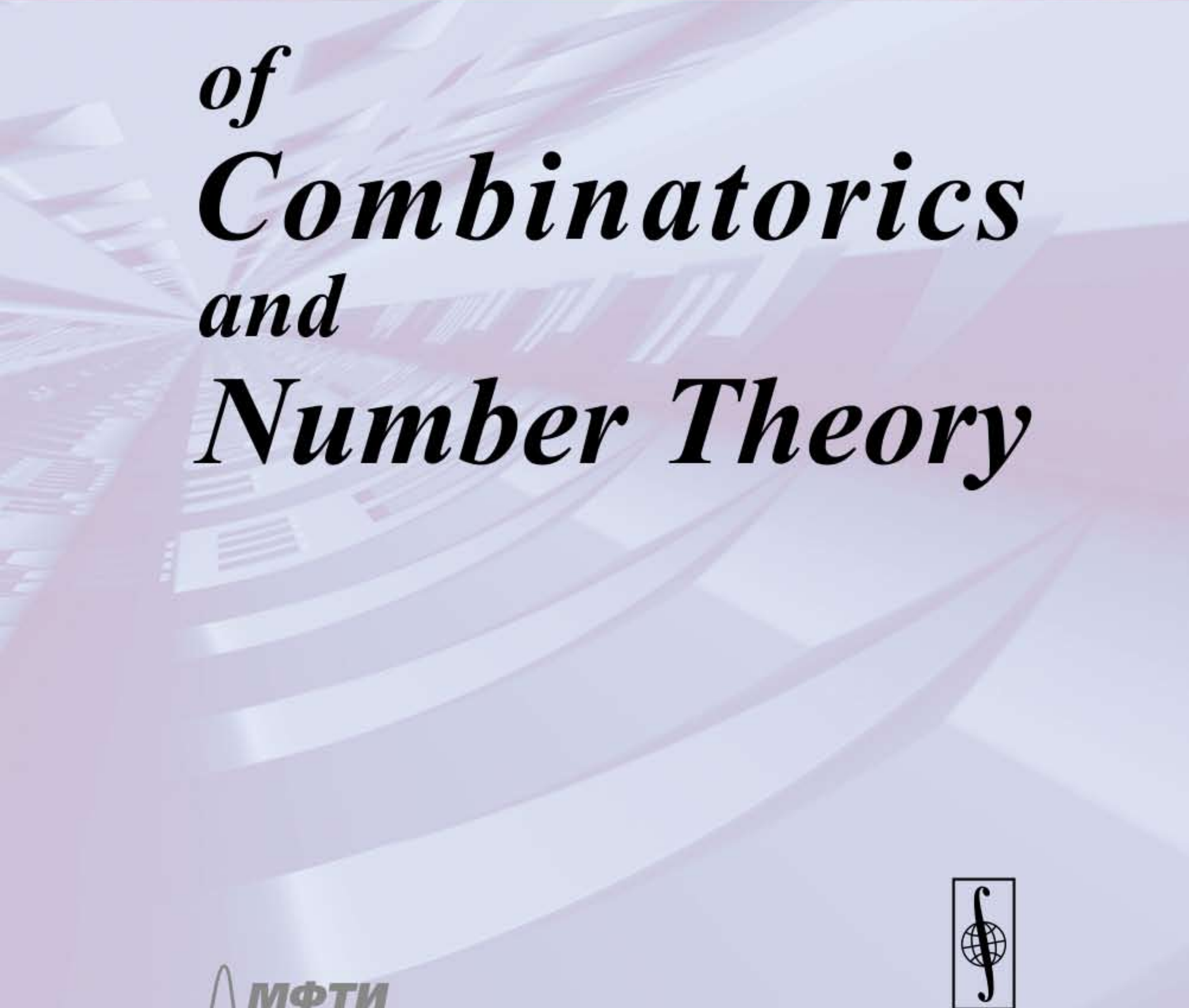
Reprint from

ISSN 2220-5438

Moscow Journal of Combinatorics and Number Theory



МФТИ

URSS

Volume 1 • Issue 2

2011

Moscow Journal of Combinatorics and Number Theory. 2011. Vol. 1. Iss. 2. 128 p.

The journal was founded in 2010.

Published by the Moscow Institute of Physics and Technology
with the support of Yandex and Microsoft.

The aim of this journal is to publish original, high-quality research articles from a broad range of interests within combinatorics, number theory and allied areas. One volume of four issues is published annually.

Website of our journal
`http://mjcnt.phystech.edu`

Address of the Editorial Board
Moscow institute of physics
and technology (state university)
Faculty of Innovations
and High Technology,
Laboratorny Korpus, k. 209,
9, Institutskii pereulok,
Dolgoprudny,
Moscow Region,
Russia,
141700

URSS Publishers
56, Nakhimovsky Prospekt,
Moscow,
Russia,
117335

E-mail
`mjcnt@phystech.edu`

Адрес редакции
Московский физико-технический
институт (государственный университет)
Факультет инноваций
и высоких технологий
Лабораторный корпус, к. 209,
Институтский переулок, д. 9,
г. Долгопрудный,
Московская область,
Российская Федерация,
141700

Издательство «УРСС»
Нахимовский пр-т, 56
Москва,
Российская Федерация,
117335

Журнал зарегистрирован в Федеральной службе по надзору в сфере массовых коммуникаций, связи и охраны культурного наследия 3 сентября 2010 г. Свидетельство ПИ № ФС77-41900.

Формат 70 × 100/16. Печ. л. 8. Зак. № ЖО-01.
Отпечатано в ООО «ЛЕНАНД».
117312, Москва, пр-т Шестидесятилетия Октября, 11А, стр. 11.

ISSN 2220-5438

10838 ID 157682
9 785453 000241

Moscow Journal
of Combinatorics
and Number Theory
2011, vol. 1, iss. 2,
pp. 3–26, [pp. 83–106]

Arithmetical results on certain q-series, III

Peter Bundschuh (Köln), Keijo Väänänen (Oulu)

Abstract: As in Parts I and II, entire transcendental solutions f of functional equations of type $f(q^m z) = R_0(z)f(z) + R_1(z)$ with polynomial coefficients R_0, R_1 are arithmetically investigated. The values of those solutions and of their derivatives up to a given order at m successive powers of q are considered. The purpose of this paper is to produce lower bounds for the dimension of the K-vector space generated by 1 and the values mentioned above, where K is the rational or an imaginary quadratic field. In some cases, we can prove linear independence, even in a quantitative version.

Keywords: Nesterenko-type dimension estimates; linear independence; quantitative versions

AMS Subject classification: 11J72, 11J82

Received: 14.12.2010; **revised:** 17.02.2011

1. Introduction and results

Let K be the field of rational numbers or an imaginary quadratic field. Let us denote by O_K its ring of integers.

The main aim of the present paper is to generalize earlier linear independence results on the values of entire functions satisfying a functional equation of the form

$$f(q^m z) = R_0(z)f(z) + R_1(z), \tag{1}$$

where $m \in \mathbb{N} := \{1, 2, \ldots\}$, $R_0, R_1 \in K[z]$ and $q \in K$. It is shown in [5] that $f(z)$ can be expressed as a linear combination of 1 and certain basic hypergeometric series.

In [2] and [5], estimates for the dimension of the K-vector space spanned by 1 and $f(\alpha q^{-\mu})$, $0 \leqslant \mu < m$, with $\alpha \in K^{\times}$ were obtained, and in [9] the corresponding dimension estimate was presented for 1 and $f^{(\sigma)}(\alpha)$, $0 \leqslant \sigma < k$, in the case $m = 1$. Further, there are several papers considering in particular the special cases $\deg R_0 = 1$ or 2, see the references in [2] and [9]. Note also that a p-adic analogue of [5] is given in [4]. The main result of the present paper is the following theorem.

