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On Dirichlet spectrum for two-dimensional simultaneous Diophantine approximation

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Abstract: We give a complete description of the Dirichlet spectrum for simultaneous Diophantine approximation to two real numbers. It turnes out that this Dirichlet spectrum is just the segment $\left[0, 2/\sqrt{3}\right]$.

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Our paper is devoted to the description of the complete structure of two-dimensional Dirichlet spectrum. In Section 1 we recall well-known results concerning one-dimensional Dirichlet spectrum. In Section 2 we formulate our results on two-dimensional spectrum. The rest of the paper deals with proofs of our results.

1. One-dimensional Diophantine approximation

Let $\alpha \in \mathbb{R}$ be an irrational number. We consider the irrationality measure function

$$\psi_\alpha(t) = \min_{1 \leqslant q \leqslant t} ||q\alpha||$$

(here min is taken over integers q and $||\cdot||$ stands for the distance to the nearest integer). The *one-dimensional Dirichlet spectrum*

$$\mathbb{D} = \{\lambda \in \mathbb{R} \mid \exists\, \alpha \in \mathbb{R} : \limsup_{t\to\infty} t \cdot \psi_\alpha(t) = \lambda\}$$

was considered by many mathematicians. It has a quite difficult structure. Szekeres [9] showed that

$$\mathbb{D} \subset \left[\frac{5+\sqrt{5}}{10}, 1\right]$$

(see also a paper by Davenport and Schmidt [1]). It is known that close to the point $5+\sqrt{5}/10$ Dirichlet spectrum is discrete and there exists $d_* < 1$ such that $[d_*, 1] \subset \mathbb{D}$ (see works by Lesca [6] and Diviš and Nowak [2]). Ivanov [3–5] showed that

$$\operatorname{mes}\left(\mathbb{D} \cap \left[0, \frac{4+3\sqrt{3}}{11}\right)\right) = 0,$$

and

$$d^* \in \left[\frac{3\sqrt{5}-5}{2}, \frac{38+6\sqrt{2}}{49}\right]$$

for

$$d^* = \inf\{d \mid \mathbb{D} \supset [d, 1]\}.$$

The complete structure of $\mathbb{D}$ is probably not known yet.

All the results mentioned above are obtained by means of continued fractions theory. For α we consider its continued fraction representation $\alpha = [a_0; a_1, a_2, \ldots]$ and convergents $\dfrac{p_n}{q_n} = [a_0; a_1, a_2, \ldots, a_n]$. In terms of continued fractions one has

$$\psi_\alpha(t) = ||q_n\alpha|| \text{ for } q_n \leqslant t < q_{n+1},$$

and

$$\mathbb{D} = \{\lambda \in \mathbb{R} \mid \exists\, \alpha \in \mathbb{R} : \limsup_{n\to\infty} q_{n+1} \cdot ||q_n\alpha|| = \lambda\}.$$

The main tool for the study of Dirichlet spectrum's structure is the equality

$$q_{n+1}\cdot||q_n\alpha|| = \frac{1}{1+\dfrac{1}{\alpha_{n+2}\alpha^*_{n+1}}},$$

where $\alpha_n = [a_n; a_{n+1}, a_{n+2}, \ldots]$ and $\alpha^*_n = [a_n; a_{n-1}, a_{n-2}, \ldots, a_1]$.

Some one-dimensional and multi-dimensional results related to Dirichlet spectrum are discussed in a survey by Moshchevitin [8].

2. Two-dimensional Dirichlet spectrum: formulations and notation

In the present paper we consider the Euclidean norm, for simplicity. Probably the similar results are true for other norms.

Definition 1. *For a vector* $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ *we define the irrationality measure function with respect to Euclidean norm as*

$$\psi_{\mathbf{v}}(t) = \min_{1\leqslant q\leqslant t,\ q\in\mathbb{Z}} \sqrt{||qv_1||^2+||qv_2||^2}.$$

Definition 2. *Two-dimensional Dirichlet spectrum with respect to Euclidean norm is defined as*

$$\mathbb{D}_2 = \left\{\lambda\in\mathbb{R} \mid \exists\, \mathbf{v}\in\mathbb{R}^2 :\ \limsup_{t\to\infty} t\cdot\psi^2_{\mathbf{v}}(t) = \lambda\right\}.$$

From Minkowski's convex body theorem it follows that

$$\mathbb{D}_2 \subset \left[0, \frac{4}{\pi}\right].$$

Mahler's theorem on the critical determinant of the three-dimensional cylinder [7] gives

$$\mathbb{D}_2 \subset \left[0, \frac{2}{\sqrt{3}}\right].$$

In the present paper we prove the following statement.

THEOREM 1. *We have*

$$\mathbb{D}_2 = \left[0, \frac{2}{\sqrt{3}}\right].$$

DEFINITION 3. *The sequence*

$$\mathcal{Z} : \mathbf{w}_n = (q_n, \mathbf{p}_n) \in \mathbb{Z}^3, \quad n \geqslant 0,$$

is the sequence of the best approximation vectors for $\mathbf{v} \in \mathbb{R}^2$ *with respect to Euclidean norm if*

1) $q_0 = 1$;

2) $q_{n+1} = \min\left\{q \in \mathbb{N} \mid \sqrt{||qv_1||^2 + ||qv_2||^2} < \sqrt{||q_n v_1||^2 + ||q_n v_2||^2}\right\} \quad (n \geqslant 0)$;

3) $\mathbf{p}_n = (p_{n,1}, p_{n,2}) : ||q_n v_1|| = |q_n v_1 - p_{n,1}|, ||q_n v_2|| = |q_n v_2 - p_{n,2}| \quad (n \geqslant 0)$.

Remark 2.1. The sequence of the best approximation vectors may be not defined uniquely, in general. However the sequence $\{q_n\}_{n=0}^{\infty}$ is defined uniquely, whereas the sequence $\{\mathbf{p}_n\}_{n=0}^{\infty}$ may be not uniquely defined.

