaum and Wu [11]) investigated the partial sums of general positive-mean-valued multiplicative arithmetic functions over certain smooth numbers.

The purpose of this paper is to establish some analogues of Theorem 1.1 when the measures in (1.3) are replaced by some special signed measures. This problem originates from our study of the Möbius function and the Mertens function. Recall that the Mertens function is the partial sum of the Möbius function given by

$$M(N) = \sum_{n \le N} \mu(n).$$

Our analysis is based on the following Euler-type product for $\mu(n)$

$$\mu(n) = \prod_{\substack{p \ge 2, \\ p \text{ is prime}}} (1 - 2\chi_p(n) + \chi_{p^2}(n)). \tag{1.5}$$

Here χ_p and χ_{p^2} are indicator functions defined by:

$$\chi_p(n) = \begin{cases} 1, & \text{if } p|n, \\ 0, & \text{otherwise;} \end{cases}$$

$$\chi_{p^2}(n) = \begin{cases} 1, & \text{if } p^2|n, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that the definition (1.5) coincides with the standard definition (1.1). Alternatively, one can think of (1.5) as expressing (1.1) through an inclusion-exclusion principle. Note that for any given n the product in formula (1.5) extends over only finitely many primes so there is no issue of convergence.

Opening the brackets in (1.5), we have

$$\begin{aligned} \mu(n) = & \sum_{\substack{P,Q \\ P \cap Q = \varnothing}} (-2)^{\omega(Q)} \chi_{P^2 Q}(n) = \\ & = 1 + \sum_{Q \ne \varnothing} (-2)^{\omega(Q)} \chi_Q(n) + \sum_{\substack{P \ne \varnothing, \\ P \cap Q \ne \varnothing}} (-2)^{\omega(Q)} \chi_{P^2 Q}(n). \end{aligned} \tag{1.6}$$

Here $P,\ Q \in \mathcal{Q}$. For $Q = q_1 \cdot \ldots \cdot q_s$, $\omega(Q) = s$ denotes the number of distinct prime factors of Q. The notation $P \cap Q \ne \varnothing$ means that P and Q have no common prime factors. Similarly, $P \ne \varnothing$ simply means $P \ne 1$ or P has at least one prime in it.

Our notation here is consistent with the identification map which was mentioned earlier, namely for any $Q = q_1 \cdot \ldots \cdot q_s$ we write (by a slight abuse of notation) $Q = \xi = \{q_1, \ldots, q_s\}$. The indicator function $\chi_{P^2Q}(n)$ is defined as

$$\chi_{P^2Q}(n) = \begin{cases} 1, & \text{if } P^2Q|n, \\ 0, & \text{otherwise.} \end{cases}$$

The notation χ_Q is similarly defined.

By using the decomposition (1.6) for $\mu(n)$, we then have the following expression for the Mertens function $M(N) = \sum_{n \le N} \mu(n)$:

$$\begin{aligned} \frac{M(N)}{N} = \frac{1}{N}\sum_{n\le N}\mu(n) &= 1 + \frac{1}{N}\sum_{n\le N}\sum_{Q\ne\varnothing}(-2)^{\omega(Q)}\chi_Q(n) + \text{ terms involving } P \;= \\ &= 1 + \frac{1}{N}\sum_{\substack{Q\ne\varnothing,\\ Q\le N}}(-2)^{\omega(Q)}\left[\frac{N}{Q}\right] + \text{ terms involving } P = \\ &= 1 + \sum_{\substack{Q\ne\varnothing,\\ Q\le N}}\frac{(-2)^{\omega(Q)}}{Q} + \text{ other terms.} \end{aligned} \tag{1.7}$$

Here in the last equality we have replaced the integer part of N/Q by $N/Q - \{N/Q\}$ and put the fractional part $\{N/Q\}$ into "other terms". The main part of (1.7) is the series

$$1 + \sum_{\substack{Q\ne\varnothing,\\ Q\le N}}\frac{(-2)^{\omega(Q)}}{Q}. \tag{1.8}$$

It is important to understand the series (1.8) from a probabilistic point of view. In terms of the ensemble $\Omega(N)$ (or the set of square-free numbers $\mathcal{Q}$) introduced earlier, we replace (1.3) by a new generalized signed measure defined as

$$\widetilde{u_N^t}(\xi) = \frac{t^s}{q_1 \cdot \ldots \cdot q_s}, \quad \text{or equivalently, } \widetilde{u_N^t}(Q) = \frac{t^{\omega(Q)}}{Q}. \tag{1.9}$$

Here $t \in \mathbb{R}$ is a general real parameter. We are especially interested in the case $t = -n$ where $n \geq 1$ is an integer. The case $t = -2$ would correspond to the series (1.8). The normalization factor (partition function) Z_N^t is given by

$$Z_N^t := \sum_{Q \in \mathcal{Q}\colon\, q_i \leq N} \frac{t^{\omega(Q)}}{Q} = \prod_{\text{prime } q \leq N} \left(1 + \frac{t}{q}\right) =$$

$$= \text{const} \cdot (\log N)^t \cdot (1 + o_N(1)), \quad \text{as } N \to \infty.$$

Note that if $t = -p_0$ and p_0 is a prime number, then Z_N^t could vanish identically. In this case we need to consider the set of primes with p_0 removed. For example, in the case $t = -2$ we will just consider odd square-free numbers. In this way we can make sure that Z_N^t always stays nonzero.

We shall normalize $\widetilde{u_N^t}$ and consider the normalized measure

$$u_N^t(Q) = \frac{1}{Z_N^t} \widetilde{u_N^t}(Q).$$

The properties of the measures u_N^t change drastically as we vary the parameter t from positive to negative values. Heuristically this can be seen from the value of the normalization factor Z_N^t. For example, if $t = 1$ then $Z_N^t \sim \text{const} \cdot \log N$. In this case we can afford $O(1)$ (or even bigger) error terms since we will be dividing by a large factor $O(\log N)$. On the other hand, if $t = -2$ then $Z_N^t \sim \text{const} \cdot \frac{1}{\log^2 N}$. In this case we have to carry out a more refined analysis and control the error terms within precision $o\big(1/\log^2 N\big)$.

Now recall the random variable $\eta_N(Q) = \frac{\log Q}{\log N}$ for $Q \in \mathcal{Q}$ and 0 for non-squarefree Q. Let μ_N^t be the distribution on $\mathbb{R}$ induced by η_N on $(\Omega^t(N), u_N^t)$. For $x \geq 0$,

$$\mu_N^t[0, x] = \frac{1}{Z_N^t} \sum_{\substack{q_i \leq N, \\ 0 \leq \frac{\log Q}{\log N} \leq x}} \frac{t^{\omega(Q)}}{Q} = \frac{1}{Z_N^t} \sum_{\substack{q_i \leq N, \\ Q \leq N^x}} \frac{t^{\omega(Q)}}{Q}. \tag{1.10}$$

We will be particularly interested in the above expression for negative integer t's and its asymptotics in the limit $N \to \infty$. To get some idea of what the limiting object might be, one can investigate the limit of the characteristic functions associated with the random variables η_N. The following theorem provides some clue.

THEOREM 1.2 (CONVERGENCE OF THE CHARACTERISTIC FUNCTIONS). *Let $t \in \mathbb{R}$. If $t = -q_0$ for some prime number q_0, then we consider the restricted ensemble $\Omega_{q_0,N}$ (see (1.2)). As $N \to \infty$, we have*

$$\mathbb{E}_N e^{i\lambda\eta_N} \longrightarrow \exp\left(t \int_0^1 \left(e^{i\lambda s} - 1\right) \frac{ds}{s} \right) =: \phi^t(\lambda).$$

We shall call the distributions in Theorem 1.2 the generalized Dickman—de Bruijn distributions. As it turns out, for negative integers t, these distributions are in general tempered distributions which are no longer finite Radon measures. For example, for $t = -2$ we have $\phi^{(-2)}(\lambda) \sim \lambda^2$ as $\lambda \to \infty$ which corresponds to the second order derivative of a Dirac function. More precise information is contained in the following theorem.

THEOREM 1.3 (STRUCTURE OF DICKMAN—DE BRUIJN DISTRIBUTION). *Let $t \in \mathbb{Z}^-$ (the set of negative integers). Then the density ρ^t_{DB} of the generalized Dickman—de Bruijn distribution with parameter t admits the following representation:*

$$\rho^t_{DB}(x) = \rho^t_c(x) + \rho^t_s(x),$$

where ρ^t_c is a continuous and bounded function on $\mathbb{R}$. The notation ρ^t_s denotes the singular part of the density which takes the form

$$\rho^t_s(x) = \frac{e^{|t|\gamma}}{2} \left(\sum_{k=0}^{|t|+1} \sum_{m=0}^{|t|+1-k} \operatorname{sgn}^{(m)}(x-k) \cdot (-1)^{|t|+1-k-m} P_{k,|t|+1-k-m} \right), \qquad (1.11)$$

where γ is the Euler—Mascheroni constant and the coefficients $P_{k,\widetilde{m}}$ are given by

$$P_{k,\widetilde{m}} := \sum_{\substack{j_1+\ldots+j_k=\widetilde{m}, \\ j_i \geq 0, \forall\ 1 \leq i \leq k}} j_1!\, j_2! \ldots j_k!, \quad \text{if } k \geq 1 \text{ and } \widetilde{m} \geq 0.$$

In the case $k = 0$, $P_{0,0} = 1$ and $P_{0,\widetilde{m}} = 0$ for $\widetilde{m} > 0$.

In (1.11) $\operatorname{sgn}(x)$ *is the usual sign function defined as*

$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0; \\ 0, & x = 0; \\ -1, & x < 0. \end{cases}$$

The notation $\operatorname{sgn}^{(m)}$ *denotes the* m^{th} *derivative of the sign function in the distributional sense.*

Remark. The formula (1.11) gives a full description of the singularities in the generalized Dickman—de Bruijn distribution. The singularities only occur at $x = 0, 1, \dots, |t|$ (we do not regard $\operatorname{sgn}(x)$ itself as a singularity when we describe the density). The "worst" singularity takes place at $x = 0$ where it contains the tempered distribution $\delta^{(|t|)}(x)$. For $x = l$, the most singular part is of the form $\delta^{(|t|-l)}(x-l)$.

Theorem 1.3 quantifies the fact that the generalized Dickman—de Bruijn distributions are in general tempered distributions. For example, for $t = -2$ we have

$$\begin{aligned} \frac{1}{2}e^{-2\gamma}\cdot \rho_{DB}^{(-2)}(x) &= \frac{1}{2}\delta''(x) + \delta'(x-1) - \delta(x-1) + \operatorname{sgn}(x-1) + \delta(x-2) - \\ &\quad - \operatorname{sgn}(x-2) + \frac{1}{3}\operatorname{sgn}(x-3) + \rho_c^{(-2)}(x). \end{aligned} \tag{1.12}$$

Due to the singular terms $\delta''(x)$ and $\delta'(x-1)$, $\rho_{DB}^{(-2)}$ cannot be the limiting density of the cumulative distribution functions μ_N^t as defined in (1.10). The correct form of the limiting distribution is given by the following theorem.