THEOREM 1. *Assume that* $q \in K$ *with* $|q| > 1$ *is the quotient* u/v *of* $u, v \in O_K^* := O_K \setminus \{0\}$, *and let* $\eta := (\log|v|)/(\log|u|)$. *Suppose that* f *is an entire transcendental solution of the functional equation* (1) *with* $R_0, R_1 \in K[z]$, $\deg R_0 =: \ell \in \mathbb{N}$, *with* $f(0) = 1$ *if* $R_1(0) = 0$, *and with* $R_0(0) = q^{-t}$, $t \in \mathbb{N}$, *if* $R_1(0) \neq 0$. *Let* $\alpha \in K^{\times}$ *satisfy the conditions* $R_0(\alpha q^{-j}) \neq 0$ *for every rational integer* $j \geqslant m$. *Then we have the dimension estimate*

$$\dim_K \left\{ K + \sum_{\sigma=0}^{k-1} \sum_{\mu=0}^{m-1} K f^{(\sigma)}(\alpha q^{-\mu}) \right\} \geqslant (1-\eta) C(k, \ell, m),$$

where $C(1,1,1) := \dfrac{7}{4}$ *and*

$$C(k, \ell, m) := \frac{km((km+1)^2 - k\ell m) + \sqrt{\Delta}}{2k\ell m(km + 2 + 6\pi^{-2}(k-1)m)}$$

for $(k, \ell, m) \neq (1,1,1)$ *with*

$$\Delta := k^2 m^2 ((km+1)^2 - k\ell m)^2 + 4k\ell m (km+1)^2 (km + 2 + 6\pi^{-2}(k-1)m).$$

Remarks.

1) We may always suppose tacitly that u, v have only units from O_K^* as common divisors, cf. the beginning of Section 2.
2) The case $k = 1, \eta = 0$ gives us the Main Theorem of [5].
3) In the case $m = 1$, Theorem 1 improves upon Töpfer's Theorems 1, 2 and 3 in [10], see Section 6.
4) We shall also show that

$$C(k, \ell, m) > \frac{km}{\ell(1 + 6\pi^{-2}(1 - 1/k))} - 1.$$

In the case of homogeneity of (1), i. e. for $R_1 = 0$, we obtain

Theorem 2. *Within the assumptions of Theorem* 1, *suppose that* $R_1 = 0$. *Then we have*

$$\dim_K \left\{ \sum_{\sigma=0}^{k-1} \sum_{\mu=0}^{m-1} K f^{(\sigma)}(\alpha q^{-\mu}) \right\} \geqslant (1-\eta)\widetilde{C}(k,\ell,m),$$

where

$$\widetilde{C}(k,\ell,m) = \frac{k^2m - k\ell + \sqrt{(k^2m-k\ell)^2 + 4k\ell(k+6\pi^{-2}(k-1))}}{2\ell(k+6\pi^{-2}(k-1))}.$$

Remarks.

5) In the case $m = 1$, Theorem 2 improves the homogeneous parts of Theorems 1 and 2 in [9].

6) We also have

$$\widetilde{C}(k,\ell,m) > \frac{km}{\ell(1+6\pi^{-2}(1-1/k))} - 1.$$

7) If in the homogeneous case we have $k = \ell = 1$ and $\eta = 0$, then the preceding remark combined with Theorem 2 gives linear independence over K of the m numbers $f(\alpha q^{-\mu})$, $0 \leqslant \mu < m$. We also notice that the second-named author with the help of a different method obtained in [11] the K-linear independence of the $m+1$ numbers 1 and $f(\alpha q^{-\mu})$, $0 \leqslant \mu < m$, in the general case of (1) with k, ℓ, η as before.

8) Since $\widetilde{C}(2,1,1) = 1{,}339\ldots$, we can deduce from Theorem 2 (a bit more than) Borwein's [1] famous result on the irrationality of the q-logarithmic function $L_q(z) := \sum_{n=0}^{\infty} 1/(q^n + z)$ at every non-zero rational point $\alpha \notin -q^{\mathbb{N}_0}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, if we assume $q \in \mathbb{Z}$, $|q| > 1$. The reason is that L_q is simply the logarithmic derivative of the q-exponential function

$$E_q(z) := \prod_{n\in\mathbb{N}_0} (1 + zq^{-n}).$$

The main tool we use when proving Theorems 1 and 2 will be Nesterenko's dimension estimate [8] and its generalization to arbitrary number fields due to Töpfer [10] (and others later on). Whereas these dimension estimates led only to qualitative results, Töpfer and the first-named author showed in [3], [10] the following. Essentially, every time one can deduce from Nesterenko-type estimates linear independence of numbers over the algebraic number field under consideration, one can write down, with not much additional expense, measures of this linear independence. The corresponding two sufficient criteria, qualitative and quantitative, will be quoted in Section 2 as Lemmas 2 and 3, respectively.

To give here at least one quantitative statement, we claim the following.

THEOREM 3. *Within the assumptions of Theorem* 1, *let us suppose that* $\deg R_0 = 1$, $R_1 = 0$, $q \in O_K$, $m \geqslant 2$. *Then for every* $\varepsilon \in \mathbb{R}_+$ *there is a constant* $C \in \mathbb{R}_+$ *depending at most on* $q, m, R_0'(0), \alpha, \varepsilon$, *such that for every* $\underline{\Lambda} := (\lambda_0, \ldots, \lambda_{m-1}) \in O_K^m$ *with* $|\underline{\Lambda}| := \max(|\lambda_0|, \ldots, |\lambda_{m-1}|) \geqslant C$ *the following inequality holds*

$$\left|\sum_{\mu=0}^{m-1} \lambda_\mu f(\alpha q^{-\mu})\right| \geqslant |\underline{\Lambda}|^{-(\psi(m)+\varepsilon)},$$

$$\psi(m) := \frac{1}{2}(m-1)\left(\sqrt{(m-1)^2+4}+m-1\right).$$

Remarks.

9) If the hypothesis of Theorem 3 holds and if, moreover, $m = 2$, then the quotient $f(\alpha)/f(\alpha/q)$ is not contained in K, and has irrationality exponent not greater than $1 + \psi(2) = (3 + \sqrt{5})/2 = 2{,}618\ldots$.

10) Recently Matala-aho [7] obtained very strong quantitative linear independence results for three special functions. The third one (for $\prod_{n=0}^{\infty}(1 - tq^n)$ within the notation of Theorem 1 loc.cit.) gives a measure for 1 and the $E_{q^m}(\alpha q^{-\mu})$, $\mu = 0, \ldots, m-1$, for our q-exponential function E and every $\alpha \in K^\times \backslash (-q^{\mathbb{N}_0})$. Here not only the number 1 is included, in contrast to our result given above, but also our exponent $\psi(m)$ is asymptotically m^2, whereas Matala-aho's one is linear in m.

Finally, to give some examples of functions of our type f, we first introduce those studied in [2], namely the infinite product

$$f_0(z) := \prod_{j=0}^{\infty} P(zq^{-jm})$$

and the series

$$f_h(z) := \sum_{j=0}^{\infty} q^{-hmj} \prod_{i=0}^{j-1} P(zq^{-im}) \qquad (h \in \mathbb{N}),$$

where $P \in K[z]$ is of exact degree ℓ and $P(0) = 1$. These functions are special cases of (1), with $R_0(z) = q^{-hm}P(q^m z)$ and $R_1(z) = 1 - \delta_{h,0}$, where δ denotes the Kronecker symbol. Notice that if $\deg P = 1$, then f_0 is actually the q-exponential function we encountered already in Remark 8. More precisely, $f_0(z) = E_{q^m}(cz)$, if $P(z) = 1 + cz$ with $c \neq 0$.

More generally, every solution f of the functional equation (1) satisfies

$$f(z) = f(zq^{-\lambda m}) \prod_{\kappa=1}^{\lambda} R_0(zq^{-\kappa m}) + \sum_{\kappa=1}^{\lambda} R_1(zq^{-\kappa m}) \prod_{\iota=1}^{\kappa-1} R_0(zq^{-\iota m}) \tag{2}$$

for any[1] $\lambda \in \mathbb{N}_0$. This is easily proved by induction on λ. To construct more examples, different than $f_0, f_1, \dots$ given above, we take an arbitrary polynomial $R_1(z) = \sum_{j=0}^{r} b_j z^j \in K[z]$ and consider the cases $b_0 \neq 0$ and $b_0 = 0$ separately.

If $b_0 \neq 0$, we use the assumptions of Theorem 1 and define the polynomial $P \in K[z]$ with $P(0) = 1$ by the equation $R_0(z) = q^{-t}P(zq^m)$. Using (2) and letting $\lambda \to \infty$ we get

$$f(z) = \sum_{j=0}^{r} b_j (zq^{-m})^j F_{j,t}(z),$$

where

$$F_{j,t}(z) = \sum_{\kappa=0}^{\infty} q^{-\kappa(mj+t)} \prod_{\iota=0}^{\kappa-1} P(zq^{-\iota m}).$$

Theorem 1 applies to all transcendental functions f of this type, and in particular, the choice $t = hm$ gives a linear combination

$$f(z) = \sum_{j=0}^{r} b_j (zq^{-m})^j f_{j+h}(z).$$

---

[1] As usual, empty products and sums are always to be interpreted as 1 or 0, respectively.