Remark 2.2. The sequence of the best approximation vectors may be finite or infinite. However in the present paper we consider the case of infinite sequences only.

Remark 2.3. The function $\psi_{\mathbf{v}}(t)$ is a piecewise constant decreasing function and

$$\psi_{\mathbf{v}}(t) = \sqrt{||q_n v_1||^2 + ||q_n v_2||^2} = \sqrt{|q_n v_1 - p_{n,1}|^2 + |q_n v_2 - p_{n,2}|^2} =$$
$$= |q_n \mathbf{v} - \mathbf{p}_n|, \qquad \text{for } q_n \leqslant t < q_{n+1}.$$

PROPOSITION 1. *One has*

$$\mathbb{D}_2 = \left\{\lambda \in \mathbb{R} \mid \exists\, \mathbf{v} \in \mathbb{R}^2 : \limsup_{n\to\infty} q_{n+1} \cdot |q_n \mathbf{v} - \mathbf{p}_n|^2 = \lambda\right\},$$

where $\mathbf{w}_n = (q_n, \mathbf{p}_n) \in \mathbb{Z}^3$, $n \geqslant 0$, *is the sequence of the best approximation vectors with respect to Euclidean norm for* $\mathbf{v} \in \mathbb{R}^2$.

Given $\mathbf{v} \in \mathbb{R}^2$, $Q > 0$ and $R > 0$, we define the cylinder $\Pi = \Pi(\mathbf{v}, Q, R)$ by

$$\Pi(\mathbf{v}, Q, R) = \left\{(q, \mathbf{p}) \in \mathbb{R} \times \mathbb{R}^2 \mid q \in [0, Q],\ |q\mathbf{v} - \mathbf{p}| \leqslant R\right\}.$$

Let $\mathrm{vol}(\Pi) = \pi Q R^2$ be the volume of Π, and let $\operatorname{int} \Pi$ be the set of interior points of Π. By $\partial \Pi$ we denote the boundary of Π.
Put

$$\widehat{\Pi} = \left\{ (q, \mathbf{p}) \in \mathbb{R} \times \mathbb{R}^2 \mid q \in \mathbb{R},\ |q\mathbf{v} - \mathbf{p}| \leqslant R \right\}.$$

We define the *main facet* of Π by

$$\left\{ (q, \mathbf{p}) \in \mathbb{R} \times \mathbb{R}^2 \mid q = Q,\ |q\mathbf{v} - \mathbf{p}| < R \right\}.$$

We define the *axis vector* for Π as any vector of the form $t \cdot (1, \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^2, t > 0$. We define the *non-principal boundary* of Π as

$$\partial\widehat{\Pi} = \left\{ (q, \mathbf{p}) \in \mathbb{R} \times \mathbb{R}^2 \mid q \in (0, Q),\ |q\mathbf{v} - \mathbf{p}| = R \right\}.$$

The *continuation* of the non-principal boundary of Π is defined as

$$\left\{ (q, \mathbf{p}) \in \mathbb{R} \times \mathbb{R}^2 \mid q \in \mathbb{R},\ |q\mathbf{v} - \mathbf{p}| = R \right\}.$$

By the *length* of the cylinder Π we mean the value of Q. By the *radius* of Π we mean R.

For $\mathbf{v} \in \mathbb{R}^2$ and the sequence

$$\mathcal{Z} : \mathbf{w}_n = (q_n, \mathbf{p}_n) \in \mathbb{N} \times \mathbb{Z}^2, \quad n \geqslant 0,$$

we define $R_0 = 1$, and $R_n = |q_{n-1}\mathbf{v} - \mathbf{p}_{n-1}|$, $\Pi_n = \Pi(\mathbf{v}, q_n, R_n)$, $V_n = \mathrm{vol}(\Pi_n) = \pi q_n \cdot |q_{n-1}\mathbf{v} - \mathbf{p}_{n-1}|^2$, for $n \geqslant 1$.

PROPOSITION 2. *One has*

$$\mathbb{D}_2 = \left\{ \lambda \in \mathbb{R} \mid \exists\, \mathbf{v} \in \mathbb{R}^2 :\ \limsup_{n \to \infty} \frac{1}{\pi} V_{n+1} = \lambda \right\},$$

where the sequence of vectors $\mathbf{w}_n = (q_n, \mathbf{p}_n) \in \mathbb{Z}^3,\ n \geqslant 0$, *is the sequence of the best approximation vectors with respect to Euclidean norm for* $\mathbf{v} \in \mathbb{R}^2$.

THEOREM 2. *Take* $\lambda \in \left[0, 2/\sqrt{3}\right]$. *Then there exists an uncountable set of vectors* $\mathbf{v} \in \mathbb{R}^2$ *such that*

$$\lim_{n \to \infty} \frac{1}{\pi} V_{n+1} = \lambda.$$

Of course Theorem 1 follows from Theorem 2. In fact we prove a more general result.

Theorem 3. *Consider an arbitrary sequence* $\{\Delta_n\}_{n=1}^{\infty}$ *of subsegments for the segment* $\left[0, 2/\sqrt{3}\right]$. *Then there exists an uncountable set of vectors* $\mathbf{v} \in \mathbb{R}^2$ *such that*

$$\frac{1}{\pi} V_n \in \Delta_n, \quad \forall\, n \in \mathbb{N}.$$

It is clear that Theorem 2 follows from Theorem 3.

Proposition 3. *The sequence*

$$\mathcal{Z} : \mathbf{w}_n = (q_n, \mathbf{p}_n) \in \mathbb{Z}^3, \quad n \geqslant 0$$

is the sequence of the best approximation vectors with respect to Euclidean norm for $\mathbf{v} \in \mathbb{R}^2$, *if and only if*

1) $q_0 = 1$;
2) $(\operatorname{int} \Pi_n) \cap \mathbb{Z}^3 = \varnothing \quad (n \geqslant 1)$;
3) $q_{n+1} > q_n \quad (n \geqslant 0)$;
4) $R_{n+1} < R_n \quad (n \geqslant 0)$.