THEOREM 1.4 (LIMITING DISTRIBUTION FUNCTION FOR η_N). *Consider the sum* $S_N(y) = \sum_{\substack{n<N^y, \\ p|n \Rightarrow 2<p\le N}} \frac{(-2)^{\omega(n)}}{n}\mu^2(n)$. *Then, for* $N \to \infty$,

$$S_N(y) = \begin{cases} O\left(e^{-c\sqrt{y\log N}}\right), & \textit{for } 0 < y \le 1, \\ \frac{C_1}{(\log N)^2}\left(G(y) + o_N(1)\right), & \textit{for } y > 1. \end{cases}$$

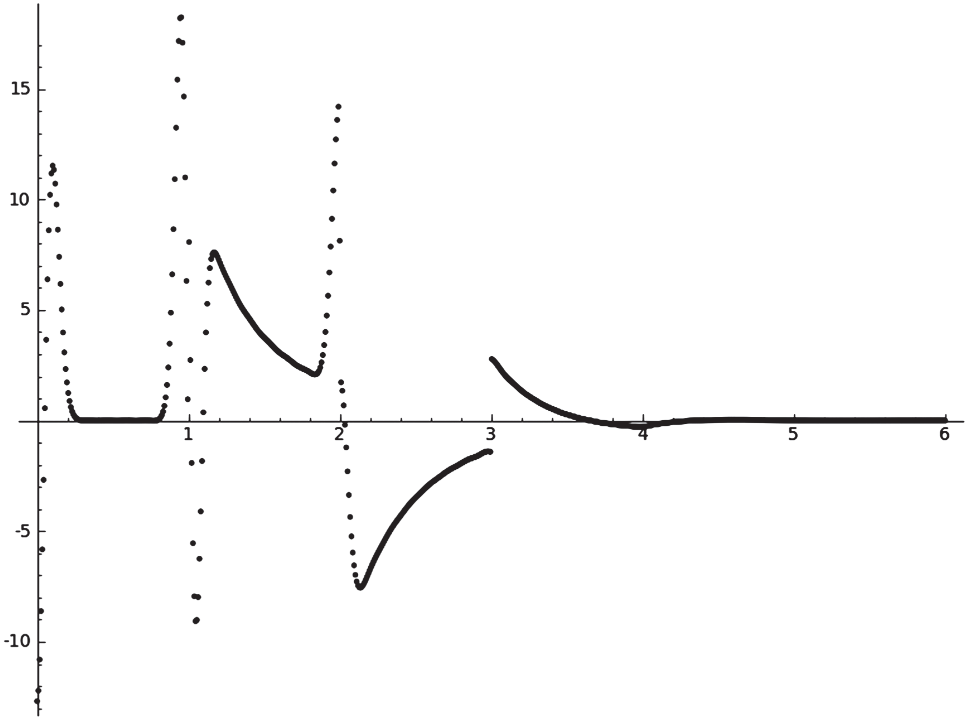

Fig. 1. The density of the generalized Dickman—de Bruijn distribution for $t = -2$

Here $c > 0$ *is a sufficiently small (absolute) constant,* $C_1 = 4 \prod_{\substack{p>2 \\ p \text{ is prime}}} \left(1 - \frac{1}{(p-1)^2}\right)$ *and* $G(y)$ *has singularities at* $y = 1, 2$; *more precisely,*

$$G(y) = G_1(y) - H(y-1) + H(y-2) + G_2(y)$$

where

$$H(y) = \begin{cases} 1, & \text{for } y > 0; \\ -1, & \text{for } y \le 0; \end{cases} \qquad G_1(y) = \frac{e^{-2\gamma}}{2\pi} \cdot \mathrm{Im}\left(\int_\Gamma e^{-2\int_0^{-s} \frac{e^t - 1}{t} dt + ys} \frac{ds}{s}\right).$$

Here γ *is the Euler—Mascheroni constant and* Γ *is the right half unit circle on* $\mathbb{C}$ *including the points* $s = \pm i$. G_2 *is a continuous and bounded function in* y.

Remark. From Theorem 1.4 we can compute the limiting distribution and the density. Denote $\mu_N^t([0,x]) \to \mu_\infty^t([0,x])$.

In the case $t=-2$, we have $\mu_\infty^{(-2)}[0,x] = e^{2\gamma}\cdot G(x)$:

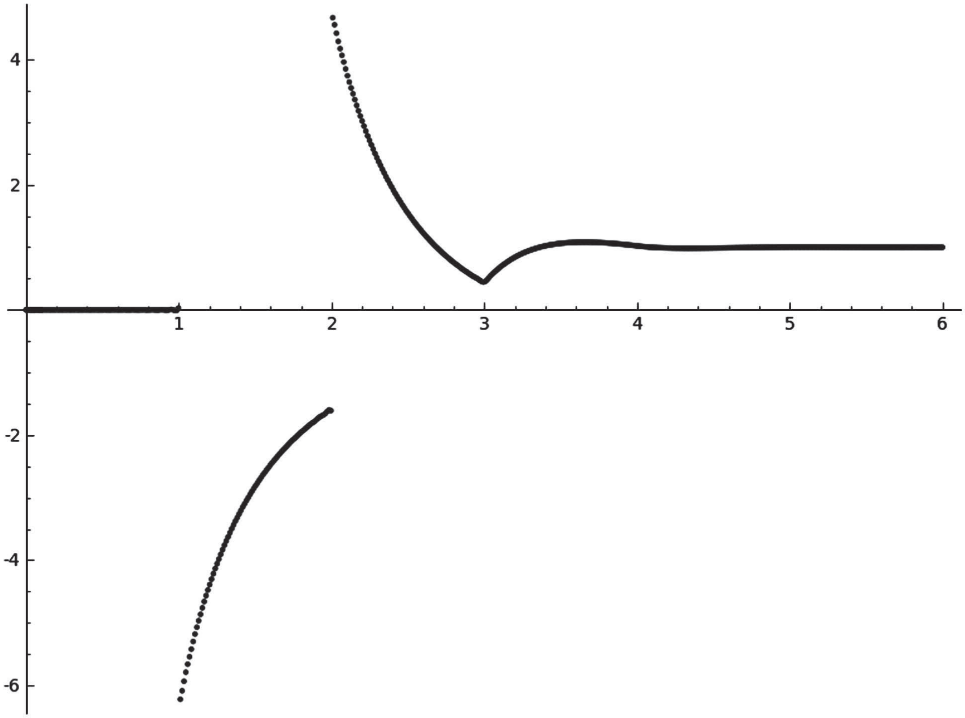

Fig. 2. The limiting distribution of η_N for $t=-2$

The density of this distribution can be calculated to be

$$\frac{1}{2}e^{-2\gamma}\rho_\infty^{(-2)} = -\delta(x-1) + \operatorname{sgn}(x-1) + \delta(x-2) - \\ - \operatorname{sgn}(x-2) + \frac{1}{3}\operatorname{sgn}(x-3) + \rho_c^{(-2)}(x).$$

Or, in terms of $\rho_{DB}^{(-2)}$ (see (1.12) and Fig. 1), we have

$$\rho_{DB}^{(-2)} = e^{2\gamma}\left(\delta''(x) + 2\delta'(x-1)\right) + \rho_\infty^{(-2)}.$$

This shows that $\rho_\infty^{(-2)}$ collects all the non-tempered part of the generalized Dickman–de Bruijn distribution.

Remark. In a recent paper [5], F. Cellarosi considered a situation different from the one considered in Theorem 1.4 and proved the limiting theorems for smooth test functions. For example, for negative t he proved that

$$\mathbb{E}_N f(\eta_N) \to \int f d\mu_{DB}^t \tag{1.13}$$

for a class of smooth functions whose Fourier transform $\widehat{f}(\lambda)$ has sufficiently fast decay as $|\lambda| \to \infty$. Note that for negative t (e.g., $t = -2$) the Fourier transform of ρ_{DB}^t grows like $|\lambda|^{|t|}$ at infinity. By the Parseval equality we have

$$\int f d\mu_{DB}^t = \int \widehat{f}(\lambda)\widehat{\rho_{DB}^t}(\lambda) d\lambda.$$

Obviously in order for the above integral to converge, we must impose at least $|\widehat{f}(\lambda)| \cdot |\lambda|^{|t|} \in L^1$ which requires $\widehat{f}$ to decay sufficiently fast as $|\lambda| \to \infty$. This is the main reason why smoothness of functions played an important role in the analysis therein. On the other hand, in our situation the limiting density is ρ_∞^t which differs from ρ_{DB}^t exactly by a part corresponding to a genuine tempered distribution. In yet other words for $f = \chi_{[0,x]}$ (indicator function of the interval $[0, x]$) (1.13) no longer holds and we have to replace the measure μ_{DB}^t by a different measure μ_∞^t. This shows the surprising change of behavior in the signed distributions.

We now list some notations and conventions used in this paper.

Notations and Conventions

- For any two quantities X and Y, we denote $X \lesssim Y$ if $X \le CY$ for some harmless constant $C > 0$. Similarly, $X \gtrsim Y$ if $X \ge CY$ for some $C > 0$. We denote $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$.
- We will use the "big O" and "little o" notations. Let n be a positive integer which tends to infinity. We shall write $f \ll g$ or $f = O(g)$ if $|f(n)| \le Ag(n)$ for some constant A independent of n. We write $f(n) = O(g(n))$ if $f(n) \ll g(n)$. We will write $f(n) = o(g(n))$ if $f(n)/g(n) \to 0$ as $n \to \infty$. To stress the dummy variable n we sometimes write $f = o_n(g)$. For example, the notation $X = o_N(1)$ means that $X \to 0$ as $N \to \infty$.

- For any complex number X, we denote by $\mathrm{Re}(X)$ and $\mathrm{Im}(X)$ the real and imaginary parts of X respectively.
- For any natural number n, $\omega(n)$ denotes the number of distinct prime divisors of n. $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ denotes the Riemann zeta function.

2. Proof of Theorem 1.2

We will frequently use the following

LEMMA 2.1 (PARTIAL SUMMATION). *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of complex numbers. Let* $0 < a < b < \infty$ *and suppose* f *is a continuously differentiable function the interval* $[a, b]$. *Set*

$$A(t) = \sum_{n \le t} a_n.$$

Then

$$\sum_{a<n\le b} a_n f(n) = A(b)f(b) - A(a)f(a) - \int_a^b A(t)f'(t)dt,$$

PROOF OF LEMMA 2.1. The proof is a standard exercise in calculus. See Theorem 1 of Chapter I.0 in [16] for a textbook proof. □

Throughout the rest of this paper, we shall not make explicit reference to Lemma 2.1 and simply mention it as partial summation (formula) wherever it is used.

We first take p_0 sufficiently large such that $p_0 > 2|t|+2$. Obviously, as $N \to \infty$, for any real λ, we have

$$\left[\prod_{\substack{q \le p_0, \\ q \text{ is prime and } q \ne q_0}} \left(1 + \frac{t}{q}\right)\right]^{-1} \prod_{\substack{q \le p_0, \\ q \text{ is prime and } q \ne q_0}} \left(1 + \frac{t}{q} e^{i\lambda \frac{\log q}{\log N}}\right) \longrightarrow 1.$$

Therefore we only need to consider

$$\left[\prod_{\substack{p_0<q\le N,\\ q \text{ is prime}}}\left(1+\frac{t}{q}\right)\right]^{-1}\prod_{\substack{p_0<q\le N,\\ q \text{ is prime}}}\left(1+\frac{t}{q}e^{i\lambda\frac{\log q}{\log N}}\right)=$$
$$=\exp\left\{\sum_{p_0<q\le N}\log\left(1+\frac{t}{q}e^{i\lambda\frac{\log q}{\log N}}\right)-\sum_{p_0<q\le N}\log\left(1+\frac{t}{q}\right)\right\}=:$$
$$=:\exp\left(\varphi_N(\lambda)-\varphi_N(0)\right).$$

We analyze $\varphi_N(\lambda)-\varphi_N(0)$ as $N\to\infty$. By using partial summation, we have

$$\varphi_N(\lambda)=\pi(N)\log\left(1+\frac{t}{N}e^{i\lambda}\right)- \tag{2.1}$$
$$-\pi(p_0)\log\left(1+\frac{t}{p_0}e^{i\lambda\frac{\log p_0}{\log N}}\right)- \tag{2.2}$$
$$-\int_{p_0}^{N}\pi(q)\frac{d}{dq}\left(\log\left(1+\frac{t}{q}e^{i\lambda\frac{\log q}{\log N}}\right)\right)dq. \tag{2.3}$$

Here $\pi(N)$ is the usual prime-counting function: $\pi(x)=\sum_{\substack{p\le x,\\ p \text{ is prime}}} 1.$

For (2.1), note that by using the Prime Number Theorem,

$$\pi(N)\log\left(1+\frac{t}{N}e^{i\lambda}\right)=O\left(\frac{N}{\log N}\right)\cdot O\left(\frac{1}{N}\right)\longrightarrow 0 \quad \text{as } N\to\infty.$$

For (2.2), note that $\frac{\log p_0}{\log N}\to 0$ as $N\to\infty$. Therefore, (2.2) converges to $-\pi(p_0)\log\left(1+\frac{t}{p_0}\right)$ as $N\to\infty$. This constant will cancel with the corresponding constant in the expression for $\varphi_N(0)$.

For (2.3), we compute

$$(2.3) = -\int_{p_0}^{N} \pi(q) \frac{-\frac{t}{q^2} e^{i\lambda \frac{\log q}{\log N}} + \frac{t}{q} \frac{i\lambda}{q \log N} e^{i\lambda \frac{\log q}{\log N}}}{1 + \frac{t}{q} e^{i\lambda \frac{\log q}{\log N}}} dq =$$

$$= \int_{p_0}^{N} \pi(q) \frac{t}{q^2} \frac{1 - \dfrac{i\lambda}{\log N}}{1 + \dfrac{t}{q} e^{i\lambda \frac{\log q}{\log N}}} e^{i\lambda \frac{\log q}{\log N}} dq.$$

Thus,

$$\varphi_N(\lambda) - \varphi_N(0) = o_N(1) + \int_{p_0}^{N} \pi(q) \frac{t}{q^2} \left[\frac{1 - \dfrac{i\lambda}{\log N}}{1 + \dfrac{t}{q} e^{i\lambda \frac{\log q}{\log N}}} e^{i\lambda \frac{\log q}{\log N}} - \frac{1}{1 + \dfrac{t}{q}} \right] dq.$$

Let us analyze the above integral denoting it by I_N. Consider first the regime $p_0 < q \leq \log N$. In this case we have

$$\left| \frac{1 - \dfrac{i\lambda}{\log N}}{1 + \dfrac{t}{q} e^{i\lambda \frac{\log q}{\log N}}} e^{i\lambda \frac{\log q}{\log N}} - \frac{1}{1 + \dfrac{t}{q}} \right| \leq \frac{1}{\log N} \left| \frac{i\lambda}{1 + \dfrac{t}{q} e^{i\lambda \frac{\log q}{\log N}}} e^{i\lambda \frac{\log q}{\log N}} \right| +$$

$$+ \left| \frac{1}{1 + \dfrac{t}{q} e^{i\lambda \frac{\log q}{\log N}}} \left(e^{i\lambda \frac{\log q}{\log N}} - 1 \right) \right| + \left| \frac{1}{1 + \dfrac{t}{q} e^{i\lambda \frac{\log q}{\log N}}} - \frac{1}{1 + \dfrac{t}{q}} \right| \leq$$

$$\leq O\left(\frac{1}{\log N} \right) + O\left(\frac{\log \log N}{\log N} \right).$$

Therefore,

$$|I_N| \leq \left| \int_{p_0}^{\log N} \frac{t}{q\log q} \frac{\log\log N}{\log N} dq \right| \leq |t| \frac{\log\log N \cdot \log\log\log N}{\log N} \longrightarrow 0,$$

as $N \to \infty$.