If $b_0 = 0$, then we have to provide the equality $R_0(0) = 1$. Defining the polynomial $P(z)$ by the equation $R_0(z) = P(zq^m)$ and using (2) as above we obtain

$$f(z) = f_0(z) + \sum_{j=1}^{r} b_j(zq^{-m})^j f_j(z).$$

In the case $b_0 = \ldots = b_r = 0$ the latter equality gives $f(z) = f_0(z)$, and Theorem 2 also applies.

Finally, considering the q-exponential function E_q mentioned above, we put

$$E_{q,m,v}(z) := E_{q^m}(zq^{-v}), \quad v \in \{0, 1, \ldots, m-1\},$$

and for $\underline{v} := (v_1, \ldots, v_\ell) \in \{0, 1, \ldots, m-1\}^\ell$ we define

$$E_{q,m,\underline{v}}(z) = \prod_{j=1}^{\ell} E_{q,m,v_j}(z).$$

Then the following functional equation holds:

$$E_{q,m,\underline{v}}(q^m z) = E_{q,m,\underline{v}}(z) \prod_{j=1}^{\ell}(1 + zq^{m-v_j}),$$

and we would like to point out that our Theorems 1 and 2 give improvements on the results of [9] concerning these functions.

For the convenience of the reader, we shall quote in the following section some lemmas needed in our proofs. Then we describe the main steps of our argument, which follow the work [5], but now with a more complicated integral construction in Section 3.

2. Some preliminaries

First of all, we explain why we may always suppose the assumptions on u, v made in Remark 1 (after Theorem 1) are satisfied, even if this is not needed for our proofs. Indeed, if we have $u' = wu, v' = wv$ with a non-unit $w \in O_K^*$, and if we put $\eta' := (\log|v'|)/(\log|u'|)$, then $1 > \eta' > \eta \geqslant 0$, whence $1 - \eta > 1 - \eta'$. Thus, our lower bounds for the dimensions in Theorems 1 and 2 become unnecessarily worse. With the convention that u, v have only units as common divisors, we have $\eta = 0$ if and only if $q \in O_K$ (under the general hypothesis on q).

Next, we quote from [5] two simplifications we can make for our proofs of Theorems 1 through 3. The first one is merely notational:

it is enough to consider the case $\alpha = 1$.

More essential (compare, e. g., (20) below, where $\deg R_1$ enters via the definition of $R_{1,\lambda}$ in (5)) is the second one saying that, without loss of generality,

it is enough to prove Theorem 1 *under the extra condition* $\deg R_1 \leqslant \ell = \deg R_0$.

Before formulating our first lemma, we briefly discuss our assumptions on f, namely, $f(0) = 1$ if $R_1(0) = 0$, and $R_0(0) = q^{-t}, t \in \mathbb{N}$, if $R_1(0) \neq 0$. In view of (1) the first part implies $R_0(0) = 1$, whence *we always have* $R_0(0) = q^{-t}$ *with* $t \in \mathbb{N}_0$, and $t = 0$ if and only if $R_1(0) = 0$.

Next, writing (1) in the form

$$f(Qz) = R_0(z)f(z) + R_1(z) \text{ with } Q := q^m,\ R_0(z) = \sum_{j=0}^{\ell} a_j z^j, R_1(z) = \sum_{j=0}^{\ell} b_j z^j,$$

we note that $a_\ell \neq 0$, and $a_0 = R_0(0) = q^{-t}$ with $t \in \mathbb{N}_0$ as we saw a moment ago. Entering now with the Taylor series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

into (1) as above, we obtain by comparing coefficients

$$c_n(Q^n - a_0) = \sum_{1 \leqslant j \leqslant \min(\ell,n)} a_j c_{n-j} + b_n \quad (n \in \mathbb{N})$$

with $b_n := 0$ for any $n > \ell$. Since $Q^n \neq a_0$ for any $n \in \mathbb{N}$, we see that all the c_n $(n > 0)$ are uniquely determined by c_0, where $c_0 = 1$ if $b_0 = 0$ and $c_0 = b_0/(1 - a_0)$ if $b_0 \neq 0$ (notice that $a_0 = q^{-t}$ with $t > 0$ in this case). Thus, under the hypothesis of Theorem 1, all the Taylor coefficients c_n of f belong to K, and the following lemma gives us information on the denominators of these coefficients.

LEMMA 1. *Let* q *and* f *satisfy the hypothesis of Theorem* 1, *let* $Q := q^m$, *and let* $s \in O_K^*$ *be such that* $sR_0, sR_1 \in O_K[z]$. *Then, for every* $n \in \mathbb{N}_0$, *one has*

$$c_j \prod_{0 \leqslant \nu \leqslant n}^{*} s(Q^\nu - R_0(0)) \in O_K[Q] \quad \text{for} \quad j = 0, 1, \ldots, n,$$

and the degrees of these polynomials with respect to Q *are bounded above by* $n(n+1)/2$. *Here* $\prod^{*}$ *means the product without the factor corresponding* $\nu = 0$ *if* $R_0(0) = 1 \Leftrightarrow \Leftrightarrow R_1(0) = 0$.

Proof. This is Lemma 1 in [4]. ☐

The next Lemma prepares directly the dimension estimates as stated in our Theorems 1 and 2.

Lemma 2. *Let* $\mathbb{E}$ *be* $\mathbb{R}$ *or* $\mathbb{C}$ *in accordance with* K *being* $\mathbb{Q}$ *or an imaginary quadratic field. Further, let* $d \in \mathbb{N} \setminus \{1\}$ *and* $\underline{\omega} = (\omega_1, \ldots, \omega_d) \in \mathbb{E}^d \setminus \{\underline{0}\}$. *Finally, assume that there exist* $N_0 \in \mathbb{N}$, $\tau \in \mathbb{R}_+$, *an unbounded increasing function* $F : \mathbb{N} \to \mathbb{R}_+$, *and a sequence* $(\Lambda_N)_{N \geqslant N_0}$ *of linear forms over* O_K *in* d *variables, such that*

(i) $\limsup_{N\to\infty} F(N+1)/F(N) \leqslant 1$,

(ii) $\log ||\underline{\Lambda}_N|| \leqslant F(N)$ *for every* $N \geqslant N_0$,

(iii) $\log |\Lambda_N(\underline{\omega})| = -(\tau + o(1))F(N)$ *for every large* $N \geqslant N_0$,

where $||\underline{\Lambda}_N||$ *denotes the euclidean (or* l_2*) norm of the coefficient vector of the linear form* Λ_N. *Then the following dimension estimate holds*

$$\dim_K K\omega_1 + \ldots + K\omega_d \geqslant 1 + \tau.$$

Proof. In the case $K = \mathbb{Q}$, i. e. $\mathbb{E} = \mathbb{R}$, this result belongs to Nesterenko [8]. If K is imaginary quadratic, it is Töpfer's Korollar 2 in [10] combined with his remark after Korollar 7 that, in case of such particular algebraic number fields, one may replace the field degree $[K : \mathbb{Q}] = 2$ by 1. ☐

Remark. Very recently Fischler and Zudilin [6] gave a new proof, which even refined in certain cases Nesterenko's main result from [8].

Our proof of Theorem 3 will be prepared by the following quantitative version of Lemma 2, which is taken from [10], Korollar 6. Notice that for $K = \mathbb{Q}$ it can already be found in [3], Korollar 2.

Lemma 3. *Let* $\mathbb{E}$, d *and* $\underline{\omega}$ *be as in Lemma 2. Assume that there exist* $N_0, N_1 \in \mathbb{N}$ *with* $N_0 < N_1$, *unbounded and monotonically increasing functions* $B^*, G, G^*, H^* : \mathbb{N} \to \mathbb{R}_+$, *and, for every* $N \in \{N_0, \ldots, N_1\}$, *a linear form* Λ_N *over* O_K *in* d *variables such that the following conditions hold for all* $N = N_0, \ldots, N_1$:

(i) $H^*(N) \leqslant -\log |\Lambda_N(\underline{\omega})| \leqslant G^*(N)$,

(ii) $G^*(N) + \log \|\underline{\Lambda}_N\| \leqslant G(N)$,

(iii) $\max\big(\log \|\underline{\Lambda}_N\|, G(N+1) - H^*(N) + \log 2d\big) \leqslant B^*(N)$.