Remark 2.4. One has $\mathbf{w}_{n-1}, \mathbf{w}_n \in \partial \Pi_n$. Moreover the point $\mathbf{w}_{n-1}$ belongs to the non-principal boundary of Π_n. The point $\mathbf{w}_n$ belongs to the main facet of Π_n.

3. Proof of Theorem 3: inductive procedure

We construct a sequence of vectors

$$\mathcal{Z} : \mathbf{w}_n = (q_n, \mathbf{p}_n) \in \mathbb{Z}^3, \quad n \geqslant 0,$$

satisfying the following properties 1)–6).

For simplicity we use the following notation.

$$\mathbf{v}_n = \frac{\mathbf{p}_n}{q_n}, \quad q_{-1} = 0, \quad \mathbf{p}_{-1} = (1, 0), \quad \mathbf{w}_{-1} = (q_{-1}, \mathbf{p}_{-1}),$$

for $0 \leqslant \nu \leqslant n$ put $R_n^{\nu} = |q_{\nu-1}/q_n \mathbf{p}_n - \mathbf{p}_{\nu-1}|$, $\Pi_n^{\nu} = \Pi\left(\mathbf{v}_n, q_\nu, R_n^{\nu}\right)$, $V_n^{\nu} = \operatorname{vol}(\Pi_n^{\nu})$.

1) $\Pi_n^\nu \cap \mathbb{Z}^3 = \begin{cases} \{(0,0,0), \mathbf{w}_{\nu-1}, \mathbf{w}_\nu, \mathbf{w}_\nu - \mathbf{w}_{\nu-1}\}, & \nu = n \geqslant 1; \\ \{(0,0,0), \mathbf{w}_{\nu-1}, \mathbf{w}_\nu\}, & 1 \leqslant \nu < n, \end{cases}$

and $\mathbf{w}_{\nu-1}$ belongs to the non-principal boundary of the cylinder Π_n^ν, and $\mathbf{w}_\nu - \mathbf{w}_{\nu-1}$ belongs to the non-principal boundary of Π_n^ν (with $\nu = n$), whereas $\mathbf{w}_\nu$ belongs to the main facet of Π_n^ν;

2) $q_n > 2q_{n-1} \quad (n \geqslant 1)$;

3) $R_n^\nu < \frac{1}{2} R_n^{\nu-1} \quad (1 \leqslant \nu \leqslant n)$;

4) $\frac{1}{\pi} V_n^\nu \in \operatorname{int}\Delta_\nu \quad (1 \leqslant \nu \leqslant n)$;

5) $|\mathbf{v}_n - \mathbf{v}_{n-1}| < \frac{1}{2^n} \quad (n \geqslant 1)$;

6) $|R_n^\nu - R_{n-1}^\nu| < \frac{1}{2^n} \quad (1 \leqslant \nu \leqslant n-1)$.

We construct the sequence $\mathcal{Z}$ by induction in n.

A. The base of induction for $n = 0$.

Define $\mathbf{w}_0$ as follows. Put $q_0 = 1$, $\mathbf{p}_0 = (0,0)$. We see that for $n = 0$ all the conditions 1)–6) are empty, and there is nothing to check.

B. The inductive assumption. We suppose that the vectors $\mathbf{w}_0, \ldots, \mathbf{w}_{n-1}$ satisfying 1)–6) are already constructed.

C. Inductive step. We construct $\mathbf{w}_n$ and prove that it satisfies conditions 1)–6).

Let π_{n-1} be the completely rational plane in $\mathbb{R}^3$ containing the points $\mathbf{w}_{n-1}$, $\mathbf{w}_{n-2}$ and the origin $\mathbf{0}$.

Let π'_{n-1} be a parallel to π_{n-1} completely rational plane, neighbouring to π_{n-1}.

Remark 3.1. There are two different completely rational parallel planes neighbouring to π_{n-1}. As π'_{n-1} we may take any of these two planes. This remark is of importance for the proof of uncountability of the set of desired vectors $\mathbf{v} \in \mathbb{R}^2$.

4. A linear transformation

Let G be the linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ defined by the following conditions **a)**–**f)**.

a) $G(\sigma) = \sigma$, where $\sigma = \{x = 1\}$;

b) G preserves Euclidean distances between points in the plane σ;

c) $G(\mathbf{w}_{n-1}) = (q_{n-1}, 0, 0)$;

d) $G(\pi_{n-1}) = \{z = 0\}$;

e) $G(\pi'_{n-1}) = \{z = h\}$ and $h > 0$;

f) $G(\mathbf{w}_{n-2}) = (q_{n-2}, d, 0)$ and $d > 0$.

For the sake of simplicity we use the following additional notation.

$$q = q_{n-1}, \widetilde{\mathbf{w}}_{n-1} = (q_{n-1}, \widetilde{\mathbf{p}}_{n-1}) = G(\mathbf{w}_{n-1}) = (q, 0, 0), \widetilde{\pi} = \widetilde{\pi}_{n-1} = G(\pi_{n-1}),$$

$$\widetilde{\pi}' = \widetilde{\pi}'_{n-1} = G(\pi'_{n-1}), \pi = \pi_{n-1}, \pi' = \pi'_{n-1}, \Lambda = \mathbb{Z}^3, \widetilde{\Lambda} = G(\Lambda), \Gamma = \pi \cap \Lambda,$$

$$\Gamma' = \pi' \cap \Lambda, \widetilde{\Gamma} = G(\Gamma) = \widetilde{\pi} \cap \widetilde{\Lambda}, \widetilde{\Gamma}' = G(\Gamma') = \widetilde{\pi}' \cap \widetilde{\Lambda}.$$

We observe the following **properties**.

(i) $\widetilde{\pi}$ and $\widetilde{\pi}'$ are neighbouring completely rational planes with respect to the lattice $\widetilde{\Lambda}$, and h is just the distance between them.