It remains to consider the regime $\log N \leq q \leq N$. First, observe that

$$\left| \int_{\log N}^{N} \pi(q) \frac{t}{q^2} \frac{\frac{\lambda}{\log N} e^{i\lambda \frac{\log q}{\log N}}}{1 + \frac{t}{q} e^{i\lambda \frac{\log q}{\log N}}} dq \right| \leq O(1) \cdot \frac{\lambda |t|}{\log N} \int_{\log N}^{N} \frac{1}{q\log q} dq \leq$$

$$\leq O(1) \cdot \frac{\lambda|t|}{\log N} \log\log N \longrightarrow 0 \quad \text{as } N \to \infty.$$

Hence we only need to consider the integral

$$\int_{\log N}^{N} \pi(q) \frac{t}{q^2} \left[\frac{1}{1 + \frac{t}{q} e^{i\lambda \frac{\log q}{\log N}}} e^{i\lambda \frac{\log q}{\log N}} - \frac{1}{1 + \frac{t}{q}} \right] dq =$$

$$= \int_{\log N}^{N} \pi(q) \frac{t}{q^2} \frac{1}{1 + \frac{t}{q}} \left(e^{i\lambda \frac{\log q}{\log N}} - 1 \right) dq +$$

$$+ \int_{\log N}^{N} \pi(q) \frac{t}{q^2} \left[\frac{1}{1 + \frac{t}{q} e^{i\lambda \frac{\log q}{\log N}}} - \frac{1}{1 + \frac{t}{q}} \right] e^{i\lambda \frac{\log q}{\log N}} dq =: I_N^{(1)} + I_N^{(2)}.$$

We first estimate $I_N^{(2)}$. Obviously,

$$\left| \frac{1}{1 + \frac{t}{q} e^{i\lambda \frac{\log q}{\log N}}} - \frac{1}{1 + \frac{t}{q}} \right| \leq \frac{O(1)}{q} \frac{\log q}{\log N}.$$

Therefore,

$$\left|I_N^{(2)}\right| \le \int\limits_{\log N}^{N} \frac{O(1)}{q\log q}\cdot\frac{1}{q}\cdot\frac{\log q}{\log N}\,dq \le \frac{O(1)}{\log N} \longrightarrow 0,$$

as $N \to \infty$.

Finally, we deal with $I_N^{(1)}$. We further decompose it as

$$I_N^{(1)} = \int\limits_{\log N}^{N} \pi(q)\frac{t}{q^2}\left(e^{i\lambda\frac{\log q}{\log N}} - 1\right) dq+ \tag{2.4}$$

$$+\int\limits_{\log N}^{N} \pi(q)\frac{t}{q^2}\left(\frac{1}{1+\frac{t}{q}} - 1\right)\left(e^{i\lambda\frac{\log q}{\log N}} - 1\right) dq. \tag{2.5}$$

Then

$$|(2.5)| \le O(1)\int\limits_{\log N}^{N} \frac{1}{q\log q}\cdot\frac{1}{q}\cdot\frac{\log q}{\log N}\,dq \le \frac{O(1)}{\log N} \longrightarrow 0,$$

as $N \to \infty$.

On the other hand, by using Prime Number Theorem (see, e.g., [16], Chapter II.4), we have

$$\frac{\pi(q)}{q} = \frac{1}{q}\int\limits_{2}^{q}\frac{1}{\log s}ds + O\left(e^{-(\log q)^{2/5}}\right)$$

and

$$\int\limits_{2}^{q}\frac{1}{\log s}ds = \frac{s}{\log s}\bigg|_2^q + \int\limits_{2}^{q}\frac{1}{(\log s)^2}\cdot\frac{1}{s}ds = \frac{q}{\log q} + O(1) + \frac{1}{\log q},$$

so

$$\frac{\pi(q)}{q} = \frac{1}{\log q} + O\left(\frac{1}{q}\right) + O\left(e^{-(\log q)^{2/5}}\right).$$

Plugging the above asymptotics into (2.4), we have

$$(2.4) = \int_{\log N}^{N} t \cdot \frac{1}{\log q} \cdot \frac{1}{q} \left(e^{i\lambda \frac{\log q}{\log N}} - 1 \right) dq +$$

$$+ \int_{\log N}^{N} \left[O\left(\frac{1}{q^2}\right) + O\left(\frac{1}{q} e^{-(\log q)^{2/5}}\right) \right] \left| e^{i\lambda \frac{\log q}{\log N}} - 1 \right| dq$$

The second integral above approaches 0 as $N \to \infty$, thus

$$(2.4) \longrightarrow t \int_0^1 \left(e^{i\lambda s} - 1 \right) \frac{ds}{s}.$$

Collecting the estimates, we have

$$\varphi_N(\lambda) - \varphi_N(0) \longrightarrow \ t \int_0^1 \left(e^{i\lambda s} - 1 \right) \frac{ds}{s}, \quad \text{as } N \to \infty.$$

Theorem 1.2 is proved.

3. Proof of Theorem 1.3 (analysis of the DB distribution)

Throughout this section we assume $t \in \mathbb{Z}^-$.

In this section we prove Theorem 1.3. We begin with a simple lemma which is needed later.

LEMMA 3.1. *For any integer $L \geq 2$ and any $\lambda \geq 1$, we have*

$$\int_{\lambda}^{\infty} s^{-1} e^{is} ds = -e^{i\lambda} \sum_{n=1}^{L} \left(\frac{-i}{\lambda} \right)^n \cdot (n-1)! + O(\lambda^{-(L+1)}).$$

PROOF OF LEMMA 3.1. For each $n \geq 1$, define

$$I_n = \int_{\lambda}^{\infty} s^{-n} e^{is} ds.$$

By using integration by parts, easy to check

$$I_n = i \cdot \lambda^{-n} e^{i\lambda} + n \cdot (-i) \cdot I_{n+1}. \tag{3.1}$$

Clearly, then

$$(-i)^n (n-1)! I_n = i \cdot (-i)^n \lambda^{-n} \cdot (n-1)! \cdot e^{i\lambda} + (-i)^{n+1} \cdot n! \cdot I_{n+1}.$$

Summing over $n = 1, \dots, L$, we get

$$(-i) I_1 = i e^{i\lambda} \sum_{n=1}^{L} \left(-\frac{i}{\lambda}\right)^n (n-1)! + (-i)^{L+1} \cdot L! \cdot I_{L+1}.$$

Note that, since $I_1 = O(\lambda^{-1})$, by (3.1), $I_{L+1} = O(\lambda^{-(L+1)})$. The desired identity then follows. ☐

By Fourier inversion, we have

$$\rho^t_{DB}(x) = \lim_{R\to\infty} \frac{1}{2\pi} \int_{-R}^{R} \exp\left\{ t \int_0^1 \left(e^{i\lambda s} - 1\right) \frac{ds}{s} \right\} e^{-i\lambda x} d\lambda.$$

First note that by a simple change of variable $\lambda \mapsto -\lambda$,

$$\rho^t_{DB}(x) = \lim_{R\to\infty} \frac{1}{\pi} \operatorname{Re} \left(\int_0^R \exp\left\{ t \int_0^1 \left(e^{i\lambda s} - 1\right) \frac{ds}{s} \right\} e^{-i\lambda x} d\lambda \right).$$

Let $\phi_{\le 1}(\lambda)$ be an even smooth cut-off function on $\mathbb{R}$ such that $\phi_{\le 1}(\lambda) = 1$ for $|\lambda| \le 1$ and $\phi_{\le 1}(\lambda) = 0$ for $|\lambda| \ge 2$. Denote $\phi_{>1}(\lambda) = 1 - \phi_{\le 1}(\lambda)$.

Now write

$$\rho^t_{DB}(x) = \lim_{R\to\infty} \frac{1}{\pi} \operatorname{Re} \left(\int_0^R \exp\left\{ t \int_0^1 \left(e^{i\lambda s} - 1\right) \frac{ds}{s} \right\} \phi_{\le 1}(\lambda) e^{-i\lambda x} d\lambda \right) + \tag{3.2}$$

$$+ \lim_{R\to\infty} \frac{1}{\pi} \operatorname{Re} \left(\int_0^R \exp\left\{ t \int_0^1 \left(e^{i\lambda s} - 1\right) \frac{ds}{s} \right\} \phi_{>1}(\lambda) e^{-i\lambda x} d\lambda \right). \tag{3.3}$$

Note that

$$(3.2) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \exp\left\{ t\int\limits_{0}^{1} \left(e^{i\lambda s}-1\right)\frac{ds}{s}\right\} \phi_{\leq 1}(\lambda) e^{-i\lambda x} d\lambda.$$

This is due to the fact that the function $\psi(\lambda) = \exp\left\{ t\int_{0}^{1} \left(e^{i\lambda s}-1\right)\frac{ds}{s}\right\}$ belongs to C^{∞} which gives rise to a smooth (and fast-decaying in x) density, thanks to the cut-off $\phi_{\leq 1}(\lambda)$.

We now only need to deal with (3.3). Note that in this case λ is localized to the regime $\lambda \geq 1$.

For $\lambda \geq 1$, we write

$$\int\limits_{0}^{1} \left(e^{i\lambda s}-1\right)\frac{ds}{s} = \int\limits_{0}^{\lambda} \frac{e^{is}-1}{s} ds = \int\limits_{0}^{1} \frac{e^{is}-1}{s} ds - \int\limits_{1}^{\lambda} \frac{1}{s} ds + \int\limits_{1}^{\lambda} \frac{e^{is}}{s} ds =$$

$$= \int\limits_{0}^{1} \frac{e^{is}-1}{s} ds - \log\lambda + \int\limits_{1}^{\infty} \frac{e^{is}}{s} ds - \int\limits_{\lambda}^{\infty} \frac{e^{is}}{s} ds.$$

Recall the trigonometric integrals (see [20]),

$$\mathrm{Ci}(x) = \gamma + \log x + \int\limits_{0}^{x} \frac{\cos t - 1}{t}\, dt, \quad \mathrm{Ci}(x) = \mathrm{ci}(x) = -\int\limits_{x}^{\infty} \frac{\cos t}{t}\, dt,$$

where γ is the Euler—Mascheroni constant.

Evaluating the above identity at $x = 1$, we get

$$\int\limits_{0}^{1} \frac{\cos t - 1}{t} dt + \int\limits_{1}^{\infty} \frac{\cos t}{t} dt = -\gamma.$$

Using this and the identity $\int_{0}^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$, we have

$$\int\limits_{0}^{1} \frac{e^{is}-1}{s} ds + \int\limits_{1}^{\infty} \frac{e^{is}}{s} ds = -\gamma + \frac{\pi}{2} i.$$

Thus,

$$(3.3) = \frac{e^{|t|\gamma}}{\pi} \cdot \lim_{R\to\infty} \operatorname{Re} \left(e^{t \cdot i \frac{\pi}{2}} \int_0^R \lambda^{-t} \, \exp \left\{ -t \int_\lambda^\infty \frac{e^{is}}{s} ds \right\} \, \phi_{>1}(\lambda) \, e^{-i\lambda x} d\lambda \right) =$$

$$= \frac{e^{|t|\gamma}}{\pi} \cdot \lim_{R\to\infty} \underbrace{\operatorname{Re} \left(i^{-|t|} \int_0^R \lambda^{-t} \, \exp \left\{ -t \int_\lambda^\infty \frac{e^{is}}{s} ds \right\} \, \phi_{>1}(\lambda) \, e^{-i\lambda x} d\lambda \right)}_{\psi_R(x)} .$$

We analyze in detail the structure of ψ_R. By Lemma 3.1,

$$\int_\lambda^\infty \frac{e^{is}}{s} ds = e^{i\lambda} \cdot \frac{i}{\lambda} \sum_{n=0}^{|t|} \left(\frac{-i}{\lambda} \right)^n \cdot n! + O\left(\frac{1}{\lambda^{|t|+2}} \right) .$$

Plugging this into $\psi_R(x)$, we obtain:

$$\psi_R(x) =$$

$$= i^{-|t|} \int_0^R \lambda^{|t|} \exp \left[|t| \left(\frac{ie^{i\lambda}}{\lambda} \sum_{n=0}^{|t|} \frac{n! \, (-i)^n}{\lambda^n} \right) + O\left(\frac{1}{\lambda^{|t|+2}} \right) \right] \phi_{>1}(\lambda) e^{-i\lambda x} d\lambda =$$

$$= \int_0^R \left(\underbrace{\lambda^{|t|} \sum_{k=0}^{|t|+1} e^{i\lambda k} \frac{i^{-|t|+k}}{\lambda^k} \Big(\sum_{n=0}^{|t|} \frac{n!(-i)^n}{\lambda^n} \Big)^k}_{H} + O\left(\frac{1}{\lambda^2} \right) \right) \phi_{>1}(\lambda) e^{-i\lambda x} d\lambda.$$

Denote for $k \geq 1$ and $m \geq 0$

$$P_{k,m} := \sum_{\substack{j_1 + \ldots + j_k = m, \\ 0 \leq j_i \leq |t|}} j_1! \, j_2! \ldots j_k! . \tag{3.4}$$

We suppress the dependence on t for simplicity. For $k = 0$ we denote $P_{0,0} = 1$ and $P_{0,m} = 0$ for $m > 0$.