If Λ *is a non-trivial linear form over* O_K *in* d *variables, then the following additional condition*

(iv) $G(N_1+1) > (d-2)B^*(N_1) + \max\big(G(N_0), B^*(N_1)\big) + \log \|\underline{\Lambda}\| + \max(0, -\log \|\underline{\omega}\|)$

implies the inequality

$$|\Lambda(\underline{\omega})| > \frac{1}{2}\|\underline{\Lambda}\| \exp(-G(N_1+1)).$$

3. Construction of linear forms

Let again $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be our entire transcendental solution of (1). For the proof of our results, we construct linear forms in 1 and $f^{(\sigma)}(q^{-\mu})$ with $0 \leqslant \sigma < k$, $0 \leqslant \mu < m$ using the integral

$$I(N) := \frac{1}{2\pi i} \oint\limits_{\Gamma(N)} \frac{f(q^G z)dz}{z^L \prod_{j=0}^{M+\beta_N+1}(z-q^j)^{k_j}}, \tag{3}$$

where $L \geqslant 0$, $M > 0$ and G are integers depending on an integer parameter N, β_N is a suitable integer satisfying $0 \leqslant \beta_N < \ell$, $k_j = k$ for $0 \leqslant j \leqslant M$, $k_j = 1$ for $j > M$, and $\Gamma(N)$ is a positively oriented circle: $|z| = R > |q|^{\widetilde{M}}$ with $\widetilde{M} := M + \beta_N + 1$. All these objects will be specified in detail later. Using the residue theorem we get

$$\begin{aligned}
I(N) = &\sum_{\sigma+\sigma_0+\ldots+\sigma_{\widetilde{M}}=L-1} (-1)^{k(M+1)+\widetilde{M}-M} c_\sigma q^{G\sigma - \sum_{j=0}^{\widetilde{M}} j(k_j+\sigma_j)} \prod_{j=0}^{M} \binom{k-1+\sigma_j}{\sigma_j} + \\
&+ \sum_{j=0}^{M} \sum_{\substack{\sigma+\sigma_0+\ldots \\ +\sigma_{\widetilde{M}}=k-1}} q^{G\sigma_j} \frac{f^{(\sigma_j)}(q^{G+j})}{\sigma_j! q^{j(L+\sigma)}} (-1)^\sigma \binom{L-1+\sigma}{\sigma} \prod_{\substack{i=0 \\ i\neq j}}^{\widetilde{M}} \binom{k_i-1+\sigma_i}{\sigma_i} \frac{(-1)^{\sigma_i}}{(q^j-q^i)^{k_i+\sigma_i}} + \\
&+ \sum_{j=M+1}^{\widetilde{M}} f(q^{G+j}) q^{-jL} \prod_{\substack{i=0 \\ i\neq j}}^{\widetilde{M}} (q^j-q^i)^{-k_i},
\end{aligned} \tag{4}$$

where the multiple sums are over all the non-negative σ's satisfying the given conditions.

Now iteration of the functional equation (1) with $Q = q^m$ gives for every $\lambda \in \mathbb{N}_0$

$$\begin{aligned} f(Q^\lambda z) &= f(z) \prod_{\kappa=0}^{\lambda-1} R_0(Q^\kappa z) + \sum_{\kappa=0}^{\lambda-1} R_1(Q^\kappa z) \prod_{\nu=\kappa+1}^{\lambda-1} R_0(Q^\nu z) =: \\ &=: R_{0,\lambda}(z) f(z) + R_{1,\lambda}(z), \end{aligned} \tag{5}$$

and

$$f(Q^{-\lambda} z) = \frac{f(z) - R_{1,\lambda}(Q^{-\lambda} z)}{R_{0,\lambda}(Q^{-\lambda} z)}. \tag{6}$$

These equalities imply

$$\begin{aligned} Q^{\lambda\sigma} f^{(\sigma)}(Q^\lambda z) &= \sum_{\tau=0}^{\sigma} \binom{\sigma}{\tau} R_{0,\lambda}^{(\sigma-\tau)}(z) f^{(\tau)}(z) + R_{1,\lambda}^{(\sigma)}(z), \\ Q^{-\lambda\sigma} f^{(\sigma)}(Q^{-\lambda} z) &= \sum_{\tau=0}^{\sigma} \binom{\sigma}{\tau} f^{(\tau)}(z) \left(\frac{1}{R_{0,\lambda}(Q^{-\lambda} z)} \right)^{(\sigma-\tau)} - \left(\frac{R_{1,\lambda}(Q^{-\lambda} z)}{R_{0,\lambda}(Q^{-\lambda} z)} \right)^{(\sigma)}. \end{aligned}$$

If $G + j = \lambda m - \mu > 0$, $0 \leqslant \mu \leqslant m - 1$, then

$$f^{(\sigma)}(q^{\lambda m - \mu}) = q^{-\lambda m\sigma} \sum_{\tau=0}^{\sigma} \binom{\sigma}{\tau} R_{0,\lambda}^{(\sigma-\tau)}(q^{-\mu}) f^{(\tau)}(q^{-\mu}) + q^{-\lambda m\sigma} R_{1,\lambda}^{(\sigma)}(q^{-\mu}), \tag{7}$$

and if $G + j = -\lambda m - \mu \leqslant 0$, then

$$\begin{aligned} &q^{-\lambda m\sigma} f^{(\sigma)}(q^{-\lambda m - \mu}) = \\ &= \sum_{\tau=0}^{\sigma} \binom{\sigma}{\tau} f^{(\tau)}(q^{-\mu}) \left(\frac{1}{R_{0,\lambda}(q^{-\lambda m} z)} \right)^{(\sigma-\tau)} \Bigg|_{z=q^{-\mu}} - \left(\frac{R_{1,\lambda}(q^{-\lambda m} z)}{R_{0,\lambda}(q^{-\lambda m} z)} \right)^{(\sigma)} \Bigg|_{z=q^{-\mu}}. \end{aligned} \tag{8}$$

If we now replace $f^{(\sigma_j)}(q^{G+j})$ in (4) by the above expressions it becomes clear that $I(N)$ is a linear form in 1 and $f^{(\sigma)}(q^{-\mu})$, $0 \leqslant \sigma < k$, $0 \leqslant \mu < m$, with coefficients from K.

Remark. It should be pointed out that for $L = 0$ the first sum on the right-hand side of (4) vanishes, whereas in case $L > 0$ it does not vanish in general independently of $R_1 \neq 0$ or $R_1 = 0$. Only if $L = 0$ *and* $R_1 = 0$, then $I(N)$ is a linear form solely in the numbers $f^{(\sigma)}(q^{-\mu})$ with $0 \leqslant \sigma < k, 0 \leqslant \mu < m$.

4. Asymptotic evaluation of the integral

According to Lemma 1 in [2], we have the asymptotic equation

$$\log|f|_R = \rho_1 \log^2 R + \rho_2 \log R + \mathrm{O}(1) \tag{9}$$

for every large R, where

$$\rho_1 = \frac{\ell}{2m\log|q|}, \quad \rho_2 = \frac{\log|a_\ell|}{m\log|q|} - \frac{\ell}{2}. \tag{10}$$

Furthermore, Lemma 3 in [2] provides that among ℓ successive Taylor coefficients of f (with large indices) there is at least one satisfying the asymptotic equation

$$\log|c_n| = -\frac{n^2}{4\rho_1} + \frac{\rho_2 n}{2\rho_1} + \mathrm{O}(1), \tag{11}$$

where $\mathrm{O}(1)$ is independent of n.