(ii) The lattice $\widetilde{\Gamma}$ is two-dimensional and $\det \widetilde{\Gamma} = 1/h$. All the points of the lattice $\widetilde{\Gamma}$ lie on parallel lines $l_k = \{(x, y, z) \mid y = kd, z = 0\}$ $(k \in \mathbb{Z})$ with the step q. That is, the distance between the neighbouring lines is equal to d, the parallelogram generated by $\mathbf{w}_{n-1}$ and $\mathbf{w}_{n-2}$ is empty.

(iii) The lattice $\widetilde{\Gamma}'$ is two-dimensional and $\det \widetilde{\Gamma}' = \frac{1}{h}$. All the points of the lattice $\widetilde{\Gamma}'$ lie on parallel lines $l'_k = \{(x, y, z) \mid y = kd + b, z = h\}$ $(k \in \mathbb{Z}$, and b is a certain real number) with the step q. The distance between the neighbouring lines is equal to d.

(iv) $\det G = \pm 1$.

(v) $\det \widetilde{\Lambda} = qhd = 1$.

(vi) G does not change the coordinate in the axis x.

(vii) G transforms any cylinder Π into a cylinder $\widetilde{\Pi} = G(\Pi)$. And the radius, the length and the volume of the cylinder remain the same.

(viii) $\widetilde{\mathbf{p}}_{n-1} = (0, 0)$.

Now we reformulate the problem of constructing the vector

$$\mathbf{w}_n = (q_n, \mathbf{p}_n) \in \mathbb{N} \times \mathbb{Z}^2.$$

We must construct the sequence of vectors

$$\widetilde{\mathbf{w}}_0 = (q_0, \widetilde{\mathbf{p}}_0), \ldots, \widetilde{\mathbf{w}}_n = (q_n, \widetilde{\mathbf{p}}_n) \in \widetilde{\Lambda},$$

with the properties 1)–6) below.

For $1 \leqslant m \leqslant n$ we use the following notation. Put

$$\widetilde{\mathbf{v}}_m = \frac{\widetilde{\mathbf{p}}_m}{q_m}, \quad q_{-1} = 0, \quad \mathbf{p}_{-1} = (1, 0), \quad (q_{-1}, \widetilde{\mathbf{p}}_{-1}) = G((q_{-1}, \mathbf{p}_{-1})).$$

For $0 \leqslant \nu \leqslant m$ put $R_m^\nu = |q_{\nu-1}/q_m \mathbf{p}_m - \mathbf{p}_{\nu-1}| = |q_{\nu-1}/q_m \widetilde{\mathbf{p}}_m - \widetilde{\mathbf{p}}_{\nu-1}|$, $\widetilde{\Pi}_m^\nu = \Pi\,(\widetilde{\mathbf{v}}_m, q_\nu, R_m^\nu)$.

1) $\widetilde{\Pi}_m^\nu \cap \widetilde{\Lambda} = \begin{cases} \{(0,0,0), \widetilde{\mathbf{w}}_{\nu-1}, \widetilde{\mathbf{w}}_\nu, \widetilde{\mathbf{w}}_\nu - \widetilde{\mathbf{w}}_{\nu-1}\}, & \nu = m; \\ \{(0,0,0), \widetilde{\mathbf{w}}_{\nu-1}, \widetilde{\mathbf{w}}_\nu\}, & 1 \leqslant \nu < m, \end{cases}$

and $\widetilde{\mathbf{w}}_{\nu-1}$ belongs to the non-principal boundary of $\widetilde{\Pi}_m^\nu$, $\widetilde{\mathbf{w}}_\nu - \widetilde{\mathbf{w}}_{\nu-1}$ belongs to the non-principal boundary of $\widetilde{\Pi}_m^\nu$ (with $\nu = m$), and $\widetilde{\mathbf{w}}_\nu$ belongs to the main facet of $\widetilde{\Pi}_m^\nu$;

2) $q_m > 2q_{m-1} \quad (m \geqslant 1)$;

3) $R_m^\nu < \frac{1}{2} R_m^{\nu-1} \quad (1 \leqslant \nu \leqslant m)$;

4) $\frac{1}{\pi} V_m^\nu \in \operatorname{int}\Delta_\nu \quad (1 \leqslant \nu \leqslant m)$;

5) $|\mathbf{v}_m - \mathbf{v}_{m-1}| < \frac{1}{2^m} \quad (m \geqslant 1)$;

6) $|R_m^\nu - R_{m-1}^\nu| < \frac{1}{2^m} \quad (1 \leqslant \nu \leqslant m-1)$.

Meanwhile the vectors

$$\widetilde{\mathbf{w}}_0 = (q_0, \widetilde{\mathbf{p}}_0) = G(\mathbf{w}_0), \ldots, \widetilde{\mathbf{w}}_{n-1} = (q_{n-1}, \widetilde{\mathbf{p}}_{n-1}) = G(\mathbf{w}_{n-1})$$

are supposed to be constructed. And we suppose that they satisfy 1)–6), due to the properties of the transformation G.

So we must construct $\widetilde{\mathbf{w}}_n = (q_n, \widetilde{\mathbf{p}}_n) \in \widetilde{\Lambda}$, such that the properties 1)–6) are satisfied.

If we construct such a vector $\widetilde{\mathbf{w}}_n$, we immediately put $\mathbf{w}_n = G^{-1}(\widetilde{\mathbf{w}}_n)$. This completes the construction of the vector $\mathbf{w}_n$.

5. Parametrization with auxiliary sets A_j

Now we define three auxiliary sets $A_0, A_1, A_2 \subset \mathbb{R}^3$. We shall denote elements of the sets A_0, A_1 and A_2 by (x_0, y_0, z_0), (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Put

$$A_0 = \{(x_0, y_0, z_0) \in \mathbb{R}^3 \mid x_0 = q, y_0 > 0, z_0 > 0\}.$$

For any point of A_0 we consider the cylinder

$$\Pi(0)(x_0, y_0, z_0) = \Pi\left(\left(\frac{y_0}{q}, \frac{z_0}{q}\right), \frac{qh}{z_0}, \sqrt{y_0^2 + z_0^2}\right).$$

Remark 5.1. The cylinder $\Pi = \Pi(0)(x_0, y_0, z_0)$ is uniquely defined by the following properties.