Observe

$$H=\sum_{k=0}^{|t|+1} e^{i\lambda k} i^{-|t|+k}\lambda^{|t|-k}\sum_{m=0}^{|t|+1-k}\left(\frac{-i}{\lambda}\right)^m P_{k,m}+O(\lambda^{-2})=$$
$$=\sum_{k=0}^{|t|+1} e^{i\lambda k}\sum_{m=0}^{|t|+1-k}(-i\lambda)^{|t|-k-m}(-1)^m P_{k,m}+O(\lambda^{-2}).$$

Plugging the above into the expression for ψ_R, we get

$$\psi_R(x)=\int_0^R\left(\sum_{k=0}^{|t|+1} e^{-i\lambda(x-k)}\sum_{m=0}^{|t|+1-k}(-i\lambda)^{|t|-k-m}(-1)^m P_{k,m}\right)\phi_{>1}(\lambda)d\lambda+$$
$$+\int_0^R O(\lambda^{-2})\cdot\phi_{>1}(\lambda)e^{-i\lambda x}d\lambda=$$
$$=\int_0^R\left(\sum_{k=0}^{|t|+1} e^{-i\lambda(x-k)}\sum_{m=0}^{|t|+1-k}(-i\lambda)^{|t|-k-m}(-1)^m P_{k,m}\right)d\lambda- \tag{3.5}$$
$$-\int_0^R\left(\sum_{k=0}^{|t|+1} e^{-i\lambda(x-k)}\sum_{m=0}^{|t|+1-k}(-i\lambda)^{|t|-k-m}(-1)^m P_{k,m}\right)\phi_{\le 1}(\lambda)d\lambda+ \tag{3.6}$$
$$+\int_0^R O(\lambda^{-2})\cdot\phi_{>1}(\lambda)e^{-i\lambda x}d\lambda. \tag{3.7}$$

Obviously, (3.7) will only contribute to the bounded continuous part of the density. For (3.6) one may worry about the singularity at $\lambda=0$ when $m=|t|+1-k$. But it turns out that the real part of this term is

$$\int_0^R \text{const}\cdot\frac{\sin\lambda(x-k)}{\lambda}\phi_{\le 1}(\lambda)d\lambda$$

which also gives rise to a bounded continuous density.

We only need to analyze the real part of (3.5).

Recall for $z\in\mathbb{R}$,

$$\int_0^\infty \frac{\sin \lambda z}{\lambda} d\lambda = \frac{\pi}{2}\operatorname{sgn}(z) = \begin{cases} \frac{\pi}{2}, & z > 0; \\ 0, & z = 0; \\ -\frac{\pi}{2}, & z < 0. \end{cases}$$

Since the multiplier $-i\lambda$ corresponds to differentiation in x, we get

$$\lim_{R\to\infty} \frac{e^{|t|\gamma}}{\pi} \operatorname{Re}((3.5)) =$$

$$= \frac{e^{|t|\gamma}}{2} \left(\sum_{k=0}^{|t|+1} \sum_{m=0}^{|t|+1-k} \operatorname{sgn}^{(|t|+1-k-m)}(x-k) \cdot (-1)^m P_{k,m} \right) =$$

$$= \frac{e^{|t|\gamma}}{2} \left(\sum_{k=0}^{|t|+1} \sum_{m=0}^{|t|+1-k} \operatorname{sgn}^{(m)}(x-k) \cdot (-1)^{|t|+1-k-m} P_{k,|t|+1-k-m} \right) .$$

Now note that for $k \geq 1$, we have $|t|+1-k-m \leq |t|$. Therefore the condition $j_i \leq |t|$ in (3.4) can be dropped. This concludes the proof of Theorem 1.3.

4. Proof of Theorem 1.4

We will consider two different regimes for the variable y:

- The regime $0 < y \leq 1$ will be handled by the Selberg—Delange method and the proof for this case can be found in Sec. 4.2.
- For the regime $y > 1$ we will use Hildebrand—Tenenbaum type of analysis – see Sec. 4.3.

In the proof we will need to recall several standard lemmas.

4.1. Useful lemmas

Define

$$\eta(x) := \begin{cases} 1, & \text{if } x > 1; \\ \frac{1}{2}, & \text{if } x = 1; \\ 0, & \text{if } 0 < x < 1. \end{cases}$$

For any $c > 0$, it is easy to show

$$\eta(x) = \lim_{R \to +\infty} \int_{c-iR}^{c+iR} \frac{x^s}{s} ds.$$

The following truncated version is most useful.

Lemma 4.1. *For any* $\kappa > 0$, $T > 0$,

$$\left| \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} \frac{x^s}{s} ds - \eta(x) \right| < \begin{cases} x^\kappa \min\left\{1, \dfrac{1}{T\,|\log x|}\right\} & \text{if } x \neq 1 \\ \dfrac{\kappa}{T} & \text{if } x = 1. \end{cases}$$

Proof of Lemma 4.1 . See Theorem 4.1.4 in [15] for a textbook proof. □

Lemma 4.1 is the key to the following well-known Perron's formula. One can prove it directly or see Theorem 2 (p. 132) of Chapter II.2 in [16] for details.

Lemma 4.2 (Perron's formula). *Suppose* $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *is a Dirichlet series which converges for any* $\mathrm{Re}(s) > \sigma_0 > 0$. *Extend the definition of the function* $n \to a_n$ *by setting* $a_x = 0$ *if* $x \in \mathbb{R} \setminus \mathbb{N}$. *Then for any* $\kappa > \sigma_0$, $T \geq 1$, $x \geq 1$ *(*x *is a real number), we have*

$$\sum_{n<x} a_n + \frac{1}{2} a_x = \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} \frac{f(s)x^s}{s} ds + O\left(\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^\kappa \frac{|a_n|}{1 + T|\log(x/n)|} \right). \tag{4.1}$$

We need the following bounds on the Riemann zeta function.

Lemma 4.3. *Let* $s = \sigma + i\tau$. *There exists* $\sigma_1 > 0$ *such that for* $|\tau| \geq 3$ *and* $\sigma \geq 1 - \sigma_1/\log|\tau|$, *we have*

$$\begin{aligned} &\zeta'(s)/\zeta(s) \ll \log|\tau|, \\ &\frac{1}{\zeta(s)} \ll \log|\tau|, \\ &|\log\zeta(s)| \leq \log_2|\tau| + O(1). \end{aligned}$$

Proof of Lemma 4.3. See Theorem 16 on p. 158 of [16]. ☐

The following lemma gives bounds on higher derivatives of the zeta function.

Lemma 4.4. *For any* $c > 0$ *and any integer* $k \geq 0$,

$$\zeta^{(k)}(s) \ll (\log|\tau|)^{k+1}, \quad s = \sigma + i\tau,\ |\tau| \geq 2,\ \sigma \geq 1 - \frac{c}{\log|\tau|}.$$

Proof of Lemma 4.4. See Corollary 7.1 on p. 147 of [16]. ☐

To control certain remainder terms later, we need the following

Lemma 4.5 (Mertens formula). *There are constants* $A_1 > 0$, $A_2 \in \mathbb{R}$, *such that for any* $x \geq 2$,

$$\sum_{1\leq n\leq x} 2^{\omega(n)} = A_1 x \log x + A_2 x + \Delta^{(2)}(x), \quad \textit{and } \Delta^{(2)}(x) = O\left(x^{1/2}\log x\right).$$

Proof of Lemma 4.5. See Mertens [13, 14]. We shall not need the precise values of the constants A_1 and A_2. Mertens computed $A_1 = \dfrac{1}{\zeta(2)}$ and $A_2 = \left(\dfrac{2\gamma - 1}{\zeta(2)} - \dfrac{2\zeta'(2)}{\zeta^2(2)}\right)$. It is also conjectured that $\Delta^{(2)}(x) = O(x^{\theta+\varepsilon})$ with $\theta = \dfrac{1}{4}$. The best known result is due to Baker [1] who proved $\theta \leq \dfrac{4}{11}$ assuming the Riemann Hypothesis. ☐

4.2. Selberg—Delange method: $0 < y \leq 1$

Note that for $0 < y \leq 1$, we have $N^y \leq N$ and therefore

$$S_N(y) = \sum_{\substack{n<N^y,\\ p|n\Rightarrow 2<p\leq N}} \frac{(-2)^{\omega(n)}}{n}\mu^2(n) = \sum_{\substack{n<N^y,\\ n \text{ is odd}}} \frac{(-2)^{\omega(n)}}{n}\mu^2(n).$$

We will prove the following estimates:

$$\sum_{\substack{n\leq x\\ n \text{ is odd}}} \frac{(-2)^{\omega(n)}}{n}\mu^2(n) = O\left(e^{-\alpha_1(\log x)^{1/2}}\right), \tag{4.2}$$

$$\sum_{n\leq x} \frac{(-2)^{\omega(n)}}{n}\mu^2(n) = O\left(e^{-\alpha_2(\log x)^{\frac{1}{2}}}\right); \tag{4.3}$$

where $\alpha_1 > 0$ and $\alpha_2 > 0$ are some constants.

Note that it suffices for us to show (4.2), the bound for $S_N(y)$ is obtained by taking $x = N^y$. The estimate (4.3) follows from (4.2) by breaking the sum into odd and even n respectively.

Here and below we shall always assume x is sufficiently large. Note that in both (4.2) and (4.3), the restriction $n \leq x$ can be replaced by $n < x$ since for any $x = N \gg 1$ we have $\omega(N) = o(\log N)$ and

$$\frac{2^{\omega(N)}}{N} = \exp\{o(\log N) - \log N\} \ll \exp\{-\frac{1}{2}\log N\}. \tag{4.4}$$

By the same reason, when we apply Perron's formula (see (4.1)) to the sequence $a_n = (-2)^{\omega(n)}\mu^2(n)/n$ later, we can neglect the boundary term a_x when $x = N^y$ is an integer since the error is of order $O(\exp\{-\frac{1}{2}y\log N\})$.

We shall use Perron's formula (Lemma 4.2) for $a_n = (-2)^{\omega(n)}\mu^2(n)/n$ and choose $\kappa = 1 + \dfrac{1}{\log x}$, $T = e^{(\log x)^{1/2}}$. We begin with a simple tail estimate.

LEMMA 4.6 (TAIL ESTIMATE). *Let* $x \geq 2$ *and* $R = \exp\left\{c_1(\log x)^{c_2}\right\}$ *with* $c_1 > 0$ *and* $c_2 > 0$. *Then*

$$\sum_{n=1}^{\infty} 2^{\omega(n)} \cdot \left(\frac{x}{n}\right)^{1+\frac{1}{\log x}} \cdot \frac{1}{1+R|\log(x/n)|} = O\left(x \cdot R^{-1/2}\right).$$

PROOF OF LEMMA 4.6. We divide the summation into several regimes. Denote $c = 1 + \dfrac{1}{\log x}$ and note that $x^c = O(x)$.