We now consider the integral (3), where we assume that

$$L = [hN], \quad M = mN, \quad G = [-gM] \tag{12}$$

with a large integer parameter N and real numbers g, h satisfying $0 \leqslant g, h \leqslant 1$. Further, we assume β_N to be chosen in such a way that $n = H := L + (M+1)k + \beta_N$ satisfies equation (11) (hence $c_H \neq 0$) with $\mathrm{O}(1)$ independent of N. With this, we define

$$J(N) := q^{-GH} I(N) = \frac{q^{-GH}}{2\pi i} \oint_{|z|=R} \frac{f(q^G z)dz}{z^{H+1}\prod_{j=0}^{\widetilde{M}}(1-\frac{q^j}{z})^{k_j}},$$

where R is to satisfy $R > |q|^{\widetilde{M}}$. This leads us to

$$J(N) = c_H + \frac{q^{-GH}}{2\pi i} \oint_{|z|=R} \frac{f(q^G z)dz}{z^{H+1}} \cdot \frac{1-\prod_{j=0}^{\widetilde{M}}(1-\frac{q^j}{z})^{k_j}}{\prod_{j=0}^{\widetilde{M}}(1-\frac{q^j}{z})^{k_j}}$$

implying

$$\left|\frac{J(N)}{c_H} - 1\right| = \left|\frac{q^{-GH}}{2\pi i\, c_H} \oint_{|z|=R} \frac{f(q^G z)dz}{z^{H+1}} \cdot \frac{1-\prod_{j=0}^{\widetilde{M}}(1-\frac{q^j}{z})^{k_j}}{\prod_{j=0}^{\widetilde{M}}(1-\frac{q^j}{z})^{k_j}}\right|. \tag{13}$$

As it is easily seen, we obtain on $|z| = R$

$$\prod_{j=0}^{\widetilde{M}} \left| 1 - \frac{q^j}{z} \right| \geqslant \prod_{j=1}^{\widetilde{M}+1} (1 - |q|^{-j}) > \prod_{j=1}^{\infty} (1 - |q|^{-j}) =: \gamma_2$$

assuming $R \geqslant |q|^{\widetilde{M}+1}$, whence, for the same z, we get the inequality

$$\prod_{j=0}^{\widetilde{M}} \left| 1 - \frac{q^j}{z} \right|^{k_j} > \gamma_2^k$$

for the denominator of the integrand in (13). To prepare a good estimate for the numerator in this integrand, we note first that

$$\prod_{j=0}^{\widetilde{M}} \left(1 + \frac{|q|^j}{R} \right) \leqslant \prod_{j=1}^{\widetilde{M}+1} (1 + |q|^{-j}) < \prod_{j=1}^{\infty} (1 + |q|^{-j}) < \gamma_2^{-1}$$

under the latter assumption on R. Furthermore, using the inequality

$$\left| \prod_{\iota} (1 + w_\iota) - 1 \right| \leqslant \prod_{\iota} (1 + |w_\iota|) - 1$$

valid for finite sets of complex numbers w_ι we establish the inequality

$$\left| 1 - \prod_{j=0}^{\widetilde{M}} \left(1 - \frac{q^j}{z} \right)^{k_j} \right| \leqslant \prod_{j=0}^{\widetilde{M}} \left(1 + \frac{|q|^j}{R} \right)^k - 1 = \left(\prod_{j=0}^{\widetilde{M}} \left(1 + \frac{|q|^j}{R} \right) - 1 \right) \sum_{\kappa=0}^{k-1} \prod_{j=0}^{\widetilde{M}} \left(1 + \frac{|q|^j}{R} \right)^{\kappa}$$

on $|z| = R$. As we just saw, the latter sum is bounded from above by $\gamma_3 := \sum_{\kappa=0}^{k-1} \gamma_2^{-\kappa}$ if $|z| = R \geqslant |q|^{\widetilde{M}+1}$. On the other hand, according to Lemma 5(i) in [2], we find

$$0 < \prod_{j=0}^{\widetilde{M}} \left(1 + \frac{|q|^j}{R} \right) - 1 < \frac{\gamma_1 |q|^{\widetilde{M}}}{R}$$

if $R \geqslant \gamma_1 |q|^{\widetilde{M}}$, where $\gamma_1 := 2|q|/(|q| - 1)$.

Suppose from now on that $R \geqslant \max(|q|, \gamma_1)|q|^{\widetilde{M}}$. Then, in view of (13),

$$\left|\frac{J(N)}{c_H} - 1\right| \leqslant \frac{2\pi R}{2\pi R|c_H|} \cdot \frac{|f|_{|q|^G R}}{(|q|^G R)^H} \cdot \frac{\gamma_1\gamma_3|q|^{\widetilde{M}}}{\gamma_2^k R}. \tag{14}$$

Supposing furthermore that R is large enough, so that $|q|^G R$ is also large and we can apply (9) to bound $|f|_{|q|^G R}$ from above in (14), we deduce from (14) and (11) that

$$\log\left|\frac{J(N)}{c_H} - 1\right| \leqslant \frac{H^2}{4\rho_1} - \frac{\rho_2 H}{2\rho_1} + \mathrm{O}(1) + \rho_1 \log^2(|q|^G R) + \rho_2 \log(|q|^G R) - \\ - H\log(|q|^G R) + \widetilde{M}\log|q| - \log R. \tag{15}$$

Since the latter expression is

$$\rho_1\left(\log(|q|^G R) - \frac{H}{2\rho_1}\right)^2 + \rho_2\left(\log(|q|^G R) - \frac{H}{2\rho_1}\right) + \widetilde{M}\log|q| - \log R + \mathrm{O}(1),$$

we derive from (15), by putting

$$R := |q|^{-G}\exp\left(\frac{H}{2\rho_1}\right), \tag{16}$$

our final inequality

$$\log\left|\frac{J(N)}{c_H} - 1\right| \leqslant (M+G)\log|q| - \frac{H}{2\rho_1} + \mathrm{O}(1) = \left(M + G - \frac{m}{\ell}H\right)\log|q| + \mathrm{O}(1). \tag{17}$$

By our choices of L, M and G in (12) and by the hypothesis $(1-g)\ell < h + mk$ we shall introduce in Lemma 4 below, we see that the right-hand side of (17) tends to $-\infty$ as $N \to \infty$.

Note that $|q|^G R$ is large if N is large, cf. (16). Note also that our choice of R in (16) guarantees also that $R \geqslant \max(|q|, \gamma_1)|q|^{\widetilde{M}}$, since $(1-g)\ell < h + mk$. Thus taking (17) into account we get

$$J(N) = (1 + \mathrm{o}(1))c_H,$$

and applying (11) gives us the following asymptotic evaluation of our integral.

LEMMA 4. *Choose* L, M *and* G *as above and assume that* $(1-g)\ell < h+mk$. *Then the following asymptotic equation holds*:

$$|I(N)| = |q|^{-\left(\frac{m}{2\ell}(h+mk)^2+gm(h+mk)\right)N^2+\mathrm{O}(N)}. \tag{18}$$

5. Denominators

Above we saw that

$$I(N) \in K + \sum_{\sigma=0}^{k-1}\sum_{\mu=0}^{m-1} K f^{(\sigma)}(q^{-\mu}).$$

Our next aim is to find an $\Omega(N) \in O_K^*$ such that

$$\Lambda_N := \Omega(N)I(N) \in O_K + \sum_{\sigma=0}^{k-1}\sum_{\mu=0}^{m-1} O_K f^{(\sigma)}(q^{-\mu}). \tag{19}$$

To this effect we first construct $\widetilde{\Omega}(N)$ in such a way that $\widetilde{\Omega}(N)I(N)$ has coefficients in $O_K[q]$, and after that, we multiply away the denominators of the powers of q. As in [5], our $\widetilde{\Omega}(N)$ will be of the form $\Omega_1(N)\Omega_2(N)\Omega_3(N)$, where $\Omega_1(N)$ takes care of the terms coming from the $f^{(\sigma)}(q^{G+j})$, $\Omega_2(N)$ of pure powers of q, and $\Omega_3(N)$ of all the others.

To begin with $\Omega_1(N)$, let s be as in Lemma 1. Because of (8), we are interested in

$$R_{0,\lambda}(Q^{-\lambda}z) = R_{0,\lambda}(q^{-\lambda m}z) = \prod_{\kappa=0}^{\lambda-1} R_0(q^{(\kappa-\lambda)m}z) = \prod_{\kappa=1}^{\lambda} R_0(q^{-\kappa m}z).$$

Clearly,

$$s^{\lambda}\left(\prod_{\kappa=1}^{\lambda} q^{\ell\kappa m}\right) R_{0,\lambda}(Q^{-\lambda}z) =: \widetilde{R}_{0,\lambda}(z)$$

is a polynomial in q and z with integer coefficients, and the same holds for

$$s^{\lambda}\left(\prod_{\kappa=1}^{\lambda} q^{\ell\kappa m}\right) R_{1,\lambda}(Q^{-\lambda}z) =: \widetilde{R}_{1,\lambda}(z). \tag{20}$$