(i) (x_0, y_0, z_0) is an axis vector for Π;

(ii) the center of the main facet of the cylinder belongs to the plane $\widetilde{\pi}'$;

(iii) the point $\widetilde{\mathbf{w}}_{n-1} = (q, 0, 0)$ belongs to the continuation of the non-principal boundary of Π.

Put

$$A_1 = \{(x_1, y_1, z_1) \in \mathbb{R}^3 \mid x_1 > 0, y_1 > 0, z_1 = h\}.$$

For any point of A_1 we consider the cylinder

$$\Pi(1)(x_1, y_1, z_1) = \Pi\left(\left(\frac{y_1}{x_1}, \frac{h}{x_1}\right), x_1, \frac{q}{x_1}\sqrt{y_1^2 + h^2}\right).$$

Remark 5.2. The cylinder $\Pi = \Pi(1)(x_1, y_1, z_1)$ is uniquely defined by the following properties.

(i) Vector (x_1, y_1, z_1) is an axis vector for Π;

(ii) the center of the main facet of the cylinder belongs to the plane $\widetilde{\pi}'$;

(iii) the point $\widetilde{\mathbf{w}}_{n-1} = (q, 0, 0)$ belongs to the continuation of the non-principal boundary of Π.

Put

$$A_2 = \{(x_2, y_2, z_2) \in \mathbb{R}^3 \mid x_2 > 0, y_2 > 0, z_2 = 0\}.$$

For any point of A_2 we consider the cylinder

$$\Pi(2)(x_2, y_2, z_2) = \Pi\left(\left(\frac{x_2y_2}{x_2^2+q^2}, \frac{qy_2}{x_2^2+q^2}\right), \frac{h(x_2^2+q^2)}{qy_2}, \frac{qy_2}{\sqrt{x_2^2+q^2}}\right).$$

Remark 5.3. The cylinder $\Pi = \Pi(2)(x_2, y_2, z_2)$ is uniquely defined by the following properties.

(i) The line $\{y = y_2, z = 0\}$ is tangent to the continuation of the non-principal boundary of Π at the point (x_2, y_2, z_2);

(ii) the center of the main facet of the cylinder belongs to the plane $\widetilde{\pi}'$;

(iii) the point $\widetilde{\mathbf{w}}_{n-1} = (q, 0, 0)$ belongs to the continuation of the non-principal boundary of Π.

Now we define a triple of bijections between the sets A_0, A_1 and A_2 in the following way:

$$\begin{cases} x_0 = q, \\ y_0 = \dfrac{qy_1}{x_1}, \\ z_0 = \dfrac{qh}{x_1}; \end{cases} \qquad \begin{cases} x_1 = \dfrac{qh}{z_0}, \\ y_1 = \dfrac{hy_0}{z_0}, \\ z_1 = h; \end{cases} \qquad \begin{cases} x_1 = \dfrac{h(x_2^2+q^2)}{y_2q}, \\ y_1 = \dfrac{hx_2}{q}, \\ z_1 = h; \end{cases}$$

$$\begin{cases} x_2 = \dfrac{qy_1}{h}, \\ y_2 = \dfrac{q(y_1^2+h^2)}{x_1h}, \\ z_2 = 0; \end{cases} \qquad \begin{cases} x_0 = q, \\ y_0 = \dfrac{qx_2y_2}{x_2^2+q^2}, \\ z_0 = \dfrac{q^2y_2}{x_2^2+q^2}; \end{cases} \qquad \begin{cases} x_2 = \dfrac{qy_0}{z_0}, \\ y_2 = \dfrac{y_0^2+z_0^2}{z_0}, \\ z_2 = 0. \end{cases}$$

Remark 5.4. The diagram

$$\begin{tikzcd} A_0 \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & A_1 \\ & A_2 \arrow[ur, leftrightarrow] & \end{tikzcd}$$

related to our triple of bijections between the sets A_0, A_1 and A_2 is commutative.

Remark 5.5. The cylinder defined by the corresponding elements of the sets A_0, A_1 and A_2 remains invariant, that is

$$\Pi(0)(x_0, y_0, z_0) = \Pi(1)(x_1, y_1, z_1) = \Pi(2)(x_2, y_2, z_2).$$

Remark 5.6. In the sequel, we identify the corresponding elements of the sets A_0, A_1 and A_2, for simplicity reasons. We shall consider the sets A_0, A_1 and A_2 as different parametrizations of the *same* family of cylinders.

The following statement can be easily verified.

LEMMA 1. *Suppose that $r > 0$. Then the following conditions are equivalent*

(i) $x_1 = \dfrac{q(y_1^2 + h^2)}{2rh}$;

(ii) $y_2 = 2r$;

(iii) $y_0^2 + (z_0 - r)^2 = r^2 \left(or\ \dfrac{y_0^2 + z_0^2}{z_0} = 2r \right)$.

They define a family of cylinders of the volume $V = V(r) = 2\pi rqh$.

Remark 5.7. The equation $V = V(r) = 2\pi rqh$ may be rewritten as $V(r) = \dfrac{2\pi r}{d}$.

We take r to satisfy $\lambda^* = 2r/d \in \operatorname{int}\Delta_n$ in such a way that the number $2rh/d^2$ is irrational.

Then we define the set

$$B_2 = \{(x_2, y_2, z_2) \in \mathbb{R}^3 \,|\, x_2 > 0, y_2 > 0, z_2 > 0, \widehat{\Pi}(2)(x_2, y_2, z_2) \cap \widetilde{\Gamma} = \{\mathbf{0}, \pm\widetilde{\mathbf{w}}_{n-1}\}\}.$$

Remark 5.8. If $(x_2, y_2, z_2) \in B_2$, then $\operatorname{int}\Pi(2)(x_2, y_2, z_2) \cap \widetilde{\Gamma} = \varnothing$.