<u>Case 1</u>: $n \geq \frac{5}{4}x$. In this case we have $\left|\log\left(\frac{x}{n}\right)\right| = \left|\log\left(\frac{n}{x}\right)\right| \gtrsim 1$. Therefore,

$$\sum_{n\geq\frac{5}{4}x} 2^{\omega(n)}\left(\frac{x}{n}\right)^c \frac{1}{1+R|\log(x/n)|} \lesssim \sum_{n\geq\frac{5}{4}x} 2^{\omega(n)}\frac{x}{n^c}\frac{1}{R} \lesssim$$

$$\lesssim \frac{x}{R}\sum_{n\geq\frac{5}{4}x} 2^{\omega(n)}\frac{1}{n^c}. \tag{4.5}$$

By Lemma 4.5 and partial summation,

$$(4.5) \lesssim \frac{x}{R}\left[O(\log x) + \int_{\frac{5}{4}x}^{\infty} s\log s \cdot \frac{1}{s^{1+c}}ds\right] \lesssim$$

$$\lesssim \frac{x}{R}\left[O(\log x) + \int_{\frac{5}{4}x}^{\infty} \frac{\log s}{s^{c}}ds\right]. \tag{4.6}$$

By a change of variable $s \to sx$, we have

$$\int_{\frac{5}{4}x}^{\infty} \frac{\log s}{s^{c}}ds = \frac{1}{x^{c-1}}\int_{\frac{5}{4}}^{\infty} \frac{\log s + \log x}{s^{c}}ds \lesssim$$

$$\lesssim \int_{\frac{5}{4}}^{\infty} \frac{\log s}{s^{c}}dt + \log x \cdot \frac{1}{c-1} \lesssim$$

$$\lesssim \frac{1}{c-1} + \frac{1}{(c-1)^2} + \frac{\log x}{c-1} = O\left(\log^2 x\right)$$

Plugging the above estimate into (4.6), we obtain $(4.5) \lesssim \frac{x}{R}(\log x)^2 \ll x/\sqrt{R}$. This settles the case $n \geq \frac{5}{4}x$.

Case 2: $n \leq \frac{3}{4}x$. Note that $\left|\log\left(\frac{x}{n}\right)\right| \geq \left|\log\left(\frac{4}{3}\right)\right| \gtrsim 1$. Therefore, by Mertens formula (Lemma 4.5) and summation by parts,

$$\sum_{n\leq\frac{3}{4}x} 2^{\omega(n)}\left(\frac{x}{n}\right)^{c}\frac{1}{1+R\left|\log\left(\frac{x}{n}\right)\right|} \lesssim \frac{x}{R}\sum_{n\leq\frac{3}{4}x} 2^{\omega(n)} \cdot \frac{1}{n^{c}} \lesssim$$

$$\lesssim \frac{x}{R}\left[O\left(x(\log x)\frac{1}{x^{c}}\right) + \int_{1}^{\frac{3}{4}x} t\log t\frac{1}{t^{c+1}}dt\right] \lesssim$$

$$\lesssim \frac{x}{R}\left[\log x+\log^2 x\right] \ll x/\sqrt{R}.$$

Hence this case is settled.

Case 3: $\frac{3}{4}x \le n \le \frac{5}{4}x$. This regime requires careful analysis. Observe that $|n-x| \le \frac{1}{4}x$ or $\left|\frac{n}{x}-1\right| \le \frac{1}{4}$. Therefore $\left|\log\left(\frac{x}{n}\right)\right| = \left|\log\left(1+\frac{x}{n}-1\right)\right| \sim \left|\frac{n}{x}-1\right|$. Therefore

$$\sum_{\frac{3}{4}x\le n\le\frac{5}{4}x} 2^{\omega(n)}\left(\frac{x}{n}\right)^c \frac{1}{1+R\left|\log\left(\frac{x}{n}\right)\right|} \lesssim \sum_{\frac{3}{4}x\le n\le\frac{5}{4}x} 2^{\omega(n)} \min\left(1, \frac{1}{R\left|\frac{n}{x}-1\right|}\right) \lesssim$$

$$\lesssim \sum_{\frac{3}{4}x\le n\le\frac{5}{4}x} 2^{\omega(n)} \min\left(1, \frac{x}{R\,|n-x|}\right). \tag{4.7}$$

To estimate (4.7), we need to consider several subcases.

Subcase 3*a*: $n > x$ and $\frac{x}{R|n-x|} \ge 1$. This is equivalent to the condition $x < n \le \frac{R+1}{R}x$. In this subcase we have $\min\left(1, \frac{x}{R|n-x|}\right) = 1$. Therefore by Mertens formula (see Lemma 4.5),

$$\sum_{x<n\le\frac{R+1}{R}x} 2^{\omega(n)}\min\left(1,\frac{x}{R|n-x|}\right) = \sum_{n\le\frac{R+1}{R}x} 2^{\omega(n)} - \sum_{n\le x} 2^{\omega(n)} =$$

$$= A_1\left(\frac{R+1}{R}\,x\log\left(\frac{R+1}{R}x\right) - x\log x\right) +$$

$$+ A_2\left(\frac{R+1}{R}x - x\right) + O\left(x^{1/2}\log x\right) =$$

$$= A_1\left(\frac{1}{R}\,x\log x + \frac{R+1}{R}\,x\log\left(1+\frac{1}{R}\right)\right) + A_2\,\frac{x}{R} + O\left(x^{1/2}\log x\right) =$$

$$= O\left(\frac{x}{R}\log x\right) + O\left(x^{1/2}\log x\right) \ll x/\sqrt{R}.$$

We are done with this subcase.

Subcase $3b$: $n > x$ and $\frac{x}{R|n-x|} < 1$. This is equivalent to $n > \frac{R+1}{R}x$. Note that in this subcase $\min\left(1, \frac{x}{R|n-x|}\right) = \frac{x}{R|n-x|}$. Then

$$\sum_{\frac{R+1}{R}x<n\le\frac{5}{4}x} 2^{\omega(n)}\left(\frac{x}{n}\right)^c \min\left(1,\frac{x}{R|n-x|}\right) \lesssim \frac{x}{R}\sum_{\frac{R+1}{R}x<n\le\frac{5}{4}x} 2^{\omega(n)}\frac{1}{n-x} =: I.$$

We now analyze I in detail. By using Mertens formula (see Lemma 4.5) and partial summation again, we get

$$I = A_1\cdot\frac{5}{4}x\,\log\left(\frac{5}{4}x\right)\frac{1}{\frac{5}{4}x-x} + A_2\cdot\frac{5}{4}x\cdot\frac{1}{\frac{5}{4}x-x} + O\left(x^{1/2}\log x\right)\frac{1}{\frac{1}{4}x} -$$

$$- A_1\cdot\frac{R+1}{R}\,x\,\log\left(\frac{R+1}{R}x\right)\frac{1}{\frac{x}{R}} - A_2\frac{R+1}{R}\,x\,\frac{1}{\frac{x}{R}} + O\left(x^{1/2}\log x\right)\frac{1}{\frac{x}{R}} -$$

$$- \int_{\frac{R+1}{R}x}^{\frac{5}{4}x}(A_1 s\log s + A_2 s)\cdot\left(-\frac{ds}{(s-x)^2}\right) + \int_{\frac{R+1}{R}x}^{\frac{5}{4}x} O\left(s^{1/2}\log s\right)\frac{ds}{(s-x)^2} =$$

$$= O(\log x) - A_1(R+1)\log x - A_2(R+1) + O\left(\frac{\log x}{x^{1/2}}\right)\cdot R +$$

$$+ A_1\int_{\frac{R+1}{R}x}^{\frac{5}{4}x}\frac{s\log s}{(s-x)^2}ds + A_2\int_{\frac{R+1}{R}x}^{\frac{5}{4}x}\frac{s}{(s-x)^2}ds + O\left(x^{1/2}\log x\frac{1}{\frac{x}{R}}\right) =$$

$$= O\left(\log x\right) - A_1 R\log x - A_2 R + A_1\underbrace{\int_{\frac{R+1}{R}x}^{\frac{5}{4}x}\frac{s\log s}{(s-x)^2}ds}_{I_2} + A_2\underbrace{\int_{\frac{R+1}{R}x}^{\frac{5}{4}x}\frac{s}{(s-x)^2}ds}_{I_3}.$$

Now we shall show that $I_2 = R\log x + O\left((\log x)^2\right)$ and $I_3 = R + O\left((\log x)^2\right)$. These two estimates will settle *Subcase* $3b$.

Estimate of I_2. Rescaling, we get

$$I_2 = \int_{\frac{R+1}{R}}^{\frac{5}{4}} \frac{s\log(sx)}{(s-1)^2}ds = \int_{\frac{R+1}{R}}^{\frac{5}{4}} \frac{s\log(s)}{(s-1)^2}ds + (\log x)\int_{\frac{R+1}{R}}^{\frac{5}{4}} \frac{s}{(s-1)^2}ds;$$

$$\int_{\frac{R+1}{R}}^{\frac{5}{4}} \frac{s\log(s)}{(s-1)^2}ds = -\frac{1}{s-1}\ s\log s\Big|_{\frac{R+1}{R}}^{\frac{5}{4}} + \int_{\frac{R+1}{R}}^{\frac{5}{4}} \frac{1}{s-1}(1+\log s)ds =$$

$$= O(1) + O\left(R\log\left(1+\frac{1}{R}\right)\right) + \int_{\frac{R+1}{R}}^{\frac{5}{4}} \frac{1}{s-1}\,ds +$$

$$+ \log(s-1)\cdot\ \log s\Big|_{\frac{R+1}{R}}^{\frac{5}{4}} - \int_{\frac{R+1}{R}}^{\frac{5}{4}} \frac{\log(s-1)}{s}ds =$$

$$= O(1) + O(\log R) = O(\log R);$$

$$\int_{\frac{R+1}{R}}^{\frac{5}{4}} \frac{s}{(s-1)^2}ds = \int_{\frac{R+1}{R}}^{\frac{5}{4}} \frac{1}{s-1}ds + \int_{\frac{R+1}{R}}^{\frac{5}{4}} \frac{1}{(s-1)^2}ds =$$

$$= O(\log R) - \frac{1}{s-1}\Big|_{\frac{R+1}{R}}^{\frac{5}{4}} = O(\log R) + R.$$

By the last two estimates, we have

$$I_2 = O(\log R) + (\log x)\cdot(O(\log R) + R) = R\log x + O\left((\log x)^2\right).$$

In a similar fashion $I_3 = R + O\left((\log x)^2\right)$.

Subcase 3*c*: $n < x$ and $\dfrac{x}{R|n-x|} \le 1$. This is similar to *Subcase* 3*a*. We omit the details.

Subcase 3*d*: $n < x$ and $\dfrac{x}{R|n-x|} > 1$. This is similar to *Subcase* 3*b*. We omit the details.

After collecting all the estimates, we have proved Lemma 4.6. □

By using Perron's formula (see Lemma 4.2), the tail estimate we just derived and the discussion after the estimate (4.4), we see that for $x \gg 1$, $c = \dfrac{1}{\log x}$, and $R = \exp\{(\log x)^{1/2}\}$,

$$\sum_{\substack{n \le x \\ n \text{ is odd}}} \frac{(-2)^{\omega(n)}}{n} \mu^2(n) = \frac{1}{2\pi i} \int_{c-iR}^{c+iR} \frac{f(s)x^s}{s} ds + O\left(e^{-\frac{1}{2}c_1(\log x)^{1/2}}\right),$$

where

$$f(s) = \sum_{\substack{n \ge 1 \\ n \text{ is odd}}} \frac{(-2)^{\omega(n)}}{n^{1+s}} \mu^2(n).$$

It remains to estimate the contour integral.

First we simplify the expression of $f(s)$:

$$f(s) = \sum_{n \ge 1, n \text{ is odd}} (-2)^{\omega(n)} \mu^2(n) \frac{1}{n^{1+s}} = \prod_{\substack{p>2 \\ p \text{ is prime}}} \left(1 - \frac{2}{p^{1+s}}\right).$$

Since $1 - 2x = (1-x)^2 - x^2 = (1-x)^2\left(1 - \left(\frac{x}{1-x}\right)^2\right)$, we can write

$$1 - \frac{2}{p^{1+s}} = \left(1 - \frac{1}{p^{1+s}}\right)^2 \left(1 - \left(\frac{\frac{1}{p^{1+s}}}{1 - \frac{1}{p^{1+s}}}\right)^2\right).$$

Hence

$$f(s) = \prod_{\substack{p>2 \\ p \text{ is prime}}} \left[\left(1 - \frac{1}{p^{1+s}}\right)^2 \left(1 - \frac{1}{(p^{1+s}-1)^2}\right)\right] = \frac{1}{\zeta^2(1+s)} g(s),$$

where $\zeta(s)$ is the Riemann zeta function while

$$g(s) = (1 - 2^{-(1+s)})^{-2} \prod_{\substack{p>2 \\ p \text{ is prime}}} \left(1 - \frac{1}{(p^{1+s}-1)^2}\right).$$

Note that $g(s)$ converges absolutely for $\mathrm{Re}(s) > -\frac{1}{2}$.