Since

$$\left(\frac{1}{R_{0,\lambda}(Q^{-\lambda}z)}\right)^{(\sigma-\tau)}\Bigg|_{z=q^{-\mu}} = s^{\lambda}\left(\prod_{\kappa=1}^{\lambda} q^{\ell\kappa m}\right)\left(\frac{1}{\widetilde{R}_{0,\lambda}(z)}\right)^{(\sigma-\tau)}\Bigg|_{z=q^{-\mu}},$$

$$\left(\frac{R_{1,\lambda}(Q^{-\lambda}z)}{R_{0,\lambda}(Q^{-\lambda}z)}\right)^{(\sigma)}\Bigg|_{z=q^{-\mu}} = \left(\frac{\widetilde{R}_{1,\lambda}(z)}{\widetilde{R}_{0,\lambda}(z)}\right)^{(\sigma)}\Bigg|_{z=q^{-\mu}},$$

it follows that

$$\left(q^{\mu\ell\lambda}\widetilde{R}_{0,\lambda}(q^{-\mu})\right)^{\sigma+1}$$

works as a denominator for $f^{(\sigma)}(q^{G+j})$ with $G+j=-\lambda m-\mu\leqslant 0$. Since here $\lambda\leqslant gN+1/m$, the product $\Omega_1(N)f^{(\sigma)}(q^{G+j})$ is, for all possible $G+j$, a linear form in 1 and the $f^{(\sigma)}(q^{-\mu})$ with coefficients from $O_K[q]$ if we choose

$$\Omega_1(N)=\begin{cases} s^{\mathrm{O}(N)}q^{\mathrm{O}(N)}, & \text{if } g=0,\\ s^{\mathrm{O}(N)}q^{\mathrm{O}(N)}\displaystyle\prod_{\mu=0}^{m-1}\prod_{\kappa=1}^{gN+\mathrm{O}(1)} R_0^k(q^{-\kappa m-\mu})q^{\ell\kappa mk}, & \text{if } 0<g\leqslant 1.\end{cases}$$

To study $\Omega_2(N)$ and $\Omega_3(N)$, we consider each term in (4). In the first term, we estimate

$$\sum_{j=0}^{\widetilde{M}}(k_j+\sigma_j)j\leqslant\frac{k}{2}M(M+1)+\widetilde{M}(L-\sigma+\ell),$$

and therefore

$$G\sigma-\sum_{j=0}^{\widetilde{M}}(k_j+\sigma_j)j\geqslant-\frac{k}{2}M(M+1)-ML+\mathrm{O}(N).$$

Moreover, by Lemma 1, for all $0\leqslant\sigma\leqslant L$, the expression

$$c_\sigma\prod_{0\leqslant\nu\leqslant L}^{*}s(Q^{\nu}-a_0)$$

becomes a polynomial in $O_K[Q]$ of degree not greater than $\frac{1}{2}L(L+1)$ after omitting the factor corresponding to $\nu=0$ in case $a_0=1$. (We remind that $a_0=R_0(0)$.)

In the second sum on the right-hand side of (4), the following product appears

$$\prod_{\substack{i=0\\ i\neq j}}^{\widetilde{M}}(q^j-q^i)^{k_i+\sigma_i}=q^{j(j-1)k/2+(M-j)jk+(\widetilde{M}-M)j}\cdot\Big(\prod_{i=1}^{j}(q^i-1)\cdot\prod_{i=1}^{M-j}(1-q^i)\Big)^{k-1}\times$$

$$\times\Big(\prod_{i=1}^{j}(q^i-1)\cdot\prod_{i=1}^{\widetilde{M}-j}(1-q^i)\Big)\cdot\Big(\prod_{i=0}^{j-1}q^{i\sigma_i}(q^{j-i}-1)^{\sigma_i}\cdot\prod_{i=j+1}^{\widetilde{M}}q^{j\sigma_i}(1-q^{i-j})^{\sigma_i}\Big). \qquad (21)$$

For the pure powers of q, we note that in this case

$$j\Big(L+\sigma+\frac{k}{2}(j-1)+k(M-j)+\widetilde{M}-M\Big)\leqslant ML+\frac{k}{2}M(M+1)+\mathrm{O}(N).$$

Finally, in the third sum of (4), the following product appears

$$\prod_{\substack{i=0\\ i\neq j}}^{\widetilde{M}}(q^j-q^i)^{k_i}=q^{\frac{k-1}{2}M(M+1)+\frac{1}{2}j(j-1)+(\widetilde{M}-j)j}\prod_{i=j-M}^{j}(q^i-1)^{k-1}\Big(\prod_{i=1}^{j}(q^i-1)\cdot\prod_{i=1}^{\widetilde{M}-j}(1-q^i)\Big). \qquad (22)$$

Furthermore, we have the following inequality

$$j\Big(L+\frac{1}{2}(j-1)+\widetilde{M}-j\Big)+\frac{k-1}{2}M(M+1)\leqslant ML+\frac{k}{2}M(M+1)+\mathrm{O}(N).$$

Note also that the q-binomial coefficient

$$\prod_{i=1}^{M}(q^i-1)\Big/\Big(\prod_{i=1}^{j}(q^i-1)\cdot\prod_{i=1}^{M-j}(q^i-1)\Big)$$

lies in $\mathbb{Z}[q]$. The preceding considerations and (4) imply that we may finally choose

$$\Omega_2(N)=q^{ML+kM(M+1)/2+\mathrm{O}(N)},$$

$$\Omega_3(N)=(k-1)!\,s^{\mathrm{O}(N)}\cdot\prod_{i=1}^{M+\ell}\Phi_i(q)^{k-1}\cdot\prod_{\rho=1}^{M+w}(q^\rho-1)^k,$$

where $w:=\max(t,\ell)$ and Φ_n is the nth cyclotomic polynomial.

Now $\widetilde{\Omega}(N)I(N)$ has coefficients in $O_K[q]$, and next we consider the degree of these coefficients with respect to q. For this purpose, we note first that the following estimate holds:

$$\deg \widetilde{\Omega}(N) \leqslant \frac{1}{2}k\ell m^2 g^2 N^2 + M(kM+L) + \frac{3}{\pi^2}(k-1)M^2 + \mathrm{O}(N\log N).$$

Moreover, in the first multiple sum on the right-hand side of (4), we obtain

$$\sum_{j=0}^{\widetilde{M}}(k_j+\sigma_j)j \geqslant \frac{k}{2}M(M+1).$$

For all $0 \leqslant j \leqslant -G$, the denominator in the second term of (4) contains, by (21), the factor

$$D_j := q^{j(L+kM-(j+1)k/2+\widetilde{M}-M)}\left(\prod_{i=1}^{j}(q^i-1)\prod_{i=1}^{M-j}(1-q^i)\right)^{k-1}\left(\prod_{i=1}^{j}(q^i-1)\prod_{i=1}^{\widetilde{M}-j}(1-q^i)\right)$$

of degree

$$\deg D_j = \frac{k}{2}M(M+1) + \frac{k}{2}j(j-1) + jL + \frac{1}{2}(\widetilde{M}-M)(\widetilde{M}+M+1) \geqslant \frac{k}{2}M(M+1).$$

Finally, we have to estimate the difference of the degrees of the numerator and denominator of the coefficients in the second and in the third terms on the right-hand side of (4) in the case when $j > -G$. We shall show that this difference is also $\leqslant -kM(M+1)/2 + \mathrm{O}(N)$.

For the second term of (4) we use (7) and (21), and note that

$$\begin{aligned}\deg R_{0,\lambda}(q^{-\mu}) - \deg D_j \leqslant{} & \frac{\ell}{2m}\big((G+j)^2 - m(G+j) - \mu(\mu-m)\big) - \\ & - \frac{k}{2}M(M+1) - \frac{k}{2}j(j-1) - jL.\end{aligned}$$

By the hypothesis of Lemma 4, we have $(1-g)\ell < h+km$. Thus, the right-hand side of the inequality just obtained is decreasing in the interval $-G < j \leqslant M$, if N is sufficiently large. Furthermore, we easily see that

$$\deg R_{0,\lambda}(q^{-\mu}) - \deg D_j < -\frac{k}{2}M(M+1) + \mathrm{O}(N)$$

and

$$-\frac{k}{2}M(M+1) > \frac{\ell}{2m}(G+M)^2 - ML - kM^2 + \mathrm{O}(N). \tag{23}$$

For the third sum in (4) we apply (7) and (22). Let

$$E_j := q^{jL+\frac{k-1}{2}M(M+1)-\frac{1}{2}j(j+1)+\widetilde{M}j} \prod_{i=j-M}^{j}(q^i-1)^{k-1} \cdot \Big(\prod_{i=1}^{j}(q^i-1)\cdot\prod_{i=1}^{\widetilde{M}-j}(1-q^i)\Big).$$

Here $M < j \leqslant \widetilde{M}$ and

$$\begin{aligned}\deg R_{0,\lambda}(q^{-\mu}) - \deg E_j \leqslant \frac{\ell}{2m}\big((G+j)^2 - m(G+j) - \mu(\mu-m)\big) - \\ -\frac{k-1}{2}M(M+1) - \frac{k}{2}j(j+1) - jL - \frac{1}{2}\widetilde{M}^2 + j.\end{aligned}$$

Again, by the inequality $(1-g)\ell < h + km$, the right-hand side is decreasing in the interval $M < j \leqslant \widetilde{M}$, if N is sufficiently large, whence

$$\deg R_{0,\lambda}(q^{-\mu}) - \deg E_j \leqslant \frac{\ell}{2m}(G+M)^2 - ML - kM^2 + \mathrm{O}(N) < -\frac{k}{2}M(M+1) + \mathrm{O}(N),$$

where the latter inequality follows from (23).