Remark 5.9. B_2 is an open set in A_2.

Remark 5.10. If $(x_2, y_2, z_2) \in B_2 \cup \partial B_2$, then $\theta(x_2, y_2, z_2) \in B_2 \quad \forall \theta \in (0, 1)$.

Remark 5.11. If $(x_2, y_2, z_2) \in B_2$ then $\left(x_2 + k\dfrac{y_2 q}{d}, y_2, z_2 \right) \in B_2 \quad \forall k \geqslant 0, k \in \mathbb{Z}$. Moreover the transformation$(x_2, y_2, z_2) \to \left(x_2 + k\dfrac{y_2 q}{d}, y_2, z_2 \right) \quad \forall k \geqslant 0\ ,\ k \in \mathbb{Z}$ preserves the lattice $\widetilde{\Gamma}$ and change the ellipse $\operatorname{int}\widehat{\Pi}(2)(x_2, y_2, z_2) \cap \widetilde{\pi}$ into the ellipse $\operatorname{int}\widehat{\Pi}(2)\left(x_2 + k\dfrac{y_2 q}{d}, y_2, z_2 \right) \cap \widetilde{\pi}$.

Remark 5.12. There exists $0 < a \leqslant q$ such that $(a, d, 0) \in \widetilde{\Gamma}$.

LEMMA 2. *There exists $\varepsilon > 0$ such that for any integer $k \geqslant 1$ the set B_2 contains a rectangle bounded by the lines*

$$y_2 = \lambda^* d;$$

$$y_2 = \lambda^* d - \frac{\lambda^* d\varepsilon}{\lambda^*\left(a + \left(k + \frac{1}{2}\right) q\right) + \varepsilon};$$

$$x_2 = \lambda^*\left(a + \left(k + \frac{1}{2}\right) q\right);$$

$$x_2 = \lambda^*\left(a + \left(k + \frac{1}{2}\right) q\right) - \varepsilon.$$

PROOF OF LEMMA 2 One can easily see that

$$\frac{2}{\sqrt{3}} \frac{(a, d, 0) + (a + q, d, 0)}{2} = \frac{2}{\sqrt{3}}\left(a + \frac{1}{2}q, d, 0\right) \in \partial B_2,$$

so

$$\lambda^*\left(a + \frac{1}{2}q, d, 0\right) \in B_2.$$

The set B_2 is open. We see that

$$\exists\, \varepsilon > 0 \quad \forall\, \theta \in [-\varepsilon, \varepsilon] \quad \lambda^*\left(a + \frac{1}{2}q + \frac{\theta}{\lambda^*}, d, 0\right) \in B_2.$$

We use the properties of the set B_2 to see that

$$\lambda^*\left(a + \left(k + \frac{1}{2}\right) q + \frac{\theta}{\lambda^*}, d, 0\right) \in B_2, \quad \forall\, k \geqslant 0.$$

So the triangle with vertices $(0, 0, 0)$, $\lambda^*\left(a + \left(k + \frac{1}{2}\right) q - \frac{\varepsilon}{\lambda^*}, d, 0\right)$ and $\lambda^*\left(a + \left(k + \frac{1}{2}\right) q + \frac{\varepsilon}{\lambda^*}, d, 0\right)$ (without the vertex $(0, 0, 0)$) is contained in B_2.

One can see that this triangle contains a rectangle bounded by the lines

$$y_2 = \lambda^* d;$$

$$y_2 = \lambda^* d - \frac{\lambda^* d\varepsilon}{\lambda^* \left(a + \left(k + \frac{1}{2}\right) q\right) + \varepsilon};$$

$$x_2 = \lambda^* \left(a + \left(k + \frac{1}{2}\right) q\right);$$

$$x_2 = \lambda^* \left(a + \left(k + \frac{1}{2}\right) q\right) - \varepsilon.$$

Lemma is proved. □

The family of rectangles from Lemma 2 may be represented as a family of domains, in terms of the parametrization related to the set A_1. From this point of view it will be a family of closed domains bounded by the curves

$$x_1 = \frac{q(y_1^2 + h^2)}{\lambda^* dh};$$

$$x_1 = \frac{q(y_1^2 + h^2)}{\left(\lambda^* d - \dfrac{\lambda^* d\varepsilon}{\lambda^* \left(a + \left(k + \frac{1}{2}\right) q\right) + \varepsilon}\right) h};$$

$$y_1 = \lambda^* \left(a + \left(k + \frac{1}{2}\right) q\right) \frac{h}{q};$$

$$y_1 = \lambda^* \left(a + \left(k + \frac{1}{2}\right) q\right) \frac{h}{q} - \varepsilon \frac{h}{q}.$$

Remark 5.13. In such a domain the value of y_1 and k satisfy $y_1 \asymp k \quad (k \to +\infty)$.

The first two equations for the boundary in A_1-parametrization give parabolas without intersection and with a common axis. For k large enough the distance between the branches of these parabolas becomes arbitrary large, supposed that the scale of the value of y is fixed. That is

$$\frac{q(y_1^2 + h^2)}{\left(\lambda^* d - \dfrac{\lambda^* d\varepsilon}{\lambda^* \left(a + \left(k + \frac{1}{2}\right) q\right) + \varepsilon}\right) h} - \frac{q(y_1^2 + h^2)}{\lambda^* dh} =$$

$$= \frac{q(y_1^2+h^2)}{\lambda^* dh}\left(\frac{1}{\left(1-\dfrac{\varepsilon}{\lambda^*\left(a+\left(k+\dfrac{1}{2}\right)q\right)+\varepsilon}\right)}-1\right) \to +\infty \quad (k\to+\infty).$$

In particular, for large k it becomes larger than $2q$.