Now we estimate $\frac{1}{2\pi i}\int\limits_{c-iR}^{c+iR}\frac{f(s)x^s}{s}ds$ by deformation of contour. By Lemma 4.3, we have

$$\frac{1}{\zeta(1+s)} \ll \log|\tau|, \quad s=\sigma+i\tau,\ |\tau|\geq 3,\ \sigma \geq -\frac{\sigma_1}{\log|\tau|}. \tag{4.8}$$

Recall $c=\frac{1}{\log x}$ and $R=\exp(\sqrt{\log x})$. Choose

$$a=-\frac{\sigma_1}{\log R}=-\frac{\sigma_1}{\sqrt{\log x}}.$$

Integrating over the contour on the Fig. 3 and using the fact that f is analytic in the integration region[1], we get

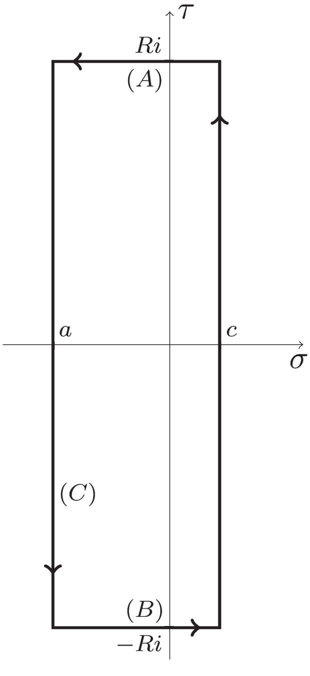

Fig. 3

$$-\frac{1}{2\pi i}\int\limits_{c-iR}^{c+iR}\frac{f(s)x^s}{s}ds=\int\limits_{(A)}+\int\limits_{(B)}+\int\limits_{(C)}.$$

Let us estimate these three integrals.

Horizontal pieces: $\int\limits_{(A)}$ and $\int\limits_{(B)}$.

Clearly,

$$\left|\int\limits_{(A)}\right| = \left|\int\limits_{a+iR}^{c+iR}\frac{f(s)}{s}x^s ds\right| \lesssim$$

$$\lesssim \int\limits_{a+iR}^{c+iR}\frac{|g(s)|}{|\zeta(1+s)|^2}\cdot\frac{1}{R}\cdot|x^s|ds \lesssim$$

$$\lesssim \int\limits_{a}^{c}(\log R)^2\cdot\frac{1}{R}\cdot x^c ds \lesssim \frac{\log x}{R}.$$

---

[1] Note the function $\zeta(1+s)$ has a simple pole at $s=0$ and hence $\frac{1}{\zeta^2(1+s)s}$ is analytic in the region depicted in Fig. 3.

The estimate of $\int\limits_{(B)}$ is done similarly.

Vertical piece: $\int\limits_{(C)}$.

Note that for $|\operatorname{Im}(s)| \leq 3$, $\operatorname{Re}(s) = a = -\sigma_1/\sqrt{\log x}$ and x sufficiently large, s will be in the region $\{s \in \mathbb{C} : \ |s| \leq 10\}$. Recall that the Riemann zeta function has a simple pole at 1. Clearly, the function $\dfrac{f(s)}{s} = \dfrac{g(s)}{\zeta(1+s)^2 s}$ is smooth and bounded by an absolute constant in the region $\{s, \ |s| \leq 10\}$. Therefore we only need to consider the piece $|\operatorname{Im}(s)| > 3$ for which (4.8) can be used.

Thus

$$\left| \int\limits_{a-iR}^{a+iR} \frac{f(s)x^s}{s} ds \right| \lesssim 1 + \int\limits_{\substack{s \in [a-iR, a+iR], \\ |\operatorname{Im}(s)|>1}} (\log R)^2 x^a \frac{ds}{|s|} \lesssim$$

$$\lesssim 1 + x^{-\frac{\sigma_1}{\log R}} \cdot (\log R)^2 \cdot \int\limits_3^R \frac{ds}{s} \lesssim 1 + e^{-\log x \cdot \frac{\sigma_1}{\sqrt{\log x}}} \cdot (\log R)^3.$$

Hence we obtain

$$\left| \int\limits_{a-iR}^{a+iR} \frac{f(s)x^s}{s} ds \right| = O\left(e^{-\frac{1}{2}\sigma_1 (\log x)^{1/2}} \right).$$

so we are done with the case $0 < y \leq 1$.

4.3. Hildebrand—Tenenbaum method: $y > 1$

By Lemma 4.1, we have for any $c > 0$ and $R > 0$,

$$S_N(y) = \sum_{\substack{n < N^y, \\ p|n \Rightarrow 2 < p \leq N}} \frac{(-2)^{\omega(n)}}{n} \mu^2(n) =$$

$$= \sum_{p|n \Rightarrow 2 < p \leq N} \frac{(-2)^{\omega(n)}}{n} \mu^2(n)\, \eta\left(\frac{N^y}{n}\right) =$$

$$= \sum_{p|n \Rightarrow 2<p\leq N} \frac{(-2)^{\omega(n)}}{n}\mu^2(n)\cdot\frac{1}{2\pi i}\int_{c-iR}^{c+iR}\frac{(N^y/n)^s}{s}ds + \text{ error,}$$

where

$$\text{error } \leq \sum_{n=1}^{\infty}\frac{2^{\omega(n)}}{n}\mu^2(n)\left(\frac{N^y}{n}\right)^c \min\left\{1, \frac{1}{R\,|\log{(N^y/n)}|}\right\}.$$

We shall take $c = \dfrac{1}{\log N}$, $R = e^{(\log N)^\delta}$ where $0 < \delta < \dfrac{3}{5}$ is some sufficiently small constant to be chosen later. By Lemma 4.6,

$$\text{error } \leq \sum_{n=1}^{\infty}\frac{2^{\omega(n)}}{n}\left(\frac{N^y}{n}\right)^{1/\log N} \min\left\{1, \frac{1}{R\left|\log\left(\frac{N^y}{n}\right)\right|}\right\} = O\left(R^{-1/2}\right).$$

Therefore the error term can be safely neglected.

Now, neglecting the (controllable) error terms, we write

$$S_N(y) = \sum_{p|n \Rightarrow 2<p\leq N} \frac{(-2)^{\omega(n)}}{n}\mu^2(n)\cdot\frac{1}{2\pi i}\int_{c-iR}^{c+iR}\frac{N^{ys}}{n^s}\frac{ds}{s} =$$

$$= \frac{1}{2\pi i}\int_{c-iR}^{c+iR}\prod_{2<p\leq N}\left(1-\frac{2}{p^{1+s}}\right)\cdot e^{ys\log N}\,\frac{ds}{s}.$$

Now write

$$\prod_{2<p\leq N}\left(1-\frac{2}{p^{1+s}}\right) = \frac{1}{[\zeta_N(1+s)]^2}\cdot g_N(s),$$

where

$$\zeta_N(1+s) = \prod_{p\leq N}\left(1-\frac{1}{p^{1+s}}\right)^{-1};$$

$$g_N(s) = \left(1-2^{-1-s}\right)^{-2}\prod_{2<p\leq N}\left[1-\left(\frac{1}{p^{1+s}-1}\right)^2\right].$$

Note that in the above expression for ζ_N, the prime $p = 2$ is included in the product.

We need the following lemma which is (after some notational changes) Lemma 9.1 from [16]. It allows us to replace the truncated Riemann zeta function by the full zeta function with a very small error term.

LEMMA 4.7. *Let* $0 < \epsilon < \frac{3}{5}$ *and denote* $L_\epsilon(N) = \exp\left((\log N)^{\frac{3}{5}-\epsilon}\right)$. *There exists* $N_0 = N_0(\epsilon) \ge 1$ *such that for any* $N \ge N_0$, $s = \sigma + i\tau$ *with* $\sigma \ge 1 - (\log N)^{-\frac{2}{5}-\varepsilon}$, $|\tau| \le L_\epsilon(N)$, *we have*

$$\zeta_N(s) = \zeta(s) \cdot (s-1)\log N \cdot \widehat{\rho}((s-1)\log N) \cdot \left(1 + O\left(\frac{1}{L_\epsilon(N)}\right)\right), \tag{4.9}$$

where

$$\widehat{\rho}(s) = e^{\gamma + I(-s)},$$

$$I(s) = \int_0^s \frac{e^t - 1}{t} dt \quad (s \in \mathbb{C}). \tag{4.10}$$

Here γ *denotes Euler's constant.*

Remark. In (4.9), the function $\widehat{\rho}(s)$ is connected with the Laplace transform of the Dickman—de Bruijn density $\rho(t)$, i.e.

$$\widehat{\rho}(s) = \int_0^\infty e^{-st} \rho(t) dt, \qquad \mathrm{Re}(s) > 0.$$

An alternative expression for $\widehat{\rho}(s)$ is given by

$$s\widehat{\rho}(s) = e^{-J(s)},$$

where

$$J(\widetilde{s}) = \int_0^\infty \frac{e^{-\widetilde{s}-t}}{\widetilde{s}+t} dt, \text{ for } \widetilde{s} > 0,$$

$$J(\widetilde{s}) = -\gamma - \log \widetilde{s} - \int_0^{-\widetilde{s}} \frac{e^t - 1}{t} dt, \text{ for } \widetilde{s} \in \mathbb{C} \setminus (-\infty, 0].$$

More discussions on this and the Dickman—de Bruijn function can be found on p. 370 of [16].

Specializing to our case, we take $\epsilon = \frac{3}{5} - \delta$ and $L_\epsilon(y) = R = e^{(\log N)^\delta}$ and get for $s = \sigma + i\tau$, $\sigma \geq c = \frac{1}{\log N}$, $|\tau| \leq R$,

$$\zeta_N(1+s) = \zeta(1+s) \cdot (s \log N) \cdot \widehat{\rho}(s \log N) \cdot \left(1 + O\left(\frac{1}{R}\right)\right) =$$
$$= \zeta(1+s) \cdot e^{-J(s \log N)} \cdot \left(1 + O\left(\frac{1}{R}\right)\right).$$

Using the above formula, we write

$$S_N(y) = \frac{1}{2\pi i} \int_{c-iR}^{c+iR} g_N(s) \frac{1}{\zeta(1+s)^2} \cdot e^{2J(s \log N)} \cdot e^{ys \log N} \cdot \frac{1}{\left(1 + O\left(\frac{1}{R}\right)\right)^2} \frac{ds}{s}.$$

We first get rid of the $O\left(\frac{1}{R}\right)$ term. To do this, we estimate

$$\text{error} = \int_{\frac{1}{\log N} - iR}^{\frac{1}{\log N} + iR} |g_N(s)| \frac{1}{|\zeta(1+s)|^2} \cdot e^{2|J(s \log N)|} \cdot \left| e^{ys \log N} \right| \cdot O\left(\frac{1}{R}\right) \frac{ds}{|s|}.$$

Since

$$\left| e^{ys \log N} \right| = \left| e^{y\left(\frac{1}{\log N} + i\tau\right) \log N} \right| = O(1),$$

$$\widetilde{s} = s \log N = \left(\frac{1}{\log N} + i\tau\right) \log N = 1 + i\tau \log N,$$

$$|J(\widetilde{s})| \lesssim \int\limits_0^\infty \frac{e^{-t}}{1+t} dt = O(1),$$

$$\left| \frac{1}{\zeta(1+s)} \right| \lesssim \log R \sim (\log N)^\delta,$$

we get

$$\begin{aligned} \text{error } &\lesssim O\left(\frac{1}{R}\right) \cdot (\log N)^{2\delta} \int\limits_1^R \frac{1}{s} ds \lesssim \\ &\lesssim e^{-(\log N)^\delta} \cdot (\log N)^{2\delta} \cdot (\log N)^\delta \lesssim \\ &= O\left(e^{-\frac{1}{2}(\log N)^\delta}\right). \end{aligned}$$

Hence we only need to treat

$$S_N(y) = \frac{1}{2\pi i} \int\limits_{\frac{1}{\log N} - iR}^{\frac{1}{\log N} + iR} \frac{g_N(s)}{\zeta(1+s)^2} \cdot e^{2J(s \log N) + ys \log N} \cdot \frac{ds}{s}.$$

Recall

$$J(\widetilde{s}) = -\gamma - \log(\widetilde{s}) - \int\limits_0^{-\widetilde{s}} \frac{e^{-t} - 1}{t} dt.$$

Therefore we have $S_N(y) = \frac{1}{2\pi} \frac{e^{-2\gamma}}{(\log N)^2} S_1$ [2], where

$$S_1 = \operatorname{Im} \left[\int\limits_{\frac{1}{\log N} - iR}^{\frac{1}{\log N} + iR} \frac{g_N(s)}{\zeta(1+s)^2} \frac{1}{s^2} \exp\left\{ -2 \int\limits_0^{(-s \log N)} \frac{e^{-t} - 1}{t} dt + ys \log N \right\} \frac{ds}{s} \right].$$

---

[2] The factor $e^{-2\gamma}$ will cancel with another $e^{2\gamma}$ in one of the main terms later, see (4.11).