Combining the above considerations, we establish the following result.

LEMMA 5. *If* $(1-g)\ell < h+km$, *then* $\widetilde{\Omega}(N)I(N)$ *is a linear form in* 1 *and* $f^{(\sigma)}(q^{-\mu})$, $0 \leqslant \sigma < k$, $0 \leqslant \mu < m$, *with coefficients in* $O_K[q]$. *For all* N *sufficiently large, the degrees* (*with respect to* q) *of these coefficient polynomials are bounded from above by*

$$D(N) := \frac{1}{2}k\ell m^2 g^2 N^2 + \frac{k}{2}M^2 + ML + \frac{3}{\pi^2}(k-1)M^2 + \mathrm{O}(N\log N).$$

By Lemma 5, we can set in (19)

$$\Omega(N) = v^{D(N)}\widetilde{\Omega}(N).$$

6. Final proof of Theorems 1 and 2

Within the assumptions of Theorem 1, let us suppose that $(1-g)\ell < h + km$, in order to be able to use the results of Sections 4 and 5. To apply Nesterenko's dimension estimate (Lemma 2) we need an upper bound for $||\underline{\Lambda}_N||$, the l_2-norm of the vector of the coefficients of (19), i. e., of

$$\Lambda_N = \Omega(N)I(N) \in O_K + \sum_{\sigma=0}^{k-1}\sum_{\mu=0}^{m-1} O_K f^{(\sigma)}(q^{-\mu}).$$

Obviously, the absolute values of the coefficients of $f^{(\sigma)}(q^{-\mu})$ are bounded from above by $|u|^{D(N)}e^{O(N)}$. Note that in the second sum in (4) a bound or order $\mathrm{O}(N^k)$ would be enough to take care of summations, binomial coefficients, and derivatives. By Lemma 4, the same also holds for the 'constant' term. Thus, for all sufficiently large N, we get

$$\begin{aligned} &\log ||\underline{\Lambda}_N|| \leqslant \frac{m}{2\ell}\Gamma_0 N^2 \log|u| + \mathrm{O}(N\log N), \\ &\Gamma_0 := km\ell^2 g^2 + km\ell + 2h\ell + 6\pi^{-2}(k-1)m\ell. \end{aligned} \tag{24}$$

By Lemma 4 and by the definition of $\widetilde{\Omega}(N)$, we also have

$$\begin{aligned} \log|\Lambda_N| &= \log(|v|^{D(N)}|\widetilde{\Omega}(N)I(N)|) = D(N)\log|v| + \Big(\frac{1}{2}k\ell m^2 g^2 + \frac{3}{\pi^2}(k-1)m^2 - \\ &\quad - \frac{m}{2\ell}(h+km)^2 + (1-g)m(h+km)\Big)N^2\log|q| + \mathrm{O}(N\log N) = \\ &= -\frac{m}{2\ell}\Big((1-\eta)\big((h+km)^2 + 2\ell g(h+km) - km\ell\big) - \Gamma_0\Big)N^2\log|u| + \\ &\quad + \mathrm{O}(N\log N). \end{aligned} \tag{25}$$

Now Lemma 2 implies the following dimension estimate

$$\dim_K\Big\{K + \sum_{\sigma=0}^{k-1}\sum_{\mu=0}^{m-1} Kf^{(\sigma)}(q^{-\mu})\Big\} \geqslant (1-\eta)\frac{(h+km)^2 + 2\ell g(h+km) - km\ell}{\ell(km\ell g^2 + km + 2h + 6\pi^{-2}(k-1)m)}. \tag{26}$$

Here, for all $0 \leqslant g \leqslant 1$, the right-hand side is an increasing function of h in the interval $0 \leqslant h \leqslant 1$, and therefore we are interested in

$$\frac{(1+km)^2 + 2\ell g(1+km) - km\ell}{\ell(km\ell g^2 + km + 2 + 6\pi^{-2}(k-1)m)}.$$

The maximal value of this quotient in the interval $0 \leqslant g \leqslant 1$ is $C(k, \ell, m)$. Since $C(k, \ell, m) \leqslant 1$ for $\ell \geqslant 1+km$, Theorem 1 holds trivially in this case. If $\ell < 1+km$, then $(1-g)\ell < 1+km$, and we may again use $C(k, \ell, m)$ as the bound. This proves Theorem 1.

We note that Remark 3 after Theorem 1 follows immediately from (26), since the inhomogeneous parts of Theorems 1 through 3 of [9] are obtained by the three choices $h = 0, g = 0; \ell = 1, h = 0, g = 1; g = 0$, respectively.

As we mentioned at the end of Section 3, to make sure that in the homogeneous case $R_1 = 0$ of (1) the integral $I(N)$ becomes a linear form in $f^{(\sigma)}(q^{-\mu})$, $0 \leqslant \sigma < k$, $0 \leqslant \mu < m$, without the 'constant' term, we have to choose $L = 0 \Leftrightarrow h = 0$ (see (12)). Thus, by (26), in this case

$$\dim_K \left\{ \sum_{\sigma=0}^{k-1} \sum_{\mu=0}^{m-1} K f^{(\sigma)}(q^{-\mu}) \right\} \geqslant (1-\eta) \frac{k^2 m + 2\ell g k - k\ell}{\ell(k\ell g^2 + k + 6\pi^{-2}(k-1))}, \tag{27}$$

if $(1-g)\ell < km$. The maximum of the rational function on the right-hand side in the interval $0 \leqslant g \leqslant 1$ equals $\widetilde{C}(k, \ell, m)$. If $\ell \geqslant km$, then $\widetilde{C}(k, \ell, m) \leqslant 1$, and Theorem 2 is true, since $f(1) \neq 0$ by the assumption that $R_0(q^{-j}) \neq 0$ for every integer $j \geqslant m$. If $\ell < km$, then we also have $(1-g)\ell < km$, and we can use $\widetilde{C}(k, \ell, m)$ as a lower bound. Thus, Theorem 2 is proved. We also see that the homogeneous parts of Theorems 1 and 2 of [9] follow from (27) by setting $m = 1$, $g = 0$ and $m = \ell = 1$, $g = 1$, respectively.

By the above considerations, we clearly have $C(k, \ell, m) \geqslant \widetilde{C}(k, \ell, m)$. Further, after some minor calculations, we see that

$$\widetilde{C}(k, \ell, m) > \frac{km}{\ell(1+6\pi^{-2}(1-1/k))} - 1 \quad \text{if} \quad km > \ell\big(1+6\pi^{-2}(1-1/k)\big),$$

and the result is trivially true if $km \leqslant \ell(1+6\pi^{-2}(1-1/k))$. Thus, Remarks 4 and 6 hold.

7. Proof of Theorem 3

Due to the hypothesis of Theorem 3, we can apply our previous assumptions concerning the special situation $k = \ell = 1$, $R_1 = 0$, and $u = q$, $v = 1$, since $q \in O_K$. As explained before, we have to choose again $h = 0$, whence we get that (24) and (25) reduce to

$$\log \|\underline{\Lambda}_N\| \leqslant \frac{m}{2}\Gamma_0 N^2 \log|q| + \mathrm{O}(N\log N) \quad \text{with} \quad \Gamma_0 = m(g^2+1), \tag{28}$$

$$-\log|\Lambda_N(\underline{\omega})| = \frac{m}{2}(\Gamma_1 - \Gamma_0)N^2 \log|q| + \mathrm{O}(N\log N) \quad \text{with} \quad \Gamma_1 = m(m+2g-1). \tag{29}$$

Here $\underline{\omega}$ denotes the vector $\big(f(1), f(q^{-1}), \ldots, f(q^{-(m-1)})\big) \in \mathbb{E}^m$, $\mathbb{E}$ being as in Lemma 2. Since we shall be able later to fix $g \in [0, 1]$ in terms of m in such a way that

$$\Gamma_1 > (m-1)\Gamma_0 \tag{30}$$

(hence $\Gamma_1 > \Gamma_0$), we are now in a position to choose functions G^*, H^*, G, B^* satisfying conditions (i), (ii), (iii) in Lemma 3.