Remark 5.14. One can show that

$$\frac{q(y_1^2+h^2)}{\left(\lambda^* d-\dfrac{\lambda^* d\varepsilon}{\lambda^*\left(a+\left(k+\dfrac{1}{2}\right)q\right)+\varepsilon}\right)h}-\frac{q(y_1^2+h^2)}{\lambda^* dh} \asymp k \quad (k\to+\infty).$$

The last two equations for the boundary in A_1-parametrization determine two lines parallel to the coordinate axis $0x$. The distance between these two lines is $\varepsilon h/q$, and this distance does not depend on k.

So in terms of A_1-parametrization our rectangles are the pieces of strips of the constant width bounded by the two parabolas. They lie in the plane $\widetilde{\pi}'$ along the $0x$ axis, and they appear periodically with the period $\lambda^* h = (2rh)/d$ along the $0y$ axis.

The points of the lattice $\widetilde{\Gamma}'$ in $\widetilde{\pi}'$ should lie on the lines parallel to $0x$. They appear periodically with the period q. The distance between the neighbouring lines of this family is equal to d. (To see this we explore properties of G.)

By our choice of r we see that the ratio $\dfrac{2rh}{d}/d$ is irrational. So there are infinitely many possibilities to choose k in such a way that the corresponding piece of the strip in the set A_1 has a least one point of $\widetilde{\Gamma}'$ inside. Such a value of the parameter k we define as an *admissible value*.

Now for an admissible value of k we take one of the corresponding points of $\widetilde{\Gamma}'$. In the sequel we say that such a point from $\widetilde{\Gamma}'$ corresponds to the admissible and large k .

Also for an admissible large k we have the corresponding point from A_0, A_1 and A_2.

Remark 5.15. If k is admissible and tends to infinity then the value of x_1 for the *corresponding* $(x_1, y_1, z_1) \in A_1$ tends to infinity.

Now we consider the family of rectangles from Lemma 2 in terms of A_0-parametrization. Their boundaries are determined by the lines

$$y_0^2 + \left(z_0 - \frac{1}{2}\lambda^* d\right)^2 = \left(\frac{1}{2}\lambda^* d\right)^2;$$

$$y_0^2 + \left(z_0 - \frac{1}{2}\left(\lambda^* d - \frac{\lambda^* d\varepsilon}{\lambda^*\left(a + \left(k + \frac{1}{2}\right)q\right) + \varepsilon}\right)\right)^2 =$$

$$= \left(\frac{1}{2}\left(\lambda^* d - \frac{\lambda^* d\varepsilon}{\lambda^*\left(a + \left(k + \frac{1}{2}\right)q\right) + \varepsilon}\right)\right)^2;$$

$$y_0 = \frac{z_0}{q}\left(\lambda^*\left(a + \left(k + \frac{1}{2}\right)q\right)\right);$$

$$y_0 = \frac{z_0}{q}\left(\lambda^*\left(a + \left(k + \frac{1}{2}\right)q\right) - \varepsilon\right).$$

The rectangles from Lemma 2 in terms of A_0-parametrization are domains in the plane $\{x = q\}$, bounded by two rays (with the common origin $\mathbf{0}$) and two circles. These circles have the unique common point $\widetilde{\mathbf{w}}_{n-1}$. One of these circles lies inside another one.

6. Proof of Theorem 3: we complete the inductive step

Now we show that thee exists an admissible and large enough k such that the corresponding vector $\widetilde{\mathbf{w}}_n$ satisfies the conditions 1)–6).

For an admissible and large enough k the value of x_1 in $(x_1, y_1, z_1) \in A_1$ increases and becomes unbounded. So we get 2).

One can easily check that for an admissible and large enough k, the triple $(x_0, y_0, z_0) \in A_0$ belongs to an arbitrarily small neighbourhood of $\widetilde{\mathbf{w}}_{n-1}$. It follows that $|\widetilde{\mathbf{v}}_n - \widetilde{\mathbf{v}}_{n-1}|$ may be arbitrarily small. So we get 5).

Put $R(\mathbf{v}) = |q_{\nu-1}\mathbf{v} - \widetilde{\mathbf{p}}_{\nu-1}|$. Then $R(\widetilde{\mathbf{v}}_{n-1}) = R^{\nu}_{n-1}$, $R(\widetilde{\mathbf{v}}_n) = R^{\nu}_n$. For an admissible and large k the value $|\widetilde{\mathbf{v}}_n - \widetilde{\mathbf{v}}_{n-1}|$ becomes arbitrary small. But $R(\mathbf{v})$ depends continuously on $\mathbf{v}$. So $|R^{\nu}_n - R^{\nu}_{n-1}|$ becomes arbitrary small, and we have 6).

Now we prove 3) for $\nu = n$.

As $\widetilde{\mathbf{p}}_{n-1} = (0, 0)$ we have $\lim\limits_{\mathbf{v}\to(0,0)} R(\mathbf{v}) = 0$. So for an admissible and large k the value of R^n_n tends to zero. At the same time the value of R^{n-1}_n tends to a positive constant R^{n-1}_{n-1}. In particular it means that we have 3) with $\nu = n$.

Suppose that $\nu < n$. Then for an admissible and large k the value of R^{ν}_n tends to R^{ν}_{n-1}, and the value of $R^{\nu-1}_n$ tends to $R^{\nu-1}_{n-1}$. By the assumption we have $R^{\nu}_{n-1} < 1/2R^{\nu-1}_{n-1}$. This shows that for an admissible and large k we also have $R^{\nu}_n < 1/2R^{\nu-1}_n$. This gives 3).

The condition 4) for $\nu = n$ is true by our construction.

Suppose that $\nu < n$. Then we may take an admissible k large enough, and the value of V^{ν}_n will tend to V^{ν}_{n-1}. By the assumption we have $\frac{1}{\pi}V^{\nu}_{n-1} \in \mathrm{int}\Delta_{\nu}$. Thus, for an admissible and large k we have the inequality $\frac{1}{\pi}V^{\nu}_n \in \mathrm{int}\Delta_{\nu}$. So we get 4).

Now we prove 1) for $\nu = n$.