Now make a change of variable $s \mapsto \dfrac{s}{\log N}$ and get

$$S_1 = \operatorname{Im}\left[\int_{1-iR\log N}^{1+iR\log N} \frac{g_N\left(\frac{s}{\log N}\right)}{\zeta\left(1+\frac{s}{\log N}\right)^2\left(\frac{s}{\log N}\right)^2} \exp\left\{-2\int_0^{-s}\frac{e^{-t}-1}{t}dt + ys\right\}\frac{ds}{s}\right].$$

Since we chose $R = e^{(\log N)^\delta}$ for some small $\delta > 0$, we have

$$e^{(\log N)^\delta} \le R\log N \le e^{(\log N)^{2\delta}}, \quad N \gg 1.$$

Abusing slightly the notation, we regard $R\log N$ as R and write

$$S_1 = \operatorname{Im}\left[\int_{1-iR}^{1+iR} \frac{g_N\left(\frac{s}{\log N}\right)}{\zeta\left(1+\frac{s}{\log N}\right)^2}\frac{1}{\left(\frac{s}{\log N}\right)^2} \exp\left\{-2\int_0^{-s}\frac{e^{-t}-1}{t}dt + ys\right\}\frac{ds}{s}\right].$$

Denote

$$F_N(s) := \frac{g_N(s)}{\zeta\,(1+s)^2\,s^2}.$$

Then we write S_1 more compactly as

$$S_1 = \operatorname{Im}\left[\int_{1-iR}^{1+iR} F_N\left(\frac{s}{\log N}\right) \exp\left\{-2\int_0^{-s}\frac{e^{-t}-1}{t}dt + ys\right\}\frac{ds}{s}\right].$$

Consider the contour depicted on the right (Fig. 4).
The integrand in S_1 is holomorphic in this region. Therefore

$$S_1 = \operatorname{Im}\left(\sum_{j=1}^{5}\int_{\tilde{I}_j}\right).$$

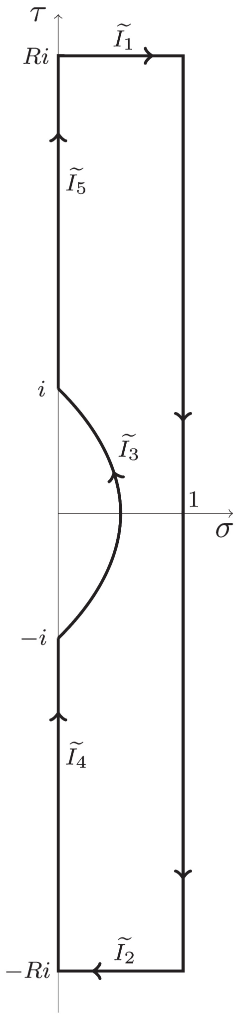

Fig. 4

On $\widetilde{I}_1$ we have $s = \sigma + iR$, $0 \le \sigma \le 1$, and

$$\left| \int_0^{-\sigma - iR} \frac{e^{-t} - 1}{t} dt \right| \lesssim O(1) + |\log R| =$$

$$= O(1) + O\left((\log N)^{\delta} \right) =$$

$$= O\left((\log N)^{\delta} \right);$$

$$|e^{ys}| = O(1).$$

Also, for $s = \sigma + iR$, $0 \le \sigma \le 1$, by (4.8),

$$\left| F_N\left(\frac{s}{\log N} \right) \right| = \left| \frac{g_N\left(\frac{s}{\log N} \right)}{\zeta\left(1 + \frac{s}{\log N} \right)^2 \left(\frac{s}{\log N} \right)^2} \right| \lesssim$$

$$\lesssim R^{-2} (\log N)^{10}.$$

Therefore

$$\int_{\widetilde{I}_1} \left| F_N\left(\frac{s}{\log N} \right) \right| \left| \exp\left\{ -2 \int_0^{-s} \frac{e^{-t} - 1}{t} dt + ys \right\} \right| \frac{ds}{|s|} \lesssim$$

$$\lesssim R^{-2} (\log N)^{11} = O\left(e^{-(\log N)^{\delta}} \right).$$

A similar estimate holds for $\widetilde{I}_2$.

Now consider $\widetilde{I}_3$. Observe that on the arc of $\widetilde{I}_3$, we have $s \sim O(1)$. Therefore we can take $N \to \infty$ and get

$$\lim_{N \to \infty} \int_{\widetilde{I}_3} F_N\left(\frac{s}{\log N} \right) \exp\left\{ -2 \int_0^{-s} \frac{e^{-t} - 1}{t} dt + ys \right\} \frac{ds}{s} =$$

$$= F(0) \int_{\widetilde{I}_3} \exp\left\{ -2 \int_0^{-s} \frac{e^{-t} - 1}{t} dt + ys \right\} \frac{ds}{s},$$

where we have denoted

$$F(s) := \lim_{N\to\infty} F_N(s) = \frac{g(s)}{\zeta\left(1+s\right)^2 s^2};$$

$$g(s) = \left(1 - 2^{-1-s}\right)^{-2} \prod_{p>2} \left(1 - \left(\frac{1}{p^{1+s}-1}\right)^2\right).$$

It remains to consider the integral on $\widetilde{I}_4$ and $\widetilde{I}_5$. It is not difficult to check that

$$\operatorname{Im}\left[\int_{\widetilde{I}_4} F_N\left(\frac{s}{\log N}\right) \exp\left\{-2\int_0^{-s} \frac{e^{-t}-1}{t}dt + ys\right\} \frac{ds}{s}\right] =$$

$$= \operatorname{Im}\left[\int_{\widetilde{I}_5} F_N\left(\frac{s}{\log N}\right) \exp\left\{-2\int_0^{-s} \frac{e^{-t}-1}{t}dt + ys\right\} \frac{ds}{s}\right].$$

Therefore we only need to look at the imaginary part of the expression

$$S_4 := \int_{\widetilde{I}_4} F_N\left(\frac{s}{\log N}\right) \exp\left\{-2\int_0^{-s} \frac{e^{-t}-1}{t}dt + ys\right\} \frac{ds}{s}.$$

To this end we write

$$S_4 = \int_{-iR}^{-i} F_N\left(\frac{s}{\log N}\right) \exp\left\{-2\int_0^{-s} \frac{e^{-t}-1}{t}dt + ys\right\} \frac{ds}{s} =$$

$$= \int_R^1 F_N\left(-\frac{i\lambda}{\log N}\right) \exp\left\{-2\int_0^{i\lambda} \frac{e^{-t}-1}{t}dt - i\lambda y\right\} \frac{d\lambda}{\lambda} =$$

$$= -\int_1^R F_N\left(-\frac{i\lambda}{\log N}\right) \exp\left\{-2\int_0^{i\lambda} \frac{e^{-t}-1}{t}dt - i\lambda y\right\} \frac{d\lambda}{\lambda}.$$

We decompose S_4 into many pieces each of which will be shown to have a limit as $N \to \infty$.

Recall the expansion (see the proof of Theorem 1.3) for $\lambda \geq 1$:

$$\int_0^1 \left(e^{i\lambda s}-1\right)\frac{ds}{s} = \int_0^\lambda \frac{e^{is}-1}{s}ds =$$

$$= \int_0^1 \frac{e^{is}-1}{s}ds - \int_1^\lambda \frac{1}{s}ds + \int_1^\lambda \frac{e^{is}}{s}ds =$$

$$= \int_0^1 \frac{e^{is}-1}{s}ds - \log\lambda + \int_1^\infty \frac{e^{is}}{s}ds - \int_\lambda^\infty \frac{e^{is}}{s}ds =$$

$$= \int_0^1 \frac{e^{is}-1}{s}ds - \log\lambda + \int_1^\infty \frac{e^{is}}{s}ds + \left(\frac{1}{i\lambda}-\frac{1}{\lambda^2}\right)e^{i\lambda} + \widetilde{a}(\lambda)$$

where $\widetilde{a}(\lambda)$ is smooth and $|\widetilde{a}(\lambda)| \lesssim \frac{1}{\lambda^3}$. Then we have

$$\exp\left\{-2\int_0^\lambda \frac{e^{it}-1}{t}dt\right\} = -e^{2\gamma}\lambda^2 \exp\left\{\left(-\frac{2}{i\lambda}+\frac{2}{\lambda^2}\right)e^{i\lambda} - 2\widetilde{a}(\lambda)\right\} =$$

$$= -e^{2\gamma}\lambda^2 \cdot \left(1+\frac{2i}{\lambda}e^{i\lambda}+\frac{2}{\lambda^2}e^{i\lambda}+\frac{1}{2}\left(-\frac{2}{i\lambda}\right)^2 e^{2i\lambda}+\widetilde{a_1}(\lambda)\right) =$$

$$= -e^{2\gamma}\lambda^2 \cdot \left(1+\frac{2i}{\lambda}e^{i\lambda}+\frac{2}{\lambda^2}e^{i\lambda}-\frac{2}{\lambda^2}e^{2i\lambda}+\widetilde{a_1}(\lambda)\right),$$

where $\widetilde{a_1}(\lambda)$ is a smooth function and $|\widetilde{a_1}(\lambda)| \lesssim \frac{1}{\lambda^3}$ for $\lambda \geq 1$.

Plugging the above expression into S_4, we get

$$S_4 = e^{2\gamma}\int_1^R F_N\left(-\frac{i\lambda}{\log N}\right)\cdot\lambda\left(1+\frac{2i}{\lambda}e^{i\lambda}+\frac{2}{\lambda^2}e^{i\lambda}-\frac{2}{\lambda^2}e^{2i\lambda}\right)e^{-i\lambda y}d\lambda +$$

$$+ e^{2\gamma}\int_1^R F_N\left(-\frac{i\lambda}{\log N}\right)\cdot\lambda\,\widetilde{a_1}(\lambda)\,e^{-i\lambda y}d\lambda =: S_4' + S_4''. \tag{4.11}$$

Consider first S_4''. Since $F_N\left(-\dfrac{i\lambda}{\log N}\right)$ is uniformly bounded for $1 \le \lambda \le R$ and $|\widetilde{a_1}(\lambda)| = O\left(\dfrac{1}{\lambda^3}\right)$, we can take $N \to \infty$ and get

$$\lim_{N\to\infty} S_4'' = e^{2\gamma}\cdot F(0)\cdot \int\limits_1^\infty \lambda\, \widetilde{a_1}(\lambda)\, e^{-i\lambda y} d\lambda.$$

Clearly, this gives rise to a bounded continuous function.

It remains to consider S_4'. Write

$$\begin{aligned} e^{-2\gamma} S_4' = & \int\limits_1^R F_N\left(-\frac{i\lambda}{\log N}\right)\cdot \lambda\, e^{-i\lambda y} d\lambda + && (a)\\ & + \int\limits_1^R F_N\left(-\frac{i\lambda}{\log N}\right)\cdot 2i\, e^{i\lambda(1-y)} d\lambda + && (b)\\ & + \int\limits_1^R F_N\left(-\frac{i\lambda}{\log N}\right)\cdot \frac{2}{\lambda}\, e^{i\lambda(1-y)} d\lambda + && (c)\\ & + \int\limits_1^R F_N\left(-\frac{i\lambda}{\log N}\right)\cdot \left(-\frac{2}{\lambda}\right)\, e^{i\lambda(2-y)} d\lambda. && (d) \end{aligned}$$

We recall that we only need to consider the imaginary part of the above expressions. We shall analyze each term separately.

We have to do several integrations by parts. For this we need a simple lemma.

Lemma 4.8. *Consider* $\widetilde{F}(\tau) = \dfrac{g_N(-i\tau)}{[\zeta(1-i\tau)\,\tau]^2}$ *for* $\dfrac{1}{\log N} < \tau < e^{(\log N)^{\delta_1}}$, $\delta_1 > 0$. *Then for* $k \le 100$ *we have*

$$\left|\widetilde{F}^{(k)}(\tau)\right| \lesssim \frac{1}{1+\tau^2}(\log N)^{c_1\delta_1}$$

for some absolute constant c_1.

Proof. If $\tau < 3$ this easily follows from the fact that $\dfrac{1}{\zeta(1-i\tau)\,\tau}$ is smooth (since $\zeta(s)$ has a simple pole at $s = 1$).