By (29), we first define

$$G^*(N) := \frac{m}{2}(\Gamma_1 - \Gamma_0)N^2 \log|q| + \theta_1 N \log N,$$

$$H^*(N) := \frac{m}{2}(\Gamma_1 - \Gamma_0)N^2 \log|q| - \theta_1 N \log N,$$

where $\theta_1 > 0$ can be suitably fixed, independently of the parameter N. Thus, the condition (i) is satisfied. Next, from (28) and the preceding definition of G^*, we can obtain

$$G^*(N) + \log \|\underline{\Lambda}_N\| \leqslant \frac{m}{2}\Gamma_1 N^2 \log|q| + \theta_2 N \log N$$

if we fix $\theta_2 > \theta_1$ appropriately. Therefore, the choice

$$G(N) := \frac{m}{2}\Gamma_1 N^2 \log|q| + \theta_2 N \log N \tag{31}$$

satisfies the condition (ii). In view of (iii), we first notice that

$$G(N+1) - H^*(N) + \log 2m = \frac{m}{2}\Gamma_0 N^2 \log|q| + (\theta_1 + \theta_2) N \log N + \mathrm{O}(N),$$

and this suggests us to fix $\theta_3 > \theta_1 + \theta_2$ and to choose then

$$B^*(N) := \frac{m}{2}\Gamma_0 N^2 \log|q| + \theta_3 N \log N.$$

Choosing here θ_3 a bit larger if necessary gives $\log ||\underline{\Lambda}_N|| \leqslant B^*(N)$, see (28). Hence the conditions (i), (ii), (iii) of Lemma 3 are satisfied, and we fix N_0 so large that all the four functions G^*, H^*, G, B^* are monotonically increasing for $N \geqslant N_0$.

Given now an arbitrary vector $\underline{\Lambda} = (\lambda_0, \ldots, \lambda_{m-1}) \in O_K^m$ with $|\underline{\Lambda}|$, or equivalently, $||\underline{\Lambda}||$ large enough (note that $|\underline{\Lambda}| \leqslant ||\underline{\Lambda}|| \leqslant \sqrt{m}\,|\underline{\Lambda}|$), we choose N_1 as the smallest integer $> N_0$ satisfying

$$G(N_1+1) > (m-1)B^*(N_1) + \log||\underline{\Lambda}|| + \theta_4, \quad \theta_4 := G(N_0) + \max(0, -\log||\underline{\omega}||). \tag{32}$$

That this is possible can be easily seen from the inequality

$$G(N+1) - (m-1)B^*(N) > \frac{m}{2}(\Gamma_1 - (m-1)\Gamma_0)N^2 \log|q| - \theta_5 N \log N$$

deduced from the above definitions of G and B^*, where we put $\theta_5 := (m-1)\theta_3 - \theta_2$ (> 0). We remind here that we shall finally provide the inequality (30). From the definition of N_1 in (32) and from the monotonicity of B^* we conclude that

$$G(N_1) \leqslant (m-1)B^*(N_1) + \log||\underline{\Lambda}|| + \theta_4,$$

which we slightly weaken to

$$\Gamma N_1^2 \log|q| < \log||\underline{\Lambda}|| + \theta_6 N_1 \log N_1, \quad \Gamma := \frac{m}{2}(\Gamma_1 - (m-1)\Gamma_0) \tag{33}$$

with θ_6 a bit larger than θ_5. Notice here that N_1 becomes large if and only if so does $||\underline{\Lambda}||$.

Now all the conditions of Lemma 3 are satisfied, so we can estimate from below the absolute value of the linear form

$$\Lambda(\underline{\omega}) = \lambda_0 f(1) + \ldots + \lambda_{m-1} f(q^{-(m-1)})$$

with the given $\underline{\Lambda} \in O_K^m$ as follows, where we first use (31).

$$\begin{aligned} \log|\Lambda(\underline{\omega})| &> -\log 2 + \log||\underline{\Lambda}|| - \frac{m}{2}\Gamma_1(N_1+1)^2\log|q| - \theta_2(N_1+1)\log(N_1+1) > \\ &> \log||\underline{\Lambda}|| - \frac{m}{2}\Gamma_1 N_1^2 \log|q| - \theta_3 N_1 \log N_1 > \left(1 - \frac{m\Gamma_1}{2\Gamma}\right)\log||\underline{\Lambda}|| - \theta_7 N_1 \log N_1. \end{aligned} \tag{34}$$

Here we applied (33) for the latter inequality, and we put $\theta_7 := \theta_3 + \theta_6/(1-(m-1)\Gamma_0/\Gamma_1)$. From (33) one can easily deduce that, given any $\varepsilon \in \mathbb{R}_+$, there exists a constant $C \in \mathbb{R}_+$ such that $\theta_7 N_1 \log N_1 \leqslant \varepsilon \log ||\underline{\Lambda}||$, if $||\underline{\Lambda}|| \geqslant C$, whence by (34) and by the definition of Γ in (33) we get the inequality

$$\log|\Lambda(\underline{\omega})| > -\left(\frac{m-1}{(\Gamma_1/\Gamma_0)-(m-1)}+\varepsilon\right)\log||\underline{\Lambda}|| \tag{35}$$

for such $\underline{\Lambda}$. Clearly, our last aim is to maximize $\Gamma_1/\Gamma_0 = (m+2g-1)/(g^2+1)$ by choosing $g \in [0,1]$ suitably. An easy computation shows that this quotient is maximal in $[0,1]$ if and only if

$$g = \frac{1}{2}\left(\sqrt{(m-1)^2+4}-(m-1)\right) = \frac{2}{\sqrt{(m-1)^2+4}+(m-1)},$$

and, for this g, the quotient under consideration becomes

$$\frac{\Gamma_1}{\Gamma_0} = \frac{1}{2}\left(\sqrt{(m-1)^2+4}+(m-1)\right),$$

so that the validity of (30) is obvious. Inserting this in the right-hand side of (35) leads to

$$|\Lambda(\underline{\omega})| > ||\underline{\Lambda}||^{-(\psi(m)+\varepsilon)} \geqslant (\sqrt{m}\,|\underline{\Lambda}|)^{-(\psi(m)+\varepsilon)}$$

with $|\underline{\Lambda}|$ and $\psi(m)$ as defined in Theorem 3, which is therefore proved.

Acknowledgements. The authors are indebted to the anonymous referee for his/her careful reading of the manuscript and for suggestions to improve the readability of the paper.

Bibliography

1. **P. B. Borwein**, *On the irrationality of* $\sum(1/(q^n+r))$, J. Number Theory **37** (1991), 253–259.
2. **P. Bundschuh**, *Arithmetical results on certain q-series, I*, Int. J. Number Theory **4** (2008), 25–43.
3. **P. Bundschuh, T. Töpfer**, *Über lineare Unabhängigkeit*, Monatsh. Math. **117** (1994), 17–32.

4. **P. Bundschuh, K. Väänänen**, *An application of Nesterenko's dimension estimate to p-adic q-series*, Int. J. Number Theory **7** (2010), 431–447.
5. **P. Bundschuh, K. Väänänen**, *Arithmetical results on certain q-series, II*, Int. J. Number Theory **5** (2009), 1231–1245.
6. **S. Fischler, W. Zudilin**, *A refinement of Nesterenko's linear independence criterion with applications to zeta values*, Math. Ann. **347** (2010), 739–763.
7. **T. Matala-aho**, *On q-analogues of divergent and exponential series*, J. Math. Soc. Japan **61** (2009), 291–313.
8. **Yu. V. Nesterenko**, *On the linear independence of numbers* (in Russian), Vestnik Moskov. Univ. Ser. I Mat. Mekh. **40** (1985), 46–49; Engl. transl.: Moscow Univ. Math. Bull. **40** (1985), 69–74.
9. **T. Töpfer**, *Arithmetical properties of functions satisfying q-difference equations*, Analysis (Munich) **15** (1995), 25–49.
10. **T. Töpfer**, *Über lineare Unabhängigkeit in algebraischen Zahlkörpern*, Results Math. **25** (1994), 139–152.
11. **K. Väänänen**, *Remarks on linear independence of certain q-series*, Fundam. Prikl. Mat. (to appear).

PETER BUNDSCHUH

Mathematisches Institut
Universität zu Köln
Weyertal 86–90
50931 Köln, Germany
pb@math.uni-koeln.de

KEIJO VÄÄNÄNEN

Department of Mathematical Sciences
University of Oulu
P. O. Box 3000
90014 Oulu, Finland
kvaanane@sun3.oulu.fi