By our construction we have $\widetilde{\Pi}^n_n \cap \widetilde{\Gamma} = \{(0,0,0), \widetilde{\mathbf{w}}_{n-1}\}$. As the cylinder $\widetilde{\Pi}^n_n$ has a symmetry with respect to the lattice $\widetilde{\Lambda}$ we have $\widetilde{\Pi}^n_n \cap \widetilde{\Gamma}' = \{\widetilde{\mathbf{w}}_n, \widetilde{\mathbf{w}}_n - \widetilde{\mathbf{w}}_{n-1}\}$. But for an admissible and large k the value of R^n_n tends to zero and $\widetilde{\Pi}^n_n \cap \widetilde{\Lambda} = \{(0,0,0), \widetilde{\mathbf{w}}_{n-1}, \widetilde{\mathbf{w}}_n, \widetilde{\mathbf{w}}_n - \widetilde{\mathbf{w}}_{n-1}\}$. So we have 1) for $\nu = n$.

Now we prove 1) for $1 \leqslant \nu = n-1$.

By the assumption we have

$$\widetilde{\Pi}^{n-1}_{n-1} \cap \widetilde{\Lambda} = \{(0,0,0), \widetilde{\mathbf{w}}_{n-2}, \widetilde{\mathbf{w}}_{n-1}, \widetilde{\mathbf{w}}_{n-1} - \widetilde{\mathbf{w}}_{n-2}\}.$$

Here $\widetilde{\mathbf{w}}_{n-2}$ belongs to the non-principal boundary of $\widetilde{\Pi}^{n-1}_{n-1}$, and $\widetilde{\mathbf{w}}_{n-1} - \widetilde{\mathbf{w}}_{n-2}$ belongs to the non-principal boundary of $\widetilde{\Pi}^{n-1}_{n-1}$. At the same time $\widetilde{\mathbf{w}}_{n-1}$ belongs to the main facet of $\widetilde{\Pi}^{n-1}_{n-1}$. By the definition of $\widetilde{\Pi}^{n-1}_n$ the point $\widetilde{\mathbf{w}}_{n-2}$ belongs to the non-principal boundary of $\widetilde{\Pi}^{n-1}_n$. For an admissible and large k the cylinder $\widetilde{\Pi}^{n-1}_n$ tends to the cylinder $\widetilde{\Pi}^{n-1}_{n-1}$, at the same time the point $\widetilde{\mathbf{w}}_{n-1}$ lies in the main facet of $\widetilde{\Pi}^{n-1}_n$. We take an admissible k and see that the point $\widetilde{\mathbf{w}}_{n-1} - \widetilde{\mathbf{w}}_{n-2}$ is not in the cylinder $\widetilde{\Pi}^{n-1}_n$, as the corresponding triple $(x_0, y_0, z_0) \in A_0$ has positive coordinates. So for admissible and large k we have $\widetilde{\Pi}^{n-1}_n \cap \widetilde{\Lambda} = \{(0,0,0), \widetilde{\mathbf{w}}_{\nu-1}, \widetilde{\mathbf{w}}_{\nu}\}$. We get 1) for $1 \leqslant \nu = n-1$.

Now we prove 1) for $1 \leqslant \nu < n-1$.
By the inductive assumption we have $\widetilde{\Pi}^{\nu}_{n-1} \cap \widetilde{\Lambda} = \{(0,0,0), \widetilde{\mathbf{w}}_{\nu-1}, \widetilde{\mathbf{w}}_{\nu}\}$. Here $\widetilde{\mathbf{w}}_{\nu-1}$ belongs to the non-principle boundary of $\widetilde{\Pi}^{\nu}_{n-1}$, and $\widetilde{\mathbf{w}}_{\nu}$ belongs to its main facet.

By the construction $\widetilde{\mathbf{w}}_{\nu-1}$ belongs to the non-principle boundary of $\widetilde{\Pi}^{\nu}_{n}$. For an admissible and large k the cylinder $\widetilde{\Pi}^{\nu}_{n}$ tends to the cylinder $\widetilde{\Pi}^{\nu}_{n-1}$, and at the same time the point $\widetilde{\mathbf{w}}_{\nu}$ lies in the main facet of $\widetilde{\Pi}^{\nu}_{n}$. Thus, for an admissible and large k we have $\widetilde{\Pi}^{\nu}_{n} \cap \widetilde{\Lambda} = \{(0,0,0), \widetilde{\mathbf{w}}_{\nu-1}, \widetilde{\mathbf{w}}_{\nu}\}$. We get 1).

The inductive construction is described completely.

By our construction of the sequence $\mathcal{Z}$ and the Cauchy criterion there exists the limit $\lim\limits_{n\to\infty} \mathbf{v}_n = \mathbf{v}$.

Now we prove that the sequence

$$\mathcal{Z} : \mathbf{w}_n = (q_n, \mathbf{p}_n) \in \mathbb{Z}^3, \quad n \geqslant 0$$

is just the sequence of all best approximation vectors to $\mathbf{v} \in \mathbb{R}^2$.

Indeed, we have

(i) $q_0 = 1$ by the construction;

(ii) $(\operatorname{int} \Pi_n) \cap \mathbb{Z}^3 = \varnothing$ $(n \geqslant 1)$ by the construction, as $\Pi_\nu = \lim\limits_{n\to\infty} \Pi^{\nu}_{n}$;

(iii) $q_{n+1} > q_n$ $(n \geqslant 0)$ by the construction;

(iv) $R_{n+1} < R_n$ $(n \geqslant 0)$ by the construction.

The uncountability of the set of the vectors $\mathbf{v} \in \mathbb{R}^2$ can be obtained as follows. At each step of the inductive process we apply Lemma 2. Here we can use both two neighbouring planes π'_{n-1}. We use both opportunities and get uncountably many ways to construct a limit vector $\mathbf{v} \in \mathbb{R}^2$. One can easily see that all the vectors $\mathbf{v} \in \mathbb{R}^2$ constructed are different.

Theorem 3 is proved. $\square$

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