If $\tau \geq 3$ then we use Lemma 4.4 to get

$$\zeta^{(k)}(s) \ll (\log|\tau|)^{k+1}, \qquad \frac{1}{\zeta(s)} \ll \log|\tau|, \qquad \text{if } |\tau| \geq 3, \sigma \geq 1 - \frac{c}{\log|\tau|}.$$

Note that $\tau < e^{(\log N)^{\delta_1}}$ implies $\log\tau < (\log N)^{\delta_1}$. The desired result follows easily. ☐

Estimate of (a). Since $y > 0$ we may integrate by parts to get

$$(a) = F_N\left(-\frac{i\lambda}{\log N}\right) \cdot \lambda \cdot \frac{e^{-i\lambda y}}{-iy}\Bigg|_1^R - \frac{1}{-iy}\int\limits_1^R \left[F_N\left(-\frac{i\lambda}{\log N}\right)\cdot\lambda\right]' \cdot e^{-i\lambda y}d\lambda =$$

$$= o_N(1) + F(0)\cdot e^{-iy}\frac{1}{iy} +$$

$$+\frac{1}{iy}\int\limits_1^R F_N'\left(-\frac{i\lambda}{\log N}\right)\cdot\left(-\frac{i\lambda}{\log N}\right)\cdot e^{-i\lambda y}d\lambda + \tag{$a1$}$$

$$+\frac{1}{iy}\int\limits_1^R F_N\left(-\frac{i\lambda}{\log N}\right)\cdot e^{-i\lambda y}d\lambda. \tag{$a2$}$$

For $(a1)$, by Lemma 4.8 and successive integration by parts one can check that $\lim\limits_{N\to\infty}(a1) = 0$.

For $(a2)$, we perform integration by parts once and this gives

$$\int\limits_1^R F_N\left(-\frac{i\lambda}{\log N}\right)e^{-i\lambda y}d\lambda = \frac{F_N\left(-\frac{i\lambda}{\log N}\right)e^{-i\lambda y}}{-iy}\Bigg|_1^R + O\left(\frac{1}{\log N}\right) \to F(0)\frac{e^{-iy}}{iy}.$$

Collecting the estimates, we get

$$\lim_{N\to\infty}(a) = F(0)\left(\frac{e^{-iy}}{iy} + \frac{1}{iy}\frac{e^{-iy}}{iy}\right) = F(0)\left(\frac{\cos y}{iy} + \frac{-i\sin y}{-y^2}\right) + \text{ real parts }.$$

Therefore $\lim\limits_{N\to\infty}\mathrm{Im}((a)) = F(0)\left(-\frac{\cos y}{y} + \frac{\sin y}{y^2}\right) = O(y)$ which gives no singularity at $y = 0$.

Estimate of (b). Consider two cases.

Case 1: $y = 1$. We have

$$(b) = 2i \int_{1}^{R} F_N\left(-\frac{i\lambda}{\log N}\right) d\lambda =$$

$$= 2i \log N \left(\int_{\frac{1}{\log N}}^{\frac{R}{\log N}} F_N(-i\lambda) d\lambda \right) =$$

$$= 2i \log N \left(\int_{0}^{\infty} F_N(-i\lambda) d\lambda - \int_{\frac{R}{\log N}}^{\infty} F_N(-i\lambda) d\lambda - \int_{0}^{\frac{1}{\log N}} F_N(-i\lambda) d\lambda \right).$$

Observe that

$$\mathrm{Re}\left(\int_{0}^{\infty} F_N(-i\lambda) d\lambda \right) = \frac{1}{2} \int_{-\infty}^{\infty} F_N(-i\lambda) d\lambda = \frac{1}{2} \int_{-\infty}^{\infty} \frac{g_N(-i\lambda)}{\zeta\left(1 - i\lambda\right)^2 (-i\lambda)^2} d\lambda.$$

Since the function $F_N(s) = \dfrac{g_N(s)}{\zeta\left(1+s\right)^2 s^2}$ is analytic on $\mathrm{Re}(s) \geq 0$ and has good decay, one can easily deform the contour and show that $\int_{-i\infty}^{i\infty} F_N(s) ds = 0$.

Therefore $\mathrm{Re}\left(\int_{0}^{\infty} F_N(-i\lambda) d\lambda \right) = 0$. By Lemma 4.3, it is easy to estimate that

$$\left| \int_{\frac{R}{\log N}}^{\infty} F_N(-i\lambda) d\lambda \right| = O\left(e^{-\frac{1}{100}(\log N)^{\delta}} \right)$$

and, obviously,

$$\lim_{N\to\infty} \left(\log N \cdot \left| \int_{\frac{R}{\log N}}^{\infty} F_N(-i\lambda) d\lambda \right| \right) = 0.$$

Thus, for $y = 1$, we have

$$\lim_{N\to\infty} \operatorname{Im}((b)) = -2 \lim_{N\to\infty} \left(\log N \int_0^{\frac{1}{\log N}} F_N(-i\lambda) d\lambda \right) = -2F(0).$$

Case 2: $y \neq 1$. In this case we may integrate by parts and obtain

$$(b) = 2i\, F_N \left(-\frac{i\lambda}{\log N} \right) \frac{e^{i\lambda(1-y)}}{i(1-y)} \Bigg|_1^R -$$

$$- \frac{2i}{i(1-y)} \int_1^R F_N' \left(-\frac{i\lambda}{\log N} \right) \cdot e^{i\lambda(1-y)} \left(-\frac{i}{\log N} \right) d\lambda =$$

$$= o_N(1) - 2iF(0) \frac{e^{i\lambda(1-y)}}{i(1-y)} =$$

$$= o_N(1) - 2iF(0) \frac{\sin(1-y)}{(1-y)} + \text{ real parts.}$$

Hence, for $y \neq 1$, $\lim\limits_{N\to\infty} \operatorname{Im}((b)) = -2F(0) \dfrac{\sin(1-y)}{(1-y)}$.

Note that the above formula also works for $y = 1$. Therefore in all cases we get $\lim\limits_{N\to\infty} \operatorname{Im}((b)) = -2F(0) \dfrac{\sin(1-y)}{(1-y)}$.

Estimate of (c). We discuss two cases.

Case 1: $y = 1$. In this case we have

$$\operatorname{Im}((c)) = 2 \operatorname{Im} \int_1^R F_N \left(-\frac{i\lambda}{\log N} \right) \frac{d\lambda}{\lambda} =$$

$$= 2 \operatorname{Im} i \int_1^R \frac{F_N \left(-\frac{i\lambda}{\log N} \right) - F_N \left(\frac{i\lambda}{\log N} \right)}{2i} \cdot \frac{d\lambda}{\lambda} =$$

$$= \int_{\frac{1}{\log N}}^{\frac{R}{\log N}} \frac{F_N(-i\lambda) - F_N(i\lambda)}{i} \cdot \frac{d\lambda}{\lambda},$$

obviously, the above integral converges absolutely and

$$\lim_{N\to\infty} \operatorname{Im}((c)) = \int_0^\infty \frac{F(-i\lambda) - F(i\lambda)}{i} \cdot \frac{d\lambda}{\lambda}.$$

In fact, we can simplify the expression $A := \int_0^\infty \frac{F(-i\lambda)-F(i\lambda)}{i}\frac{d\lambda}{\lambda}$ further. Observe that

$$A = \lim_{\varepsilon\to 0} 2 \operatorname{Im} \int_\varepsilon^\infty \frac{F(-i\lambda)}{\lambda} d\lambda =$$

$$= -2 \lim_{\varepsilon\to 0} \operatorname{Im} \int_\varepsilon^\infty \frac{F(i\lambda)}{\lambda} d\lambda =$$

$$= -2 \lim_{\varepsilon\to 0} \operatorname{Im} \int_{i\varepsilon}^{i\infty} \frac{F(s)}{s} ds =$$

$$= - \lim_{\varepsilon\to 0} \operatorname{Im} \left(\int_{-i\infty}^{-i\varepsilon} \frac{F(s)}{s} ds + \int_{i\varepsilon}^{i\infty} \frac{F(s)}{s} ds \right).$$

Since $\frac{F(s)}{s}$ is analytic away from 0, it is easy to see that

$$A = \lim_{\varepsilon\to 0} \operatorname{Im} \left(\int_{-i\varepsilon}^{i\varepsilon} \frac{F(s)-F(0)}{s} ds + F(0) \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} i d\theta \right) = \pi F(0).$$

<u>Case 2</u>: $y \neq 1$. Integrating by parts, we get

$$(c) = F_N\left(-\frac{i\lambda}{\log N}\right) \frac{2}{\lambda} \frac{e^{i\lambda(1-y)}}{i(1-y)}\Bigg|_1^R -$$

$$- \frac{2}{i(1-y)} \int_1^R F_N'\left(-\frac{i\lambda}{\log N}\right) \cdot \frac{1}{\lambda} \left(-\frac{i}{\log N}\right) e^{i\lambda(1-y)} d\lambda -$$

$$
\begin{aligned}
&-\frac{2}{i(1-y)}\int_1^R F_N\left(-\frac{i\lambda}{\log N}\right)\cdot\left(-\frac{1}{\lambda^2}\right)\ e^{i\lambda(1-y)}d\lambda=\\
&=o_N(1)-2F(0)\frac{e^{i\lambda(1-y)}}{i(1-y)}+\\
&+\frac{2}{i(1-y)}\int_1^R F_N\left(-\frac{i\lambda}{\log N}\right)\cdot\frac{1}{\lambda^2}\,e^{i\lambda(1-y)}d\lambda\longrightarrow\\
&\longrightarrow 2F(0)\frac{ie^{i(1-y)}}{(1-y)}+F(0)\frac{2}{i(1-y)}\int_1^\infty\frac{e^{i\lambda(1-y)}}{\lambda^2}d\lambda.
\end{aligned}
$$

From the above computation, we see that F_N is essentially taken as the constant $F(0)$ during the limit process $N\to\infty$.

Returning to the original expression for (c) and replacing $F_N\left(-\frac{i\lambda}{\log N}\right)$ by $F(0)$, we get for $y\neq 1$,

$$
\begin{aligned}
\lim_{N\to\infty}\operatorname{Im}((c))&=2F(0)\lim_{N\to\infty}\left(\int_1^R\frac{\sin(\lambda(1-y))}{\lambda}d\lambda\right)=\\
&=2F(0)\lim_{N\to\infty}\left(\operatorname{sgn}(1-y)\int_1^R\frac{\sin(\lambda\,|y-1|)}{\lambda}d\lambda\right)=\\
&=2F(0)\operatorname{sgn}(1-y)\lim_{N\to\infty}\left(\int_{|y-1|}^{R|y-1|}\frac{\sin\lambda}{\lambda}d\lambda\right).
\end{aligned}
$$

Since $\int_0^\infty\frac{\sin\lambda}{\lambda}d\lambda=\frac{\pi}{2}$, we get for $y\neq 1$,

$$
\begin{aligned}
\lim_{N\to\infty}\operatorname{Im}((c))&=2F(0)\operatorname{sgn}(1-y)\left(\frac{\pi}{2}-\int_0^{|y-1|}\frac{\sin\lambda}{\lambda}d\lambda\right)=\\
&=\pi F(0)\operatorname{sgn}(1-y)-2F(0)\operatorname{sgn}(1-y)\int_0^{|y-1|}\frac{\sin\lambda}{\lambda}d\lambda.
\end{aligned}
$$

Therefore

$$\lim_{N\to\infty} \operatorname{Im}((c)) = \begin{cases} F(0)\left(\pi - 2\displaystyle\int_0^{1-y} \frac{\sin\lambda}{\lambda}d\lambda\right), & y \le 1; \\ F(0)\left(-\pi + 2\displaystyle\int_0^{y-1} \frac{\sin\lambda}{\lambda}d\lambda\right), & y > 1. \end{cases}$$

Clearly, the “jump” at $y = 1$ is $-2\pi F(0)$.

Similarly, we get

$$\lim_{N\to\infty} \operatorname{Im}((d)) = \begin{cases} F(0)\left(\pi - 2\displaystyle\int_0^{2-y} \frac{\sin\lambda}{\lambda}d\lambda\right), & y \le 2; \\ F(0)\left(-\pi + 2\displaystyle\int_0^{y-2} \frac{\sin\lambda}{\lambda}d\lambda\right), & y > 2. \end{cases}$$

Collecting all the above estimates, we obtain

$$\begin{aligned} S_N(y) &= \frac{e^{-2\gamma}}{2\pi} \cdot \frac{1}{(\log N)^2} \cdot S_1 = \\ &= \frac{e^{-2\gamma}}{2\pi} \cdot \frac{1}{(\log N)^2} \cdot \left\{ F(0) \operatorname{Im}\left(\int_{\widetilde{I}_3} \exp\left\{ -2\int_0^{-s} \frac{e^{-t}-1}{t}dt + ys \right\} \frac{ds}{s} \right) + \right. \\ &\quad \left. + e^{2\gamma} \operatorname{Im}(S_4') + o_N(1) + \text{continuous part} \right\} = \\ &= \frac{F(0)}{(\log N)^2} \left\{ e^{-2\gamma} \operatorname{Im}\left(\int_{\widetilde{I}_3} \exp\left\{ -2\int_0^{-s} \frac{e^{-t}-1}{t}dt + ys \right\} \frac{ds}{s} \right) - \right. \\ &\quad \left. - H(y-1) + H(y-2) + o_N(1) + \text{continuous part} \right\}. \qquad (4.12) \end{aligned}$$

Now note that $F(0) = g(0) = 4 \prod_{\substack{p>2 \\ p \text{ is prime}}} \left(1 - \frac{1}{(p-1)^2}\right)$.

We have obtained the desired form for the limiting distribution $G(y)$.

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